Analysis of Saddle-Node Bifurcation in a Small Discrete Hopfield Neural Network

R. Marichal, J.D. Piñeiro, E. González and J. Torres

***Abstract*— A simple two-neuron model of a discrete Hopfield neural network is considered. The local stability is analyzed with the associated characteristic model. In order to study the dynamic behavior, the Fold bifurcation is examined. In the case of two neurons, one necessary condition for yielding the Fold bifurcation is found. In addition, the stability and direction of the fold bifurcation are determined by applying the normal form theory and the center manifold theorem.**

***Index Terms*— Nonlinear System, Neural Networks, Fold Bifurcation, Fixed Points.**

I. Introduction

The purpose of this paper is to present some results on the analysis of the dynamics of a discrete recurrent neural network. The particular network in which we are interested is the Hopfield network, also known as a Discrete Hopfield Neural Network in [1]. Its state evolution equation is

$$x_i(k+1) = \sum_{n=1}^{N} w_{in} f(x_i(k)) + \sum_{m=1}^{M} w'_{im} u_m(k) + w''_i \quad (1)$$

where

$x_i(k)$ is the ith neuron output.

$u_m(k)$ is the mth input of the network.

w_{in}, w'_{im} are the weight factors of the neuron outputs, network inputs and w''_i is a bias weight.

N is the neuron number.

M is the input number.

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$f(\cdot)$ is a continuous, bounded, monotonically increasing function, such as the hyperbolic tangent.

This model has the same dynamic behavior as the Williams-Zipser neural network. The relationship between the Williams-Zipser states and Hopfield states is

$$X_h = WX_{W-Z}$$

where

X_h are Hopfield states.

X_{W-Z} are the Williams-Zipser states.

W is the weight matrix without the bias and input weight factor.

We will consider the Williams-Zipser model in order to simplify the mathematical calculations.

The neural network presents different classes of equivalent dynamics. A system will be equivalent to another if its trajectories exhibit the same qualitative behavior. This is made mathematically precise in the definition of topological equivalence [2]. The simplest trajectories are those that are equilibrium or fixed points that do not change in time. Their character or stability is given by the local behavior of nearby trajectories. A fixed point can attract (sink), repel (source) or have directions of attraction and repulsion (saddle) of close trajectories [3]. Next in complexity are periodic trajectories, quasi-periodic trajectories or even chaotic sets, each with its own stability characterization. All of these features are similar in a class of topologically equivalent systems. When a system parameter is varied, the system can reach a critical point at which it is no longer equivalent. This is called a bifurcation, and the system will exhibit new behavior. The study of how these changes can be carried out will be another powerful tool in the analysis.

With respect to discrete recurrent neural networks as systems, several results on their dynamics are available in the literature. The most general result is derived using the Lyapunov stability theorem in [4], and establishes that for a symmetric weight matrix, there are only fixed points and period two limit cycles, such as stable equilibrium states. It also gives the conditions under which only fixed-point attractors exist. More recently Cao [5] proposed other, less restrictive and more complex, conditions. In [6], chaos is found even in a simple two-neuron network in a specific weight configuration by demonstrating its equivalence with a 1-dimension chaotic system (the logistic map). In [7], the same author describes another interesting type of trajectory, the quasi-periodic orbits. These are closed orbits with

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irrational periods that appear in complex phenomena, such as frequency-locking and synchronization, which are typical of biological networks. In the same paper, conditions for the stability of these orbits are given. These can be simplified, as we shall show below.

Passeman [8] obtains some experimental results, such as the coexisting of the periodic, quasi-periodic and chaotic attractors. Additionally, [9] gives the position, number and stability types of fixed points of a two-neuron discrete recurrent network with nonzero weights. In [17] are given some results of the Fold bifurcation in discrete recurrent neural networks.

There are some works that analyze the Hopfield continuous neural networks [10, 11], like [12, 13, 14, 15]. These papers show the stability of Hopf-bifurcation with two delays.

Firstly, we analyze the number and stability-type characterization of the fixed points. We then continue with an analysis of the fold bifurcation. Finally, the simulations are shown and conclusions are given.

II. DETERMINATION OF FIXED POINTS

For the sake of simplicity, we studied the two-neuron network. This allows for an easy visualization of the problem. In this model, we considered zero inputs so as to isolate the dynamics from the input action. Secondly, and without loss of generality with respect to dynamics, we used zero bias weights. The activation function is the hyperbolic tangent.

With these conditions, the network mapping function is

$$\begin{aligned} x_1(k+1) &= \tanh(w_{11}x_1(k) + w_{12}x_2(k)) \\ x_2(k+1) &= \tanh(w_{21}x_1(k) + w_{22}x_2(k)) \end{aligned} \quad (2)$$

where $x(k)$ and $y(k)$ are the neural output of step k.

The fixed points are solutions of the following equations

$$\begin{aligned} x_{1,p} &= \tanh(w_{11}x_{1,p} + w_{12}x_{2,p}) \\ x_{2,p} &= \tanh(w_{21}x_{1,p} + w_{22}x_{2,p}) \end{aligned} \quad (3)$$

The point $(0, 0)$ is always a fixed point for every value of the weights. The number of fixed points is odd because for every fixed point $(x_{1,p}, x_{2,p})$, $(-x_{1,p}, -x_{2,p})$ is also a fixed point.

To graphically determine the configuration of fixed points, we redefine the above equations as

$$\begin{aligned} x_{2,p} &= \frac{a\tanh(x_{1,p}) - w_{11}x_{1,p}}{w_{12}} = F(x_{1,p}, w_{11}, w_{12}) \\ x_{1,p} &= \frac{a\tanh(x_{2,p}) - w_{22}x_{2,p}}{w_{21}} = F(x_{2,p}, w_{22}, w_{21}) \end{aligned} \quad (4)$$

Depending on the diagonal weights, there are two qualitative behavior functions. We are going to determine the number of fixed points using the graphical representation of the above equations (4). First, we can show that the graph of the F function has a maximum and a minimum if $w_{ii}>1$ or, if the opposite condition holds, is like the hyperbolic arctangent function.

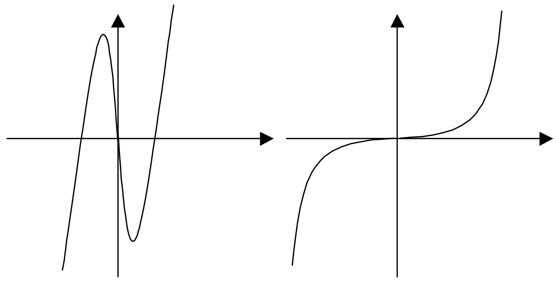

Fig. 1. The two possible behaviors of the F function. The left figure corresponds to the respective diagonal weight lower than unity. The right shows the opposite condition**.**

The combination of these two possibilities with another condition on the ratio of slopes at the origin of the two curves (4) gives the number of fixed points. The latter condition can be expressed as

$$|W| = w_{11} + w_{22} - 1$$

where $|W|$ is the weight matrix determinant.

If $w_{11}>1$, $w_{22}>1$ and $|W|>w_{11}+w_{22}-1$, then there can exist nine, seven or five fixed points. When this condition fails, there are three fixed points.

When a diagonal weight is less than one, there can be three or one fixed points.

III. LOCAL STABILITY ANALYSIS

In the process below, a two-neuron neural network is considered. It is usual for the activation function to be a sigmoid function or a tangent hyperbolic function.

Considering the fixed point equation (3), the elements of the Jacobian matrix at the fixed point (x, y) are

$$J = \begin{bmatrix} w_{11} f'(x_1) & w_{12} f'(x_1) \\ w_{21} f'(x_2) & w_{22} f'(x_2) \end{bmatrix}$$

The associated characteristic equation of the linearized system evaluated at the fixed point is

$$\lambda^2 - [w_{11}f'(x_1) + w_{22}f'(x_2)]\lambda + |W| f'(x_1) f'(x_2) = 0 \quad (5)$$

where w_{11},w_{22} and $|W|$ are the diagonal elements and the determinant of the matrix weight, respectively.

We can define new variables

$$\sigma_1 = \frac{w_{11} f'(x_1) + w_{22} f'(x_2)}{2}$$

$$\sigma_2 = |W| f'(x_1) f'(x_2)$$

The eigenvalues of the characteristic equation (5) are defined as

$$\lambda_{\pm} = \sigma_1 \pm \sqrt{\sigma_1^2 - \sigma_2}$$

The Fold bifurcation appears when two complex conjugate eigenvalues reach the unit circle. It is easy to show that this limit condition is

$$\lambda = 1$$

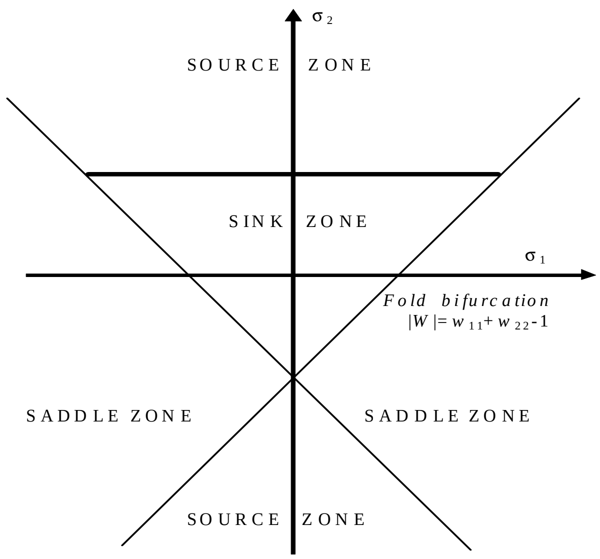

Fig. 2. The stability regions and the fold bifurcation line at the fixed point (*0, 0*).

The boundaries between the regions shown in Fig. 2 are the bifurcations. At these limit zones the fixed point changes its character. The fold bifurcation is represented by the line $|W|=w_{11}+w_{22}-1$ in Fig. 2.

IV. FOLD BIFURCATION DIRECTION

In order to determine the direction and stability of the fold bifurcation, it is necessary to use the center manifold theory [2]. The center manifold theory demonstrates that the mapping behavior in the bifurcation is equivalent to the complex mapping below:

$$\tilde{u} = u + a(0)u^2 + c(0)u^3 + O(\|u\|^4) \qquad (6)$$

The parameters $a(0)$ and $b(0)$ are [2]

$$a(0) = \frac{1}{2}\langle p, B(q,q)\rangle \qquad (7)$$

$$c(0) = \frac{1}{6}\langle p, C(q,q,q)\rangle - \frac{1}{2}\langle p, B(q,(A-E)^{-1}B(q,q))\rangle \qquad (8)$$

where E is the identity matrix, B and C are the second and third derivative terms of the Taylor development, respectively, J is the Jacobian matrix, the notation *INV* and <.,.> represent the inverse matrix and scalar product, respectively, and p, q are the eigenvector Jacobian matrix and its transpose, respectively. These vectors satisfy the normalization condition

$$\langle p, q\rangle = 1 \ .$$

The above coefficients are evaluated for the critical parameter of the system where the bifurcation takes place. In order to simplify the analytical calculation, only the quadratic term in equation (6) is considered. For the case where $a(0)$ is zero, it is necessary to calculate the parameter $c(0)$ associated with the third term of the Taylor development, and the Fold bifurcation is known as the Pitchfork bifurcation [2]. The $a(0)$ sign determines the bifurcation direction. When $a(0)$ is negative, the stable fixed point disappears and is replaced by two additional fixed points (saddle and source). In the opposite case, $a(0)$ positive, an unstable fixed point disappears and two additional unstable fixed points appear.
In the neural network mapping, p and q are

$$q = \frac{d}{e+d}\left\{\frac{e}{w_{21}X_{2,0}}, -1\right\} \qquad (9)$$

$$p = \left\{\frac{e}{w_{12}X_{1,0}}, -1\right\} \qquad (10)$$

where

$$d = w_{11}X_{1,0} - 1$$
$$e = w_{22}X_{2,0} - 1$$
$$X_{1,0} = 1 - x_{1,0}^2$$
$$X_{2,0} = 1 - x_{2,0}^2$$

$x_{1,0}$ and $x_{2.0}$ are the fixed point coordinates where the bifurcation appears.
The Taylor development terms are

$$\begin{aligned} B_i(q,q) &= \sum_{j,k=1}^{2} \frac{\partial f_i}{\partial x_j \partial x_k} q_j q_k \\ &= f''(0)\sum_{j,k=1}^{2} \delta_{ik} w_{ij} w_{ik} q_j q_k \\ &= f''(0)\sum_{j=1}^{2} w_{ij}^2 q_j^2 {}_j \end{aligned} \qquad (11)$$

$$\begin{aligned} C_i(q,q,q) &= \sum_{j,k,l=1}^{2} \frac{\partial f_i}{\partial x_j \partial x_k \partial x_l} q_j q_k q_l \\ &= f'''(0)\sum_{j,k,l=1}^{2} \delta_{ik}\delta_{il} w_{ij} w_{ik} w_{il} q_j q_k q_l \\ &= f'''(0)\sum_{j=1}^{2} w_{ij}^3 q_j^2 q_j \end{aligned} \qquad (12)$$

where δ_{ij} is the Kronecker delta.

In order to determine the parameters $a(0)$ and $c(0)$, it is necessary to calculate the second and third derivates of the mapping (1) given by equations (11) and (12), respectively.

$$\frac{\partial f_i}{\partial x_j \partial x_k} = -2x_i(1-x_i^2)w_{ij}w_{ik}$$
$$\frac{\partial f_i}{\partial x_j \partial x_k \partial x_l} = 2(1-x_i^2)(3x_i^2-1)w_{ij}w_{ik}w_{il}$$

The Taylor development terms are then

$$B_i(a,b) = -2\sum_{j,k=1}^{2} x_i(1-x_i^2)w_{ij}w_{ik}a_jb_k$$
$$C_i(a,b,c) = 2\sum_{j,k,l=1}^{2}(1-x_i^2)(3x_i^2-1)w_{ij}w_{ik}w_{il}a_jb_kc_l .$$

Taking into account the previous equations and the q autovector equation (9)

$$\mathbf{B}(q,q) = -2\begin{bmatrix} \dfrac{x_{1,0}X_{1,0}w_{12}^2}{(e+d)^2} \\ \dfrac{y_{2,0}X_{2,0}w_{12}^2}{(e+d)^2} \end{bmatrix}$$

$$\mathbf{C}(q,q,q) = 2\begin{bmatrix} \dfrac{2X_{1,0}(3x_{1.0}^2-1)w_{12}^3}{d^3} \\ \dfrac{(1-3x_{2,0}^2)}{X_{2,0}^2} \end{bmatrix} .$$

In the remainder of the section, we will distinguish between the Pitchfork bifurcation, associated with the zero fixed point, and Fold bifurcation.

A. Pitchfork Bifurcation at zero fixed point

Firstly, it can be shown in equation (3) that the zero is always a fixed point. In this case, the B coefficient, given by the expression (11), is always zero. The normal form [2] is redefined as a Pitchfork bifurcation, that is:

$$u(k+1) = u(k) + c(0)u(k)^3 + o(u(k)^4)$$

where $c(0)$ is redefined as

$$c(0) = \frac{1}{6}\langle p, C(q,q,q)\rangle$$

because

$$B(a,b) \equiv 0 .$$

Replacing the expressions for $C(q,q,q)$, q and p given by equations (12), (9) and (10), respectively, and evaluating them at the zero fixed point, the previous $c(0)$ coefficient is

$$c(0) = \frac{1}{6}\langle p, C(q,q,q)\rangle = \frac{w_{12}^2(w_{22}-1)+(w_{11}-1)^3}{3(1-w_{11})^2(2-w_{11}-w_{22})}$$

The previous expression is not defined for the following cases

a) $w_{11} = 1$

b) $w_{11} + w_{22} = 2$.

In this paper we only consider condition a) since condition b) shows the presence of another bifurcation, known as Neimark-Sacker and which is analyzed in another paper [16].

Taking into account condition a) and the bifurcation parameter equation

$$|W| = w_{11} + w_{22} - 1$$

then

$$w_{12}w_{21} = 0 .$$

In this particular case, the eigenvalues match with the diagonal elements of the weight matrix

$$\lambda_1 = w_{11}$$
$$\lambda_2 = w_{22} .$$

The new q and p eigenvectors are given by

$$q = \{1,0\}$$
$$p = \left\{1, -\frac{w_{12}}{w_{22}-1}\right\}$$

The $c(0)$ coefficient is

$$c(0) = \frac{1}{6}\langle p, C(q,q,q)\rangle = -\frac{1}{3}w_{11}^3 = -\frac{1}{3} .$$

Therefore, in this particular case, the coefficient of the normal form c (0) is negative, a stable fixed point becomes a saddle fixed point and two additional stable symmetrical fixed points appear.

B. Fold Bifurcation

In the normal form (6), the fixed point is assumed to be zero. In general, the normal form of a fixed point different from zero is

$$\eta(k+1) = \beta + [1+\lambda]\eta(k) + a(0)\eta(k)^2 + o(\eta(k)^3)$$

where

$$\beta(w_{21}) = |a(0)|\frac{\partial u_0}{\partial w_{21}}(w_{21} - w_{21}^{+}) + O(|w_{21} - w_{21}^{+}|^2) \qquad (13)$$

and

$$u_0 = \langle p, x_0 \rangle = x_{1,0}p_1 + x_{2,0}p_2$$
$$\lambda = eigenvalue - 1$$

w_{21}^{+} is the parameter where the bifurcation is produced. $x_{1,0}$ and $x_{2,0}$ are the coordinates of the fixed point.

In order to determine the direction variation of β, it is necessary to calculate the partial derivative given by equation

$$\frac{\partial u_0}{\partial w_{21}} = -\frac{e x_0}{w_{12} w_{21} X_0}$$

The normal form parameter $a(0)$ given by the equation (7) is

$$a(0) = \frac{1}{2}\langle p, B(q,q) \rangle = \frac{\left(d^2 x_{2,0} - e w_{12} x_{1,0} X_{2,0}\right)}{2(e+d)^2 X_{2,0}} \quad .$$

V. SIMULATIONS

In order to examine the results obtained, two examples have been considered. The first simulation shows the Pitchfork bifurcation, Fig. 3. The Pitchfork bifurcation is produced by the diagonal element weight matrix w_{11}. Fig. 3.a shows the dynamic configuration before the bifurcation is produced, with only one stable fixed point. Subsequently, when the bifurcation is produced, Fig. 3.b, two additional stable fixed points appear and the zero fixed point changes its stability from stable to unstable (the normal form coefficient c is negative). The second simulation, Fig. 4, shows the Fold bifurcation, which takes the non-diagonal weight matrix element w_{12} as the bifurcation parameter. Fig. 4.a. shows the dynamic configuration before the bifurcation is produced, with only one stable fixed point. When the bifurcation is produced, Fig. 4.b, four new stable fixed points appear (two stable and two saddle) and the zero fixed point disappears (the normal form coefficient a is negative).

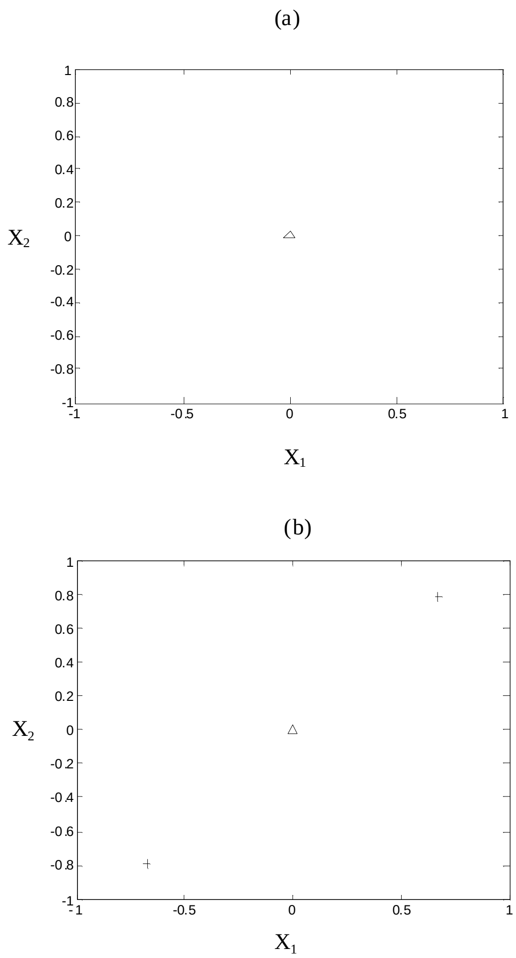

Fig. 3. The dynamic behavior when the Pitchfork bifurcation is produced. + and △ are the saddle and source fixed points, respectively. (a): w_{11}=0.9, w_{12}=0.1, w_{21}=1 and w_{22}=0.5 ; (b): w_{11}=1.1, w_{12}=0.1, w_{21}=1 and w_{22}=0.5 .

VI. CONCLUSION

In this paper we considered the Hopfield discrete two-neuron network model. We discussed the number of fixed points and the type of stability. We showed the bifurcation Fold direction and the dynamical behavior associated with the bifurcation.

The two-neuron networks discussed above are quite simple, but they are potentially useful since the complexity found in these simple cases might be carried over to larger Hopfield discrete neural networks. There exists the possibility of generalizing some of these results to higher dimensions and of using them to design training algorithms that avoid the problems associated with the learning process.

(a)

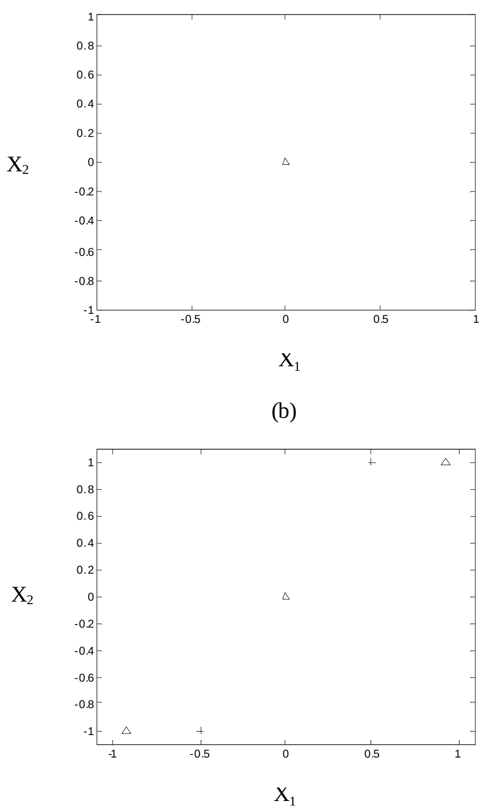

Fig. 4. The dynamic behavior when the Fold bifurcation is produced. + and △ are the saddle and source fixed points, respectively. (a): w_{11}=2.5, w_{12}=-1, w_{21}=4 and w_{22}=3 ; (b): w_{11}=2.5, w_{12}=-0.7, w_{21}=4 and w_{22}=3 .

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