# Two Fuzzy Approaches for Multiobjective Stochastic Programming and Multiobjective Fuzzy Random Programming Through a Probability Maximization Model

Hitoshi Yano and Kota Matsui

***Abstract*—In this paper, two kinds of fuzzy approaches are proposed for not only multiobjective stochastic linear programming problems, but also multiobjective fuzzy random linear programming problems through a probability maximization model. In a probability maximization model, it is necessary for the decision maker to specify permissible values of objective functions in advance, which have a great influence on the corresponding distribution function values. In our proposed methods, the decision maker does not specify permissible values of objective functions, but sets his/her membership functions for permissible values. By assuming that the decision maker adopts the fuzzy decision as an aggregation operator of fuzzy goals for not only the permissible objective levels but also the permissible probability levels, a satisfactory solution of the decision maker is easily obtained based on linear programming technique. Two kinds of numerical examples are illustrated to show the feasibility of the proposed methods.**

***Index Terms*—multiobjective stochastic linear programming, multiobjective fuzzy random linear programming, fuzzy decision, a probability maximization model.**

## I. Introduction

In the real world decision making situations, we often have to make a decision under uncertainty. In order to deal with decision problems involving uncertainty, stochastic programming approaches and fuzzy programming approaches have been developed. In stochastic programming approaches [1],[2],[4],[7], two stage problems and chance constrained programming models have been investigated in various ways, and they were extended to multiobjective stochastic programming problems [13],[17]. In fuzzy programming approaches, various types of fuzzy programming problems have been formulated and investigated [10],[12],[21]. As a natural extension, multiobjective fuzzy programming technique first proposed bu Zimmermann [20], and many methods have been proposed [14],[21].

From a different point of view, mathematical programming problems with fuzzy random variables have been proposed [8],[11],[18], whose concept includes both probabilistic uncertainty and fuzzy one simultaneously. Since such fuzzy random programming problems are usually ill-defined, it is necessary to utilize not only stochastic programming technique but also fuzzy programming technique to construct a decision making model.

H. Yano is with Graduate School of Humanities and Social Sciences, Nagoya City University, Nagoya, 467-8501, Japan, e-mail: yano@hum.nagoya-cu.ac.jp

K. Matsui is with Department of Computer Science and Mathematical Informatics, Graduate School of Information Science, Nagoya University, Nagoya, 464-8601, Japan, e-mail: matsui@math.cm.is.nagoya-u.ac.jp

Recently, in order to deal with probabilistic uncertainty and fuzzy one simultaneously, the hybrid approaches of stochastic programming and fuzzy programming have been proposed [6]. Especially, Sakawa et al. [15], [16] proposed an interactive method for multiobjective fuzzy linear programming problem with random variable coefficients. Katagiri et al. [9] proposed an interactive method for multiobjective linear programming problem with fuzzy random variable coefficients. They also adopted a probability maximization model to transform stochastic programming problems into well-defined mathematical programming ones. However, using a probability maximization model, it is necessary that, in advance, the decision maker specifies permissible levels for objective functions in his/her subjective manner. It seems to be very difficult to specify such values in advance, because there exist conflicts among permissible levels and the corresponding distribution function values.

From such a point of view, in this paper, assuming that the decision maker adopts the fuzzy decision to integrate membership functions, two types of fuzzy approaches [19] are proposed for both multiobjective fuzzy linear programming problem with random variable coefficients and fuzzy random variable coefficients. In section II, a fuzzy approach is proposed for multiobjective stochastic linear programming problems through a probability maximization model, where the coefficients of the objective functions are random variables. Section III provides a numerical example to demonstrate the proposed fuzzy approach for multiobjective stochastic linear programming problems. In section IV, a fuzzy approach is proposed for multiobjective fuzzy random linear programming problems through a probability maximization model, where the coefficients of the objective functions are fuzzy random variables. Section V provides a numerical example to demonstrate the proposed fuzzy approach for multiobjective fuzzy random linear programming problems. Finally, in section VI, we conclude this paper.

## II. A Fuzzy Approach for Multiobjective Stochastic Linear Programming Problems

In this section, we focus on multiobjective programming problem involving random variable coefficients in objective functions. Such a problem can be formally formulated as follows.

**[MOSP1]**

$$\min_{\boldsymbol{x}\in X} \overline{\boldsymbol{z}}(\boldsymbol{x}) = (\overline{z}_1(\boldsymbol{x}), \cdots, \overline{z}_k(\boldsymbol{x})) \quad (1)$$

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where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ is an $n$ dimensional decision variable column vector, $\bar{z}_i(\boldsymbol{x}) = \bar{\boldsymbol{c}}_i\boldsymbol{x} + \bar{\alpha}_i, i = 1, \cdots, k$, are objective functions involving random variable coefficients, $\bar{\boldsymbol{c}}_i$ is a $n$ dimensional random variable row vector expressed by $\bar{\boldsymbol{c}}_i = \boldsymbol{c}_i^1 + \bar{t}_i\boldsymbol{c}_i^2$, where $\bar{t}_i$ is a random variable, $\bar{\alpha}_i$ is a random variable row vector expressed by $\bar{\alpha}_i = \bar{\alpha}_i^1 + \bar{t}_i\alpha_i^2$. In the following, we assume that $T_i(\cdot)$ is a distribution function of a random variable $\bar{t}_i$, which is strictly monotone increasing and continuous. $X$ is a linear constraint set with respect to $\boldsymbol{x}$.

Since MOSP1 contains random variable coefficients in objective functions, mathematical programming techniques can not be directly applied. In order to deal with such multiobjective stochastic programming problems, we make use of a probability maximization model, which aims to maximize the probability that each objective function $\bar{z}_i(\boldsymbol{x})$ is less than or equal to a certain permissible objective level $f_i$. Such a probability $p_i(\boldsymbol{x}, f_i)$ can be defined as follows.

$$p_i(\boldsymbol{x}, f_i) \stackrel{\text{def}}{=} \Pr(\omega \mid \bar{z}_i(\boldsymbol{x}, \omega) \leq f_i), i = 1, \cdots, k \qquad (2)$$

where $\Pr(\cdot)$ denotes a probability measure, $\omega$ is an event, and $\bar{z}_i(\boldsymbol{x}, \omega)$ is a realization of the random objective function $\bar{z}_i(\boldsymbol{x})$ under the occurrence of each elementary event $\omega$. The decision maker subjectively specifies a certain permissible objective level $f_i$ for each objective function $\bar{z}_i(\boldsymbol{x}, \omega)$. Let us denote a $k$ dimensional vector of certain permissible objective levels as $\boldsymbol{f} = (f_1, \cdots, f_k)$.

Then, MOSP1 can be transformed into the traditional multiobjective programming problem MOP1($\boldsymbol{f}$), where probability functions $p_i(\boldsymbol{x}, f_i), i = 1, \cdots, k$ are adopted as objective functions instead of $\bar{z}_i(\boldsymbol{x}, \omega)$, and each of them is maximized.

**[MOP1($\boldsymbol{f}$)]**

$$\max_{\boldsymbol{x} \in X} (p_1(\boldsymbol{x}, f_1), \cdots, p_k(\boldsymbol{x}, f_k)) \qquad (3)$$

Under the assumption that $\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2 \geq 0, i = 1, \cdots, k$ for any $\boldsymbol{x} \in X$, using distribution functions $T_i(\cdot), i = 1, \cdots, k$ we can rewrite the objective function $p_i(\boldsymbol{x}, f_i)$ as the following form.

$$\begin{aligned} p_i(\boldsymbol{x}, f_i) &= \Pr(\omega \mid z_i(\boldsymbol{x}, \omega) \leq f_i) \\ &= \Pr(\omega \mid \boldsymbol{c}_i(\omega)\boldsymbol{x} + \alpha_i(\omega) \leq f_i) \\ &= \Pr\left(\omega \mid t_i(\omega) \leq \frac{f_i - (\boldsymbol{c}_i^1\boldsymbol{x} + \alpha_i^1)}{\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2}\right) \\ &= T_i\left(\frac{f_i - (\boldsymbol{c}_i^1\boldsymbol{x} + \alpha_i^1)}{\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2}\right) \end{aligned}$$

In order to deal with MOP1($\boldsymbol{f}$), we consider the feasible region $P(\boldsymbol{f}) = \{(p_1(\boldsymbol{x}, f_1), \cdots, p_k(\boldsymbol{x}, f_k) \in \mathrm{R}^k \mid \boldsymbol{x} \in X\}$. In the feasible region $P(\boldsymbol{f})$, we can define Pareto optimal solution to MOP1($\boldsymbol{f}$).

**Definition 1.**

$\boldsymbol{x}^* \in X$ is said to be a Pareto optimal solution to MOP1($\boldsymbol{f}$), if and only if there does not exist another $\boldsymbol{x} \in X$ such that $p_i(\boldsymbol{x}, f_i) \geq p_i(\boldsymbol{x}^*, f_i)$, $i = 1, \cdots, k$, with strict inequality holding for at least one $i$.

Sakawa et al. [15] formulated a probability maximization model for MOSP1, and proposed an interactive method to obtain the satisfactory solution of the decision maker. In their interactive method, after the decision maker specifies permissible objective levels $f_i, i = 1, \cdots, k$ for each objective function $\bar{z}_i(\boldsymbol{x}, \omega)$, the candidate of the satisfactory solution is obtained from among M-Pareto optimal solution set which is Pareto optimal solutions in membership space. However, in general, the decision maker seems to prefer not only the less value of permissible objective level $f_i$, but also the larger value of probability function $p_i(\boldsymbol{x}, f_i)$. Since these values conflict with each other, the less values of permissible objective level $f_i$ results in the less value of probability function $p_i(\boldsymbol{x}, f_i)$. From such a point of view, we consider the following multiobjective programming problem which can be regarded as a natural extension of MOP1($\boldsymbol{f}$).

**[MOP2]**

$$\max_{\boldsymbol{x} \in X, f_i, i=1,\cdots,k} (p_1(\boldsymbol{x}, f_1), \cdots, p_k(\boldsymbol{x}, f_k), -f_1, \cdots, -f_k) \qquad (4)$$

Considering the imprecise nature of the decision maker's judgment, it is natural to assume that the decision maker have fuzzy goals for each objective function in MOP2. In this section, it is assumed that such fuzzy goals can be quantified by eliciting the corresponding membership functions. Let us denote a membership function of probability function $p_i(\boldsymbol{x}, f_i)$ as $\mu_{p_i}(p_i(\boldsymbol{x}, f_i))$, and a membership function of permissible objective level $f_i$ as $\mu_{f_i}(f_i)$ respectively. Then, MOP2 can be transformed as the following multiobjective programming problem.

**[MOP3]**

$$\begin{aligned} \max_{\boldsymbol{x} \in X, f_i, i=1,\cdots,k} \quad & (\mu_{p_1}(p_1(\boldsymbol{x}, f_1)), \cdots, \mu_{p_k}(p_k(\boldsymbol{x}, f_k)), \\ & \mu_{f_1}(f_1), \cdots, \mu_{f_k}(f_k)) \end{aligned} \qquad (5)$$

Throughout this section, we make the assumptions that $\mu_{f_i}(f_i), i = 1, \cdots, k$ are strictly monotone decreasing and continuous with respect to $f_i$, and $\mu_{p_i}(p_i(\boldsymbol{x}, f_i)), i = 1, \cdots, k$ are strictly monotone increasing and continuous with respect to $p_i(\boldsymbol{x}, f_i)$.

For example, we can define the domain of $\mu_{p_i}(p_i(\boldsymbol{x}, f_i))$ as follows. Considering the individual minimum and maximum of $E(\bar{z}_i(\boldsymbol{x}))$, the decision maker subjectively specifies the sufficiently satisfactory maximum value $f_{i\min}$ and the acceptable minimum value $f_{i\max}$. Then, the domain of $\mu_{f_i}(f_i)$ is defined as:

$$F_i = [f_{i\min}, f_{i\max}]. \qquad (6)$$

Corresponding to the domain $F_i$, denote the domain of $\mu_{p_i}(p_i(\boldsymbol{x}, f_i))$ as:

$$P_i(F_i) = [p_{i\min}, p_{i\max}]. \qquad (7)$$

$p_{i\max}$ can be obtained by solving the following problem.

$$p_{i\max} = \max_{\boldsymbol{x} \in X} \; p_i(\boldsymbol{x}, f_{i\max}), i = 1, \cdots, k, \qquad (8)$$

It should be noted here that the above problem is equivalent to the following linear fractional programming problem [3] because distribution function $T(\cdot)$ is strictly monotone increasing and continuous.

$$\max_{\boldsymbol{x} \in X} \; \left(\frac{f_{i\max} - (\boldsymbol{c}_i^1\boldsymbol{x} + \alpha_i^1)}{\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2}\right) \qquad (9)$$

On the other hand, in order to obtain $p_{i\min}$, we first solve

$$\max_{\boldsymbol{x} \in X} p_i(\boldsymbol{x}, f_{i\min}), i = 1, \cdots, k, \qquad (10)$$

and denote the corresponding optimal solution as $\boldsymbol{x}_i$. Using the optimal solutions $\boldsymbol{x}_i, i = 1, \cdots, k$, $p_{i\min}$ can be obtained as follows.

$$p_{i\min} = \min_{\ell=1,\cdots,k,\ell\neq i} p_i(\boldsymbol{x}_\ell, f_{i\min}) \tag{11}$$

If the decision maker adopts the fuzzy decision as aggregation operator for MOP3, the satisfactory solution is obtained by solving the following maxmin problem.

**[MAXMIN1]**

$$\max_{\boldsymbol{x}\in X, f_i\in F_i, i=1,\cdots,k, \lambda\in[0,1]} \lambda \tag{12}$$

subject to

$$\begin{aligned} \mu_{p_i}(p_i(\boldsymbol{x}, f_i)) &\geq \lambda, i = 1, \cdots, k & (13)\\ \mu_{f_i}(f_i) &\geq \lambda, i = 1, \cdots, k & (14) \end{aligned}$$

According to the assumption for $\mu_{p_i}(p_i(\boldsymbol{x}, f_i))$ and $\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2 > 0$, the constraints (13) can be transformed as:

$$\begin{aligned} & \mu_{p_i}(p_i(\boldsymbol{x}, f_i)) \geq \lambda, \\ \Leftrightarrow \quad & p_i(\boldsymbol{x}, f_i) \geq \mu_{p_i}^{-1}(\lambda), \\ \Leftrightarrow \quad & T_i\left(\frac{f_i - (\boldsymbol{c}_i^1\boldsymbol{x} + \alpha_i^1)}{\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2}\right) \geq \mu_{p_i}^{-1}(\lambda), \\ \Leftrightarrow \quad & f_i - (\boldsymbol{c}_i^1\boldsymbol{x} + \alpha_i^1) \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda)) \cdot (\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2), \end{aligned} \tag{15}$$

where $\mu_{p_i}^{-1}(\cdot)$ and $T_i^{-1}(\cdot)$ are pseudo-inverse functions with respect to $\mu_{p_i}(\cdot)$ and $T_i(\cdot)$ respectively. Moreover, from the constraints (14) and the assumption for $\mu_{f_i}(f_i)$, it holds that $f_i \leq \mu_{f_i}^{-1}(\lambda)$. Therefore, the constraint (15) can be reduced to the following inequality where a permissible objective level $f_i$ is removed.

$$\mu_{f_i}^{-1}(\lambda) - (\boldsymbol{c}_i^1\boldsymbol{x} + \alpha_i^1) \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda)) \cdot (\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2) \tag{16}$$

Then, MAXMIN1 is equivalently transformed into the following problem.

**[MAXMIN2]**

$$\max_{\boldsymbol{x}\in X, \lambda\in[0,1]} \lambda \tag{17}$$

subject to

$$\begin{aligned} \mu_{f_i}^{-1}(\lambda) - (\boldsymbol{c}_i^1\boldsymbol{x} + \alpha_i^1) \quad \geq \quad & T_i^{-1}(\mu_{p_i}^{-1}(\lambda)) \cdot (\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2), \\ & i = 1, \cdots, k \end{aligned} \tag{18}$$

It should be noted here that the constraints (18) can be reduced to a set of linear inequalities for some fixed value $\lambda$. This means that an optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MAXMIN2 is obtained by combined use of the bisection method with respect to $0 \leq \lambda \leq 1$ and the first-phase of the two-phase simplex method of linear programming.

The relationship between the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MAXMIN2 and Pareto optimal solutions to MOP1($\boldsymbol{f}$) can be characterized by the following theorem.

**Theorem 1.**

If $(\boldsymbol{x}^*, \lambda^*)$ is a unique optimal solution of MAXMIN2, then $\boldsymbol{x}^* \in X$ is a Pareto optimal solution to MOP1($\boldsymbol{f}^*$), where $\boldsymbol{f}^* = (\mu_{f_1}^{-1}(\lambda^*), \cdots, \mu_{f_k}^{-1}(\lambda^*))$.

(Proof)

TABLE I

PARAMETERS OF OBJECTIVE FUNCTIONS AND CONSTRAINTS IN MOSLP

| $\boldsymbol{x}$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{c}_1^1$ | 19 | 48 | 21 | 10 | 18 | 35 | 46 | 11 | 24 | 33 |
| $\boldsymbol{c}_1^2$ | 3 | 2 | 2 | 1 | 4 | 3 | 1 | 2 | 4 | 2 |
| $\boldsymbol{c}_2^1$ | 12 | -46 | -23 | -38 | -33 | -48 | 12 | 8 | 19 | 20 |
| $\boldsymbol{c}_2^2$ | 1 | 2 | 4 | 2 | 2 | 1 | 2 | 1 | 2 | 1 |
| $\boldsymbol{c}_3^1$ | -18 | -26 | -22 | -28 | -15 | -29 | -10 | -19 | -17 | -28 |
| $\boldsymbol{c}_3^2$ | 2 | 1 | 3 | 2 | 1 | 2 | 3 | 3 | 2 | 1 |
| $\boldsymbol{a}_1$ | 12 | -2 | 4 | -7 | 13 | -1 | -6 | 6 | 11 | -8 |
| $\boldsymbol{a}_2$ | -2 | 5 | 3 | 16 | 6 | -12 | 12 | 4 | -7 | -10 |
| $\boldsymbol{a}_3$ | 3 | -16 | -4 | -8 | -8 | 2 | -12 | -12 | 4 | -3 |
| $\boldsymbol{a}_4$ | -11 | 6 | -5 | 9 | -1 | 8 | -4 | 6 | -9 | 6 |
| $\boldsymbol{a}_5$ | -4 | 7 | -6 | -5 | 13 | 6 | -2 | -5 | 14 | -6 |
| $\boldsymbol{a}_6$ | 5 | -3 | 14 | -3 | -9 | -7 | 4 | -4 | -5 | 9 |
| $\boldsymbol{a}_7$ | -3 | -4 | -6 | 9 | 6 | 18 | 11 | -9 | -4 | 7 |

Since an optimal solution $(\boldsymbol{x}^*, \lambda^*)$ satisfies the constraints (18), it holds that

$$\begin{aligned} & \mu_{f_i}^{-1}(\lambda^*) - (\boldsymbol{c}_i^1\boldsymbol{x}^* + \alpha_i^1) \\ & \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*)) \cdot (\boldsymbol{c}_i^2\boldsymbol{x}^* + \alpha_i^2), \\ \Leftrightarrow \quad & T_i\left(\frac{\mu_{f_i}^{-1}(\lambda^*) - (\boldsymbol{c}_i^1\boldsymbol{x}^* + \alpha_i^1)}{\boldsymbol{c}_i^2\boldsymbol{x}^* + \alpha_i^2}\right) = p_i(\boldsymbol{x}^*, \mu_{f_i}^{-1}(\lambda^*)) \\ & \geq \mu_{p_i}^{-1}(\lambda^*), i = 1, \cdots, k. \end{aligned}$$

Assume that $\boldsymbol{x}^* \in X$ is not a Pareto optimal solution to MOP1($\boldsymbol{f}^*$), where $\boldsymbol{f}^* = (\mu_{f_1}^{-1}(\lambda^*), \cdots, \mu_{f_k}^{-1}(\lambda^*))$, then there exists $\boldsymbol{x} \in X$ such that

$$\begin{aligned} & p_i(\boldsymbol{x}, \mu_{f_i}^{-1}(\lambda^*)) \geq p_i(\boldsymbol{x}^*, \mu_{f_i}^{-1}(\lambda^*)) \geq \mu_{p_i}^{-1}(\lambda^*), \\ & i = 1, \cdots, k. \end{aligned}$$

Then there exists $\boldsymbol{x} \in X$ such that

$$\begin{aligned} \mu_{f_i}^{-1}(\lambda^*) - (\boldsymbol{c}_i^1\boldsymbol{x} + \alpha_i^1) \quad \geq \quad & T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*)) \cdot (\boldsymbol{c}_i^2\boldsymbol{x} + \alpha_i^2), \\ & i = 1, \cdots, k, \end{aligned}$$

which contradicts the fact that $(\boldsymbol{x}^*, \lambda^*)$ is a unique optimal solution of MAXMIN2.

## III. A Numerical Example for MOSLP

In this section, in order to demonstrate the feasibility of our proposed method, we consider the following three-objective stochastic linear programming problem (MOSLP) which is the modified version of the numerical example formulated by Sakawa et al. [15].

**[MOSLP]**

$$\min \bar{z}_1(\boldsymbol{x}) = (\boldsymbol{c}_1^1 + \bar{t}_1\boldsymbol{c}_1^2)\boldsymbol{x} + (\alpha_1^1 + \bar{t}_1\alpha_1^2)$$

$$\min \bar{z}_2(\boldsymbol{x}) = (\boldsymbol{c}_2^1 + \bar{t}_2\boldsymbol{c}_2^2)\boldsymbol{x} + (\alpha_2^1 + \bar{t}_2\alpha_2^2)$$

$$\min \bar{z}_3(\boldsymbol{x}) = (\boldsymbol{c}_3^1 + \bar{t}_3\boldsymbol{c}_3^2)\boldsymbol{x} + (\alpha_3^1 + \bar{t}_3\alpha_3^2)$$

subject to $\boldsymbol{x} \in X = \{\boldsymbol{a}_i\boldsymbol{x} \leq b_i, i = 1, \cdots, 7, \boldsymbol{x} \geq \boldsymbol{0}\}$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_{10})^T$ is a 10-dimensional decision vector, $\boldsymbol{c}_1^1, \boldsymbol{c}_1^2, \boldsymbol{c}_2^1, \boldsymbol{c}_2^2, \boldsymbol{c}_3^1, \boldsymbol{c}_3^2, \boldsymbol{a}_i, i = 1, \cdots, 7$ are parameter vectors of the objective functions and the constraints as shown in Table I. $\alpha_1^1 = -18, \alpha_1^2 = 5, \alpha_2^1 = -27, \alpha_2^2 = 6, \alpha_3^1 = -10, \alpha_3^2 = 4$, are parameters of the objective functions. $\bar{t}_i, i = 1,2,3$ are Gaussian random variables, that is,

$$\begin{aligned} \bar{t}_1 &\sim \mathrm{N}(4, 2^2), \\ \bar{t}_2 &\sim \mathrm{N}(3, 3^2), \\ \bar{t}_3 &\sim \mathrm{N}(3, 2^2). \end{aligned}$$

The right-hand-side parameters of the constraints $(b_1, b_2, b_3, b_4, b_5, b_6, b_7)$ are set as $(164, -190, -184, 99, -150, 154, 142)$.

Considering the individual minimum and maximum of $E(\bar{z}_i(\boldsymbol{x}))$, let us assume that the hypothetical decision maker subjectively specifies the sufficiently satisfactory maximum value and the acceptable minimum value as follows.

$$\begin{aligned} F_1 &= [f_{1\min}, f_{1\max}] = [2100, 2200] \\ F_2 &= [f_{2\min}, f_{2\max}] = [400, 500] \\ F_3 &= [f_{3\min}, f_{3\max}] = [-1000, -900] \end{aligned}$$

Then, the hypothetical decision maker sets his/her membership functions of fuzzy goals for the permissible objective levels as follows.

$$\begin{aligned} \mu_{f_1}(f_1) &= \frac{f_1 - 2200}{(2100 - 2200)} \\ \mu_{f_2}(f_2) &= \frac{f_2 - 500}{(400 - 500)} \\ \mu_{f_3}(f_3) &= \frac{f_3 - (-900)}{((-1000) - (-900))} \end{aligned}$$

Corresponding to the domain $F_i$, the domain $P_i(F_i) = [p_{i\min}, p_{i\max}], i = 1,2,3$ can be obtained by solving the optimization problems (8), (10) and (11) as follows.

$$\begin{aligned} P_1(F_1) &= [p_{1\min}, p_{1\max}] = [0.00390, 0.99989] \\ P_2(F_2) &= [p_{2\min}, p_{2\max}] = [0.00704, 0.99783] \\ P_3(F_3) &= [p_{3\min}, p_{3\max}] = [0.07331, 0.99351] \end{aligned}$$

Then, the hypothetical decision maker sets his/her membership functions of fuzzy goals for the permissible probability levels as follows.

$$\begin{aligned} \mu_{p_1}(p_1(\boldsymbol{x}, f_1)) &= \frac{p_1(\boldsymbol{x}, f_1) - 0.0039}{(0.99989 - 0.0039)} \\ \mu_{p_2}(p_2(\boldsymbol{x}, f_2)) &= \frac{p_2(\boldsymbol{x}, f_2) - 0.00704}{(0.99783 - 0.00704)} \\ \mu_{p_3}(p_3(\boldsymbol{x}, f_3)) &= \frac{p_3(\boldsymbol{x}, f_3) - 0.07331}{(0.99351 - 0.07331)} \end{aligned}$$

For these membership functions $\mu_{p_i}(p_i(\boldsymbol{x}, f_i))$, $\mu_{f_i}(f_i)$, $i = 1,2,3$, MAXMIN2 is formulated and solved by combined use of the bisection method with respect to $0 \leq \lambda \leq 1$ and the first-phase of the two-phase simplex method of linear programming. The optimal solution is obtained as $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*, x_9^*, x_{10}^*, \lambda^*) = (3.3833, 3.2987, 0, 4.6295, 0.1135, 4.7246, 0, 7.5564, 2.6569, 20.9413, 0.6807)$. Since three constraints (18) of MAXMIN2 are active at the optimal solution, we can get the following optimal values of the corresponding membership function.

$$\begin{aligned} \mu_{p_1}(p_1(\boldsymbol{x}^*, \mu_{f_1}^{-1}(\lambda^*))) &= 0.6807 \\ \mu_{p_2}(p_2(\boldsymbol{x}^*, \mu_{f_2}^{-1}(\lambda^*))) &= 0.6807 \\ \mu_{p_3}(p_3(\boldsymbol{x}^*, \mu_{f_3}^{-1}(\lambda^*))) &= 0.6807 \\ \mu_{f_1}(\mu_{f_1}^{-1}(\lambda^*)) &= 0.6807 \\ \mu_{f_2}(\mu_{f_2}^{-1}(\lambda^*)) &= 0.6807 \\ \mu_{f_3}(\mu_{f_3}^{-1}(\lambda^*)) &= 0.6807 \end{aligned}$$

At the optimal solution, the proper balance between the membership functions $\mu_{p_i}(p_i(\boldsymbol{x}, \mu_{f_i}^{-1}(\lambda))), i = 1,2,3$ and $\mu_{f_i}((\mu_{f_i}^{-1}(\lambda))), i = 1,2,3$ in a probability maximization model is attained through the fuzzy decision.

## IV. A Fuzzy Approach for Multiobjective Fuzzy Random Linear Programming Problems

In this section, we focus on multiobjective programming problems involving fuzzy random variable coefficients in objective functions called multiobjective fuzzy random linear programming problem (MOFRLP).

**[MOFRLP]**

$$\min_{\boldsymbol{x} \in X} \tilde{\bar{\boldsymbol{C}}}\boldsymbol{x} = (\tilde{\bar{\boldsymbol{c}}}_1\boldsymbol{x}, \cdots, \tilde{\bar{\boldsymbol{c}}}_k) \tag{19}$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ is an $n$ dimensional decision variable column vector, $X$ is a linear constraint set with respect to $\boldsymbol{x}$. $\tilde{\bar{\boldsymbol{c}}}_i = (\tilde{\bar{c}}_{i1}, \cdots, \tilde{\bar{c}}_{in}), i = 1, \cdots, k$, are coefficient vector of objective function $\tilde{\bar{\boldsymbol{c}}}_i\boldsymbol{x}$, whose elements are fuzzy random variables (The symbols "-" and "~" mean randomness and fuzziness respectively, and the concept of fuzzy random variable in this section is defined precisely in [9],[16]). Under the occurrence of each elementary event $\omega$, $\tilde{\bar{c}}_{ij}(\omega)$ is a realization of the fuzzy random variable $\tilde{\bar{c}}_{ij}$, which is a fuzzy number whose membership function is defined as follows.

$$\mu_{\tilde{\bar{c}}_{ij}(\omega)}(s) = \begin{cases} L\left(\frac{\bar{d}_{ij}(\omega) - s}{\bar{\alpha}_{ij}(\omega)}\right) & (s \leq \bar{d}_{ij}(\omega) \ \forall \omega), \\ R\left(\frac{s - \bar{d}_{ij}(\omega)}{\bar{\beta}_{ij}(\omega)}\right) & (s > \bar{d}_{ij}(\omega) \ \forall \omega), \end{cases}$$

where the function $L(t) \stackrel{\text{def}}{=} \max\{0, l(t)\}$ is a real-valued continuous function from $[0, \infty)$ to $[0, 1]$, and $l(t)$is a strictly decreasing continuous function satisfying $l(0) = 1$. Also, $R(t) \stackrel{\text{def}}{=} \max\{0, r(t)\}$ satisfies the same conditions. Let us assume that the parameters $\bar{d}_{ij}, \bar{\alpha}_{ij}, \bar{\beta}_{ij}$ are random variables expressed as $\bar{d}_{ij} = d_{ij}^1 + \bar{t}_i d_{ij}^2$, $\bar{\alpha}_{ij} = \alpha_{ij}^1 + \bar{t}_i \alpha_{ij}^2$, $\bar{\beta}_{ij} = \beta_{ij}^1 + \bar{t}_i \beta_{ij}^2$ respectively, where $\bar{t}_i$ is a random variable whose distribution function $T(\cdot)$ is continuous and strictly monotone increasing, and $d_{ij}^\ell$, $\alpha_{ij}^\ell, \beta_{ij}^\ell$, $\ell = 1, 2$ are constants. It should be noted that $\bar{\alpha}_{ij}(\omega), \bar{\beta}_{ij}(\omega)$ are positive for any $\omega$ because of a property of spread parameters of LR-type fuzzy numbers. Therefore, let us give the assumptions that $\alpha_{ij}^1 + \bar{t}_i(\omega)\alpha_{ij}^2 > 0, \beta_{ij}^1 + \bar{t}_i(\omega)\beta_{ij}^2 > 0$, for any $\omega$.

As shown in [9], the realizations $\tilde{\bar{\boldsymbol{c}}}_i(\omega)\boldsymbol{x}$ becomes a fuzzy number characterized by the following membership functions.

$$\mu_{\tilde{\bar{\boldsymbol{c}}}_i(\omega)\boldsymbol{x}}(y) = \begin{cases} L\left(\frac{\bar{\boldsymbol{d}}_i(\omega)\boldsymbol{x} - y}{\bar{\boldsymbol{\alpha}}_i(\omega)\boldsymbol{x}}\right) & (y \leq \bar{\boldsymbol{d}}_i(\omega)\boldsymbol{x} \ \forall \omega), \\ R\left(\frac{y - \bar{\boldsymbol{d}}_i(\omega)\boldsymbol{x}}{\bar{\boldsymbol{\beta}}_i(\omega)\boldsymbol{x}}\right) & (y > \bar{\boldsymbol{d}}_i(\omega)\boldsymbol{x} \ \forall \omega), \end{cases}$$

Similar to the previous section, it is assumed that the decision maker has fuzzy goals for the objective functions in MOFRLP, whose membership functions $\mu_{\widetilde{G}_i}(y)$, $i = 1, \cdots, k$ are continuous and strictly monotone decreasing. By using a concept of possibility measure [5], the degree of possibility that the objective function value $\widetilde{\overline{\boldsymbol{c}}}_i\boldsymbol{x}$ satisfies the fuzzy goal $\widetilde{G}_i$ is expressed as follows.

$$\Pi_{\widetilde{\overline{\boldsymbol{c}}}_i\boldsymbol{x}}(\widetilde{G}_i) \stackrel{\mathrm{def}}{=} \sup_y \min\{\mu_{\widetilde{\overline{\boldsymbol{c}}}_i\boldsymbol{x}}(y), \mu_{\widetilde{G}_i}(y)\} \tag{20}$$

Using the above possibility measure, MOFRLP can be transformed to the following multiobjective stochastic programming problem MOSP2.

**[MOSP2]**

$$\max_{\boldsymbol{x}\in X}(\Pi_{\widetilde{\overline{\boldsymbol{c}}}_1\boldsymbol{x}}(\widetilde{G}_1), \cdots, \Pi_{\widetilde{\overline{\boldsymbol{c}}}_k\boldsymbol{x}}(\widetilde{G}_k)) \tag{21}$$

Katagiri et al. [9] first formulated MOFRLP as the following multiobjective programming problem through a probability maximization model.

**[MOP4($\boldsymbol{h}$)]**

$$\begin{aligned}\max_{\boldsymbol{x}\in X} \quad & (\Pr(\omega \mid \Pi_{\widetilde{\overline{\boldsymbol{c}}}_1(\omega)\boldsymbol{x}}(\widetilde{G}_1) \ge h_1), \cdots, \\ & \Pr(\omega \mid \Pi_{\widetilde{\overline{\boldsymbol{c}}}_k(\omega)\boldsymbol{x}}(\widetilde{G}_k) \ge h_k))\end{aligned} \tag{22}$$

where $\boldsymbol{h} = (h_1, \cdots, h_k)$ are permissible degrees of possibility measure specified by the decision maker. In MOP4($\boldsymbol{h}$), the constraint $\Pi_{\widetilde{\overline{\boldsymbol{c}}}_i(\omega)\boldsymbol{x}}(\widetilde{G}_i) \ge h_i$ can be transformed as follows.

$$\begin{aligned} & \sup_y \min\{\mu_{\widetilde{\overline{\boldsymbol{c}}}_i(\omega)\boldsymbol{x}}(y), \mu_{\widetilde{G}_i}(y)\} \ge h_i, \\ \Leftrightarrow \quad & \exists y : \mu_{\widetilde{\overline{\boldsymbol{c}}}_i(\omega)\boldsymbol{x}}(y) \ge h_i, \mu_{\widetilde{G}_i}(y) \ge h_i, \\ \Leftrightarrow \quad & \exists y : L(\frac{\overline{\boldsymbol{d}}_i(\omega)\boldsymbol{x} - y}{\overline{\boldsymbol{\alpha}}_i(\omega)\boldsymbol{x}}) \ge h_i, R(\frac{y - \overline{\boldsymbol{d}}_i(\omega)\boldsymbol{x}}{\overline{\boldsymbol{\beta}}_i(\omega)\boldsymbol{x}}) \ge h_i, \\ & \mu_{\widetilde{G}_i}(y) \ge h_i, \\ \Leftrightarrow \quad & \exists y : (\overline{\boldsymbol{d}}_i(\omega) - L^{-1}(h_i)\overline{\boldsymbol{\alpha}}_i)\boldsymbol{x} \le y \\ & \le (\overline{\boldsymbol{d}}_i(\omega) + R^{-1}(h_i)\overline{\boldsymbol{\beta}}_i)\boldsymbol{x}, y \le \mu^{-1}_{\widetilde{G}_i}(h_i), \\ \Leftrightarrow \quad & (\overline{\boldsymbol{d}}_i(\omega) - L^{-1}(h_i)\overline{\boldsymbol{\alpha}}_i(\omega))\boldsymbol{x} \le \mu^{-1}_{\widetilde{G}_i}(h_i) \end{aligned}$$

where $L^{-1}(\cdot)$ and $R^{-1}(\cdot)$ are pseudo-inverse function corresponding to $L(\cdot)$ and $R(\cdot)$. Using a distribution function $T_i(\cdot)$ of $\bar{t}_i$, each objective function of MOP4($\boldsymbol{h}$) is transformed as below.

$$\begin{aligned} & \Pr(\omega \mid \Pi_{\widetilde{\overline{\boldsymbol{c}}}_i(\omega)\boldsymbol{x}}(\widetilde{G}_i) \ge h_i) \\ = \quad & \Pr\left(\omega \mid (\overline{\boldsymbol{d}}_i(\omega) - L^{-1}(h_i)\overline{\boldsymbol{\alpha}}_i(\omega))\boldsymbol{x} \le \mu^{-1}_{\widetilde{G}_i}(h_i)\right) \\ = \quad & \Pr\big(\omega \mid (\boldsymbol{d}_i^1 + \bar{t}_i(\omega)\boldsymbol{d}_i^2)\boldsymbol{x} \\ & -L^{-1}(h_i)(\boldsymbol{\alpha}_i^1 + \bar{t}_i(\omega)\boldsymbol{\alpha}_i^2)\boldsymbol{x} \le \mu^{-1}_{\widetilde{G}_i}(h_i)\big) \\ = \quad & \Pr\big(\omega \mid (\boldsymbol{d}_i^1\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^1\boldsymbol{x}) \\ & +\bar{t}_i(\omega)(\boldsymbol{d}_i^2\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^2\boldsymbol{x}) \le \mu^{-1}_{\widetilde{G}_i}(h_i)\big) \\ = \quad & \Pr\left(\omega \mid \bar{t}_i(\omega) \le \frac{\mu^{-1}_{\widetilde{G}_i}(h_i) - (\boldsymbol{d}_i^1\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^1\boldsymbol{x})}{\boldsymbol{d}_i^2\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^2\boldsymbol{x}}\right) \\ = \quad & T_i\left(\frac{\mu^{-1}_{\widetilde{G}_i}(h_i) - (\boldsymbol{d}_i^1\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^1\boldsymbol{x})}{\boldsymbol{d}_i^2\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^2\boldsymbol{x}}\right) \\ \stackrel{\mathrm{def}}{=} \quad & p_i(\boldsymbol{x}, h_i) \end{aligned} \tag{23}$$

where $(\boldsymbol{d}_i^2 - L^{-1}(0)\boldsymbol{\alpha}_i^2)\boldsymbol{x} > 0$, $i = 1, \cdots, k$ for any $\boldsymbol{x} \in X$. Using $p_i(\boldsymbol{x}, h_i), i = 1, \cdots, k$, MOP4($\boldsymbol{h}$) can be expressed as the following simple form.

**[MOP5($\boldsymbol{h}$)]**

$$\max_{\boldsymbol{x}\in X} \ (p_1(\boldsymbol{x}, h_1), \cdots, p_k(\boldsymbol{x}, h_k)) \tag{24}$$

In order to deal with MOP5($\boldsymbol{h}$), we define Pareto optimal solutions in the feasible set $P(\boldsymbol{h}) = \{(p_1(\boldsymbol{x}, h_1), \cdots, p_k(\boldsymbol{x}, h_k) \in [0,1]^k \mid \boldsymbol{x} \in X\}$.

**Definition 2.**

$\boldsymbol{x}^* \in X$ is said to be a Pareto optimal solution to MOP5($\boldsymbol{h}$), if and only if there does not exist another $\boldsymbol{x} \in X$ such that $p_i(\boldsymbol{x}, h_i) \ge p_i(\boldsymbol{x}^*, h_i)$, $i = 1, \cdots, k$, with strict inequality holding for at least one $i$.

Katagiri et al. [9] proposed an interactive method to obtain a satisfactory solution from among Pareto optimal solution set to MOP5($\boldsymbol{h}$), where permissible values of possibility measure $\boldsymbol{h} = (h_1, \cdots, h_k)$ must be set in advance by the decision maker in his/her subjective manner. However, in general, the decision maker seems to prefer not only the larger value of permissible value of possibility measure $h_i$ but also the larger value of probability function $p_i(\boldsymbol{x}, h_i)$. From such a point of view, we consider the following multiobjective programming problem which can be regarded as a natural extension of MOP5($\boldsymbol{h}$).

**[MOP6]**

$$\max_{\boldsymbol{x}\in X, h_i\in[0,1], i=1,\cdots,k} (p_1(\boldsymbol{x}, h_1), \cdots, p_k(\boldsymbol{x}, h_k), h_1, \cdots, h_k) \tag{25}$$

Similar to the previous section, we assume that the decision maker has fuzzy goals for $p_i(\boldsymbol{x}, h_i), i = 1. \cdots, k$, and such fuzzy goals can be quantified by eliciting the corresponding membership functions $\mu_{p_i}(p_i(\boldsymbol{x}, h_i))$. Then MOP6 can be replaced by the following form.

**[MOP7]**

$$\begin{aligned}\max_{\boldsymbol{x}\in X, h_i\in[0,1], i=1,\cdots,k} \quad & (\mu_{p_1}(p_1(\boldsymbol{x}, h_1)), \cdots, \mu_{p_k}(p_k(\boldsymbol{x}, h_k)), \\ & h_1, \cdots, h_k)\end{aligned} \tag{26}$$

Throughout this section, we assume that $\mu_{p_i}(p_i(\boldsymbol{x}, h_i)), i = 1, \cdots, k$ are strictly monotone increasing and continuous with respect to $p_i(\boldsymbol{x}, h_i)$.

For example, we can define the domain of $\mu_{p_i}(p_i(\boldsymbol{x}, h_i))$ as follows. Considering the individual minimum and maximum of $E(\overline{\boldsymbol{d}}_i)\boldsymbol{x}$, the decision maker subjectively specifies the sufficiently satisfactory maximum value and the acceptable minimum value for the original objective functions in MOFRLP, and defines membership function $\mu_{\widetilde{G}_i}(y)$. For the possibility measure (20) based on $\mu_{\widetilde{G}_i}(y)$, the decision maker subjectively specifies the sufficiently satisfactory maximum value $h_{i\max}$ and the acceptable minimum value $h_{i\min}$. Then, the interval for permissible value $h_i$ is defined as:

$$H_i = [h_{i\min}, h_{i\max}]. \tag{27}$$

Corresponding to the interval $H_i$, let us denote the domain of $\mu_{p_i}(p_i(\boldsymbol{x}, h_i))$ as:

$$P_i(H_i) = [p_{i\min}, p_{i\max}]. \tag{28}$$

$p_{i\max}$ can be obtained by solving the following problem.

$$p_{i\max} = \max_{\boldsymbol{x}\in X} p_i(\boldsymbol{x}, h_{i\min}) \tag{29}$$

In order to obtain $p_{i\min}$, we first solve

$$\max_{\boldsymbol{x}\in X} p_i(\boldsymbol{x}, h_{i\max}), i=1,\cdots,k, \tag{30}$$

and denote the optimal solution as $\boldsymbol{x}_i$. Using the optimal solutions $\boldsymbol{x}_i, i=1,\cdots,k$, we can obtain $p_{i\min}$ as follows.

$$p_{i\min} = \min_{\ell=1,\cdots,k,\ell\neq i} p_i(\boldsymbol{x}_\ell, h_{i\max}) \tag{31}$$

If the decision maker adopts the fuzzy decision as an aggregation operator for MOP7, a satisfactory solution is obtained by solving the following maxmin problem.

**[MAXMIN3]**

$$\max_{\boldsymbol{x}\in X, h_i\in H_i, i=1,\cdots,k,\lambda\in[0,1]} \lambda \tag{32}$$

subject to

$$\begin{aligned} \mu_{p_i}(p_i(\boldsymbol{x},h_i)) &\geq \lambda, i=1,\cdots,k & (33)\\ h_i &\geq \lambda, i=1,\cdots,k & (34) \end{aligned}$$

Since there exist pseudo-inverse functions $\mu_{p_i}^{-1}(\cdot)$ and $T_i^{-1}(\cdot)$ with respect to $\mu_{p_i}(\cdot)$ and $T_i(\cdot)$, the constraints (33) can be transformed as:

$$\begin{aligned} & \mu_{p_i}(p_i(\boldsymbol{x},h_i)) \geq \lambda, \\ \Leftrightarrow\ & p_i(\boldsymbol{x},h_i) \geq \mu_{p_i}^{-1}(\lambda), \\ \Leftrightarrow\ & T_i\left(\frac{\mu_{\widetilde{G}_i}^{-1}(h_i) - (\boldsymbol{d}_i^1\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^1\boldsymbol{x})}{\boldsymbol{d}_i^2\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^2\boldsymbol{x}}\right) \geq \mu_{p_i}^{-1}(\lambda), \\ \Leftrightarrow\ & \mu_{\widetilde{G}_i}^{-1}(h_i) \geq (\boldsymbol{d}_i^1\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^1\boldsymbol{x}) \\ & \quad +T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\cdot(\boldsymbol{d}_i^2\boldsymbol{x} - L^{-1}(h_i)\boldsymbol{\alpha}_i^2\boldsymbol{x}) \\ \Leftrightarrow\ & \mu_{\widetilde{G}_i}^{-1}(h_i) \geq (\boldsymbol{d}_i^1\boldsymbol{x} + T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\boldsymbol{d}_i^2\boldsymbol{x}) \\ & \quad -L^{-1}(h_i)(\boldsymbol{\alpha}_i^1\boldsymbol{x} + T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\boldsymbol{\alpha}_i^2\boldsymbol{x}) \end{aligned} \tag{35}$$

On the other hand, because of the constraint (34), it holds that $\mu_{\widetilde{G}_i}^{-1}(h_i) \leq \mu_{\widetilde{G}_i}^{-1}(\lambda)$, $L^{-1}(h_i) \leq L^{-1}(\lambda)$. Since it is guaranteed that $(\boldsymbol{\alpha}_i^1\boldsymbol{x} + T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\ \boldsymbol{\alpha}_i^2\boldsymbol{x}) > 0$, the right hand side of the constraint (35) can be transformed as the following form.

$$\begin{aligned} & (\boldsymbol{d}_i^1\boldsymbol{x} + T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\boldsymbol{d}_i^2\boldsymbol{x}) \\ & -L^{-1}(h_i)(\boldsymbol{\alpha}_i^1\boldsymbol{x} + T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\boldsymbol{\alpha}_i^2\boldsymbol{x}) \\ \geq\ & (\boldsymbol{d}_i^1\boldsymbol{x} + T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\boldsymbol{d}_i^2\boldsymbol{x}) \\ & -L^{-1}(\lambda)(\boldsymbol{\alpha}_i^1\boldsymbol{x} + T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\boldsymbol{\alpha}_i^2\boldsymbol{x}) \\ =\ & (\boldsymbol{d}_i^1\boldsymbol{x} - L^{-1}(\lambda)\boldsymbol{\alpha}_i^1\boldsymbol{x}) \\ & +T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\cdot(\boldsymbol{d}_i^2\boldsymbol{x} - L^{-1}(\lambda)\boldsymbol{\alpha}_i^2\boldsymbol{x}) \end{aligned} \tag{36}$$

From the inequalities (35) and (36), the following inequality can be easily obtained.

$$\begin{aligned} & \mu_{\widetilde{G}_i}^{-1}(\lambda) \geq \mu_{\widetilde{G}_i}^{-1}(h_i) \geq (\boldsymbol{d}_i^1\boldsymbol{x} - L^{-1}(\lambda)\boldsymbol{\alpha}_i^1\boldsymbol{x}) \\ & +T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\cdot(\boldsymbol{d}_i^2\boldsymbol{x} - L^{-1}(\lambda)\boldsymbol{\alpha}_i^2\boldsymbol{x}) \end{aligned}$$

As a result, MAXMIN3 can be transformed into MAXMIN4 where permissible degrees of possibility measure $h_i$, $i = 1,\cdots,k$ have disappeared.

**[MAXMIN4]**

$$\max_{\boldsymbol{x}\in X, 0\leq\lambda\leq 1} \lambda \tag{37}$$

subject to

$$\begin{aligned} \mu_{\widetilde{G}_i}^{-1}(\lambda) \geq\ & (\boldsymbol{d}_i^1\boldsymbol{x} - L^{-1}(\lambda)\boldsymbol{\alpha}_i^1\boldsymbol{x}) \\ & +T_i^{-1}(\mu_{p_i}^{-1}(\lambda))\cdot(\boldsymbol{d}_i^2\boldsymbol{x} - L^{-1}(\lambda)\boldsymbol{\alpha}_i^2\boldsymbol{x}), \\ & i=1,\cdots,k \end{aligned} \tag{38}$$

It should be noted here that the constraints (38) can be reduced to a set of linear inequalities for some fixed value $\lambda$. This means that an optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MAXMIN4 is obtained by combined use of the bisection method with respect to $0 \leq \lambda \leq 1$ and the first-phase of the two-phase simplex method of linear programming.

The relationship between the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MAXMIN4 and Pareto optimal solutions to MOP5($\boldsymbol{h}$) can be characterized by the following theorem.

**Theorem 2.**

If $(\boldsymbol{x}^*, \lambda^*)$ is a unique optimal solution of MAXMIN4, then $\boldsymbol{x}^* \in X$is a Pareto optimal solution of MOP5($\boldsymbol{\lambda}^*$), where $\boldsymbol{\lambda}^* = (\lambda^*, \cdots, \lambda^*)$.

(Proof)

Since an optimal solution $(\boldsymbol{x}^*, \lambda^*)$ satisfies the constraints (38), it holds that

$$\begin{aligned} & \mu_{\widetilde{G}_i}^{-1}(\lambda^*) \geq (\boldsymbol{d}_i^1\boldsymbol{x}^* - L^{-1}(\lambda^*)\boldsymbol{\alpha}_i^1\boldsymbol{x}^*) \\ & +T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*))\cdot(\boldsymbol{d}_i^2\boldsymbol{x}^* - L^{-1}(\lambda^*)\boldsymbol{\alpha}_i^2\boldsymbol{x}^*) \\ \Leftrightarrow\ & T_i\left(\frac{\mu_{\widetilde{G}_i}^{-1}(\lambda^*) - (\boldsymbol{d}_i^1\boldsymbol{x}^* - L^{-1}(\lambda^*)\boldsymbol{\alpha}_i^1\boldsymbol{x}^*)}{\boldsymbol{d}_i^2\boldsymbol{x}^* - L^{-1}(\lambda^*)\boldsymbol{\alpha}_i^2\boldsymbol{x}^*}\right) \\ & = p_i(\boldsymbol{x}^*, \lambda^*) \\ & \geq \mu_{p_i}^{-1}(\lambda^*), i=1,\cdots,k \end{aligned}$$

Assume that $\boldsymbol{x}^* \in X$ is not a Pareto optimal solution of MOP5($\boldsymbol{\lambda}^*$), where $\boldsymbol{\lambda}^* = (\lambda^*, \cdots, \lambda^*)$, then there exists $\boldsymbol{x} \in X$ such that

$$p_i(\boldsymbol{x}, \lambda_i^*) \geq p_i(\boldsymbol{x}^*, \lambda_i^*) \geq \mu_{p_i}^{-1}(\lambda^*), i=1,\cdots,k.$$

Then there exists $\boldsymbol{x} \in X$ such that

$$\begin{aligned} \mu_{\widetilde{G}_i}^{-1}(\lambda^*) \geq\ & (\boldsymbol{d}_i^1\boldsymbol{x} - L^{-1}(\lambda^*)\boldsymbol{\alpha}_i^1\boldsymbol{x}) \\ & +T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*))\cdot(\boldsymbol{d}_i^2\boldsymbol{x} - L^{-1}(\lambda^*)\boldsymbol{\alpha}_i^2\boldsymbol{x}), \\ & i=1,\cdots,k \end{aligned}$$

which contradicts the fact that $(\boldsymbol{x}^*, \lambda^*)$ is a unique optimal solution of MAXMIN4.

## V. A Numerical Example for MOFRLP

In this section, in order to demonstrate the feasibility of our proposed method, we consider the following three-objective fuzzy random linear programming problem (MOFRLP) which is the modified version of the numerical example formulated by Sakawa et al. [15].

**[MOFRLP]**

$$\begin{aligned} \min_{\boldsymbol{x}\in X} \quad & \widetilde{\bar{\boldsymbol{c}}}_1\boldsymbol{x} = \sum_{\ell=1}^{10} \widetilde{\bar{c}}_{1\ell}x_\ell \\ \min_{\boldsymbol{x}\in X} \quad & \widetilde{\bar{\boldsymbol{c}}}_2\boldsymbol{x} = \sum_{\ell=1}^{10} \widetilde{\bar{c}}_{2\ell}x_\ell \\ \min_{\boldsymbol{x}\in X} \quad & \widetilde{\bar{\boldsymbol{c}}}_3\boldsymbol{x} = \sum_{\ell=1}^{10} \widetilde{\bar{c}}_{3\ell}x_\ell \end{aligned}$$

TABLE II
PARAMETERS OF OBJECTIVE FUNCTIONS AND CONSTRAINTS IN MOFRLP

| $\boldsymbol{x}$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{d}_1^1$ | 19 | 48 | 21 | 10 | 18 | 35 | 46 | 11 | 24 | 33 |
| $\boldsymbol{d}_1^2$ | 3 | 2 | 2 | 1 | 4 | 3 | 1 | 2 | 4 | 2 |
| $\boldsymbol{d}_2^1$ | 12 | -46 | -23 | -38 | -33 | -48 | 12 | 8 | 19 | 20 |
| $\boldsymbol{d}_2^2$ | 1 | 2 | 4 | 2 | 2 | 1 | 2 | 1 | 2 | 1 |
| $\boldsymbol{d}_3^1$ | -18 | -26 | -22 | -28 | -15 | -29 | -10 | -19 | -17 | -28 |
| $\boldsymbol{d}_3^2$ | 2 | 1 | 3 | 2 | 1 | 2 | 3 | 3 | 2 | 1 |
| $\boldsymbol{\alpha}_1$ | 0.51 | 0.54 | 0.53 | 0.55 | 0.48 | 0.57 | 0.47 | 0.52 | 0.5 | 0.53 |
| $\boldsymbol{\alpha}_2$ | 0.43 | 0.48 | 0.46 | 0.37 | 0.44 | 0.46 | 0.39 | 0.41 | 0.48 | 0.42 |
| $\boldsymbol{\alpha}_3$ | 0.59 | 0.56 | 0.58 | 0.62 | 0.6 | 0.52 | 0.65 | 0.57 | 0.64 | 0.63 |
| $\boldsymbol{\beta}_1$ | 0.51 | 0.54 | 0.53 | 0.55 | 0.48 | 0.57 | 0.47 | 0.52 | 0.5 | 0.53 |
| $\boldsymbol{\beta}_2$ | 0.43 | 0.48 | 0.46 | 0.37 | 0.44 | 0.46 | 0.39 | 0.41 | 0.48 | 0.42 |
| $\boldsymbol{\beta}_3$ | 0.59 | 0.56 | 0.58 | 0.62 | 0.6 | 0.52 | 0.65 | 0.57 | 0.64 | 0.63 |
| $\boldsymbol{a}_1$ | 12 | -2 | 4 | -7 | 13 | -1 | -6 | 6 | 11 | -8 |
| $\boldsymbol{a}_2$ | -2 | 5 | 3 | 16 | 6 | -12 | 12 | 4 | -7 | -10 |
| $\boldsymbol{a}_3$ | 3 | -16 | -4 | -8 | -8 | 2 | -12 | -12 | 4 | -3 |
| $\boldsymbol{a}_4$ | -11 | 6 | -5 | 9 | -1 | 8 | -4 | 6 | -9 | 6 |
| $\boldsymbol{a}_5$ | -4 | 7 | -6 | -5 | 13 | 6 | -2 | -5 | 14 | -6 |
| $\boldsymbol{a}_6$ | 5 | -3 | 14 | -3 | -9 | -7 | 4 | -4 | -5 | 9 |
| $\boldsymbol{a}_7$ | -3 | -4 | -6 | 9 | 6 | 18 | 11 | -9 | -4 | 7 |

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_{10})^T$ is a 10-dimensional decision vector and $X = \{\boldsymbol{a}_j \boldsymbol{x} \leq b_j, j = 1, \cdots, 7, \boldsymbol{x} \geq \boldsymbol{0}\}$. Under the occurrence of each elementary event $\omega$, $\widetilde{c}_{i\ell}(\omega)$ is a realization of the fuzzy random variable $\widetilde{\bar{c}}_{i\ell}$, which is a fuzzy number whose membership function is defined as follows.

$$\mu_{\widetilde{c}_{i\ell}(\omega)}(s) = \begin{cases} L\left(\frac{\bar{d}_{i\ell}(\omega)-s}{\alpha_{i\ell}}\right) & (s \leq \bar{d}_{i\ell}(\omega)\ \forall\omega), \\ R\left(\frac{s-\bar{d}_{i\ell}(\omega)}{\beta_{i\ell}}\right) & (s > \bar{d}_{i\ell}(\omega)\ \forall\omega), \end{cases}$$

where the function $L(t)(= R(t)) \stackrel{\text{def}}{=} \max\{0, 1- \mid t \mid\}$. The parameters $\bar{d}_{i\ell}, i = 1, 2, 3, j = 1, \cdots, 10$ are random variables expressed as:

$$\bar{d}_{i\ell} = d_{i\ell}^1 + \bar{t}_i d_{i\ell}^2, \tag{39}$$

where $\bar{t}_i, i = 1, 2, 3$ are Gaussian random variables as follows:

$$\begin{aligned} \bar{t}_1 &\sim \mathrm{N}(4, 2^2), \\ \bar{t}_2 &\sim \mathrm{N}(3, 3^2), \\ \bar{t}_3 &\sim \mathrm{N}(3, 2^2). \end{aligned}$$

$d_{i\ell}^1$, $d_{i\ell}^2$, $\alpha_{i\ell}$, and $\beta_{i\ell}, i = 1, 2, 3, j = 1, \cdots, 10$ are constants as shown in Table II. Considering the individual minimum and maximum of $E(\bar{\boldsymbol{d}}_i)\boldsymbol{x}, i = 1, 2, 3$, the hypothetical decision maker sets his/her linear membership functions of fuzzy goals $\widetilde{G}_i, i = 1, 2, 3$ for the original objective functions $\widetilde{\bar{\boldsymbol{c}}}_i \boldsymbol{x}, i = 1, 2, 3$ as follows.

$$\begin{aligned} \mu_{\widetilde{G}_1}(y_1) &= \frac{y_1 - 1800}{1700 - 1800} \\ \mu_{\widetilde{G}_2}(y_2) &= \frac{y_2 - 700}{600 - 700} \\ \mu_{\widetilde{G}_3}(y_3) &= \frac{y_3 - (-900)}{(-1000) - (-900)} \end{aligned}$$

For the elicited membership functions $\mu_{\widetilde{G}_i}(y_i)$,$i = 1, 2, 3$, the hypothetical decision maker sets the intervals $H_i = [h_{i\min}, h_{i\max}], i = 1, 2, 3$ as follows in his/her subjective manner.

$$\begin{aligned} H_1 &= [h_{1\min}, h_{1\max}] = [0.3, 0.7] \\ H_2 &= [h_{2\min}, h_{2\max}] = [0.3, 0.7] \\ H_3 &= [h_{3\min}, h_{3\max}] = [0.3, 0.7] \end{aligned}$$

Then, using (29), (30) and (31), the corresponding domains $P_i(H_i) = [p_{i\min}, p_{i\max}], i = 1, 2, 3$ are calculated as follows.

$$\begin{aligned} P_1(H_1) &= [p_{1\min}, p_{1\max}] = [0.0003, 0.9510] \\ P_2(H_2) &= [p_{2\min}, p_{2\max}] = [0.1099, 0.9996] \\ P_3(H_3) &= [p_{3\min}, p_{3\max}] = [0.0754, 0.9906] \end{aligned}$$

Then, the hypothetical decision maker sets his/her membership functions of fuzzy goals for $p_i(\boldsymbol{x}, h_i), i = 1, 2, 3$ as follows.

$$\begin{aligned} \mu_{p_1}(p_1(\boldsymbol{x}, h_1)) &= \frac{p_1(\boldsymbol{x}, h_1) - 0.0003}{(0.9510 - 0.0003)} \\ \mu_{p_2}(p_2(\boldsymbol{x}, h_2)) &= \frac{p_2(\boldsymbol{x}, h_2) - 0.1099}{(0.9996 - 0.1099)} \\ \mu_{p_3}(p_3(\boldsymbol{x}, h_3)) &= \frac{p_3(\boldsymbol{x}, h_3) - 0.0754}{(0.9906 - 0.0754)} \end{aligned}$$

For these membership functions $\mu_{\widetilde{G}_i}(y_i)$, $\mu_{p_i}(p_i(\boldsymbol{x}, h_i))$,$i = 1, 2, 3$, MAXMIN4 is formulated and solved by combined use of the bisection method with respect to $0 \leq \lambda \leq 1$ and the first-phase of the two-phase simplex method of linear programming. The optimal solution is obtained as $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*, x_9^*, x_{10}^*, \lambda^*) = (11.7424, 0, 0, 3.3132, 0, 3.0463, 0, 12.2617, 2.9396, 21.144, 0.4704)$. Since three constraints for (38) are active at the optimal solution, we can get the following optimal values of the corresponding membership function.

$$\begin{aligned} \mu_{p_1}(p_1(\boldsymbol{x}^*, \lambda^*)) &= 0.4704 \\ \mu_{p_2}(p_2(\boldsymbol{x}^*, \lambda^*)) &= 0.4704 \\ \mu_{p_3}(p_3(\boldsymbol{x}^*, \lambda^*)) &= 0.4704 \\ \mu_{\widetilde{G}_1}(\mu_{\widetilde{G}_1}^{-1}(\lambda^*)) &= 0.4704 \\ \mu_{\widetilde{G}_2}(\mu_{\widetilde{G}_2}^{-1}(\lambda^*)) &= 0.4704 \\ \mu_{\widetilde{G}_3}(\mu_{\widetilde{G}_3}^{-1}(\lambda^*)) &= 0.4704 \end{aligned}$$

At the optimal solution, the proper balance between the membership functions $\mu_{p_i}(p_i(\boldsymbol{x}, \lambda)), i = 1, 2, 3$ and $\mu_{\widetilde{G}_i}(\mu_{\widetilde{G}_i}^{-1}(\lambda)), i = 1, 2, 3$ in a probability maximization model is attained through the fuzzy decision.

## VI. Conclusion

In this paper, two kinds of fuzzy approaches are proposed to obtain a satisfactory solution of the decision maker, where the first one is for multiobjective stochastic linear programming problems, and the second one is for multiobjective fuzzy random linear programming problems. Both of them are formulated on the basis of a probability maximization model. In our proposed methods for such two kinds of multiobjective programming problems, it is not necessary that the decision maker specifies permissible levels in a probability maximization model. Instead of that, by adopting the fuzzy decision as an aggregation operator of fuzzy goals for both permissible levels and distribution functions, a satisfactory solution of the decision maker is easily obtained based on linear programming technique. Although a probability maximization model is one of the most efficient tool to transform stochastic programming problems into well-defined mathematical programming ones, appropriate permissible levels are not known for the decision maker in advance. In order to resolve such a problem, we have proposed fuzzy approaches for both multiobjective stochastic linear programming problems and multiobjective fuzzy random linear programming problems under the assumption that the decision maker adopts the fuzzy decision as an aggregation operator of fuzzy goals.

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