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Intrinsic Order, Lexicographic Order, Vector Order and Hamming Weight

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Abstract—**To compare binary n-tuple probabilities with no need to compute them, we have defined a partial order relation on the set $\{0,1\}^n$ of all binary n-tuples: The so-called intrinsic order relation. In this paper, some properties of the intrinsic ordering are derived. These properties involve the lexicographic (truth-table) order in $\{0,1\}^n$, the vector order defined between the vectors of positions of 1-bits of the binary n-tuples, and the number of 1-bits in the binary n-tuples (i.e., the Hamming weights). These results are illustrated through simple examples and the intrinsic order graph.**

Index Terms—**complex stochastic Boolean system, Hamming weight, intrinsic order, intrinsic order graph, lexicographic order, vector order.**

I. Introduction

THIS paper analyzes the behavior of those complex systems which depend on a large number n of random Boolean variables: The so-called complex stochastic Boolean systems (hereafter, CSBSs). That is, the n basic Boolean variables of the system are stochastic (non-deterministic) and they only take two possible values, 0 or 1. Using the statistical terminology, a stochastic Boolean variable can be considered as a Bernoulli variable.

Each one of the 2^n outcomes for a CSBS is given by a binary n-tuple $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ of 0s and 1s. In the following, we assume that the n Bernoulli variables $x_1, x_2, \ldots, x_n$ of the CSBS are statistically independent, so that the occurrence probability of a given binary string of length n, $u = (u_1, \ldots, u_n) \in \{0,1\}^n$, is given by

$$\Pr\{u\} = \prod_{i=1}^{n} p_i^{u_i} (1-p_i)^{1-u_i}, \tag{1}$$

that is, $\Pr\{u\}$ is the product of factors p_i if $u_i = 1$, $1 - p_i$ if $u_i = 0$.

Example 1.1: Let $n = 4$ and $u = (1,0,1,0) \in \{0,1\}^4$. Let $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.3$, $p_4 = 0.4$. Then using (1), we have

$$\Pr\{(1,0,1,0)\} = p_1(1-p_2)\,p_3(1-p_4) = 0.0144.$$

One of the main questions in the analysis of CSBSs consists of determining the ordering between the current values of the 2^n associated binary n-tuple probabilities $\Pr\{u\}$. The simplest answer to this question, namely computing all these 2^n probabilities –by using (1)– and ordering them in decreasing or increasing order of their values, is only possible in practice for small values of n. However, for large values of n, we need alternative procedures for comparing the binary string probabilities overcoming the exponential nature of this problem. For this purpose, in [2] we have defined a partial order relation on the set $\{0,1\}^n$ of all the 2^n binary n-tuples, the so-called *intrinsic order* between binary n-tuples.

Using the intrinsic ordering, we can compare (order) two given binary n-tuple probabilities $\Pr\{u\}, \Pr\{v\}$, with no need to compute them, simply looking at the relative positions of the 0s and 1s in the binary n-tuples u, v to be compared. In this way, for those pairs (u,v) of binary n-tuples comparable by intrinsic order, the ordering between their occurrence probabilities is always the same for all sets of basic probabilities $\{p_i\}_{i=1}^n$. On the contrary, for those pairs (u,v) of binary n-tuples incomparable by intrinsic order, the ordering between their occurrence probabilities depends on the current values of the basic probabilities $\{p_i\}_{i=1}^n$.

The lexicographic order on the set $\{0,1\}^n$ is the usual truth-table order between binary strings of length n. The Hamming weight of a binary n-tuple $u \in \{0,1\}^n$ is the sum of all its bits, that is, the number of 1-bits in u. The vector order is a total order relation defined between the vectors of positions of 1-bits of the binary n-tuples with the same Hamming weight.

The purpose of this paper is to present the relations between the intrinsic ordering and the three above concepts (lexicographic order, Hamming weight, and vector order). Some of these relations, especially those dealing with the Hamming weight, can be found in [9]. For this purpose, this paper has been organized as follows. In Section II, we present all the background about the intrinsic order required to make this paper self-contained. Section III is devoted to present the relations between the intrinsic ordering and the lexicographic order. In Section IV, the relation between the intrinsic ordering and the Hamming weight is presented. The relation between the intrinsic ordering and the vector order is analyzed in Section V. Finally, conclusions are presented in Section VI.

II. The Intrinsic Ordering

A. Intrinsic Order Relation on $\{0,1\}^n$

Throughout this paper, the decimal numbering of a binary string u is denoted by the symbol $u_{(10}$, and we use the symbol "≡" to denote the equivalence between the binary and decimal representations of a binary string, i.e.,

$$u = (u_1, \ldots, u_n) \equiv u_{(10} = \sum_{i=1}^{n} 2^{n-i} u_i,$$

e.g., for $n = 6$ we have

$$(1,0,1,1,0,1) \equiv 2^0 + 2^2 + 2^3 + 2^5 = 45.$$

According to (1), the ordering between two given binary string probabilities $\Pr(u)$ and $\Pr(v)$ depends, in general, on the parameters p_i, as the following simple example shows.

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Example 2.1: Let $n = 3,\ u = (0, 1, 1)$ and $v = (1, 0, 0)$. Using (1), we get the following inequalities.
For $p_1 = 0.1,\ p_2 = 0.2,\ p_3 = 0.3$:

$$\Pr\{u\} = 0.054 < \Pr\{v\} = 0.056,$$

while, for $p_1 = 0.2,\ p_2 = 0.3,\ p_3 = 0.4$:

$$\Pr\{u\} = 0.096 > \Pr\{v\} = 0.084.$$

As mentioned in Section I, to overcome the exponential complexity inherent to the task of computing and sorting the 2^n binary string probabilities (associated to a CSBS with n Boolean variables), we have introduced the following intrinsic order criterion [2], denoted from now on by the acronym IOC.

Theorem 2.1 (The intrinsic order theorem): Let $n \geq 1$. Suppose that $x_1, \ldots, x_n$ are n mutually independent Bernoulli variables whose parameters $p_i = \Pr\{x_i = 1\}$ satisfy

$$0 < p_1 \leq p_2 \leq \cdots \leq p_n \leq 0.5. \tag{2}$$

Then the probability of the binary n-tuple $v = (v_1, \ldots, v_n)$ is *intrinsically* less than or equal to the probability of the binary n-tuple $u = (u_1, \ldots, u_n)$ (that is, for all set $\{p_i\}_{i=1}^n$ satisfying (2)) if and only if the matrix

$$M_v^u := \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

either has no $\binom{1}{0}$ columns, or for each $\binom{1}{0}$ column in M_v^u there exists (at least) one corresponding preceding $\binom{0}{1}$ column (IOC).

Remark 2.1: In the following, we assume that the parameters p_i always satisfy condition (2). Note that this hypothesis is not restrictive for practical applications because, if for some $i : p_i > 0.5$, then we only need to consider the variable $\overline{x_i} = 1 - x_i$, instead of x_i. Next, we order the n Bernoulli variables by increasing order of their probabilities.

Remark 2.2: The $\binom{0}{1}$ column preceding to each $\binom{1}{0}$ column is not required to be necessarily placed at the immediately previous position, but just at previous position.

Remark 2.3: The term *corresponding,* used in Theorem 2.1, has the following meaning: For each two $\binom{1}{0}$ columns in matrix M_v^u, there must exist (at least) two *different* $\binom{0}{1}$ columns preceding to each other. In other words: For each $\binom{1}{0}$ column in matrix M_v^u, the number of preceding $\binom{0}{1}$ columns must be strictly greater than the number of preceding $\binom{1}{0}$ columns.

Remark 2.4: IOC can be equivalently reformulated in the following way, involving only the 1-bits of u and v (with no need to use their 0-bits). Matrix M_v^u satisfies IOC if and only if either u has no 1-bits (i.e., u is the zero n-tuple) or for each 1-bit in u there exists (at least) one corresponding 1-bit in v placed at the same or at a previous position. In other words, either u has no 1-bits or for each 1-bit in u, say $u_i = 1$, the number of 1-bits in $(v_1, \ldots, v_i)$ must be greater than or equal to the number of 1-bits in $(u_1, \ldots, u_i)$.

The matrix condition IOC, stated by Theorem 2.1 or by Remark 2.4, is called the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it *intrinsically* depends on the relative positions of the 0s and 1s in the binary n-tuples u, v. Theorem 2.1 or Remark 2.4 naturally lead to the following partial order relation on the set $\{0,1\}^n$ [2].

The so-called intrinsic order will be denoted by "$\preceq$", and we shall write $v \preceq u$, or $u \succeq v$, to indicate that v is intrinsically less than or equal to u, or that u is intrinsically greater than or equal to v.

Definition 2.1: For all $u, v \in \{0,1\}^n$

$$v \preceq u \text{ iff } \Pr\{v\} \leq \Pr\{u\} \text{ for all set } \{p_i\}_{i=1}^n \text{ s.t. (2)}$$

$$\text{iff } M_v^u \text{ satisfies IOC.}$$

From now on, the partially ordered set (poset, for short) $(\{0,1\}^n, \preceq)$ will be denoted by I_n.

Example 2.2: For $n = 3$, we have

$$3 \equiv (0, 1, 1) \npreceq 4 \equiv (1, 0, 0)$$

and

$$4 \equiv (1, 0, 0) \npreceq 3 \equiv (0, 1, 1),$$

because the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

do not satisfy IOC (Remark 2.3). Thus, $(0, 1, 1)$ and $(1, 0, 0)$ are incomparable by intrinsic order, i.e., the ordering between $\Pr\{(0, 1, 1)\}$ and $\Pr\{(1, 0, 0)\}$ depends on the parameters $\{p_i\}_{i=1}^3$, as Example 2.1 has shown.

Example 2.3: For $n = 6$, we have

$$56 \equiv (1, 1, 1, 0, 0, 0) \preceq 5 \equiv (0, 0, 0, 1, 0, 1)$$

because matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

satisfies IOC (Remark 2.2).
Thus, for all $\{p_i\}_{i=1}^6$ s.t. (2)

$$\Pr\{(1, 1, 1, 0, 0, 0)\} \leq \Pr\{(0, 0, 0, 1, 0, 1)\}.$$

Example 2.4: For all $n \geq 1$, the binary n-tuples

$$\left(0, \overset{n}{\ldots}, 0\right) \equiv 0 \quad \text{and} \quad \left(1, \overset{n}{\ldots}, 1\right) \equiv 2^n - 1$$

are the maximum and minimum elements, respectively, in the poset I_n. Indeed, both matrices

$$\begin{pmatrix} 0 & \ldots & 0 \\ u_1 & \ldots & u_n \end{pmatrix} \text{ and } \begin{pmatrix} u_1 & \ldots & u_n \\ 1 & \ldots & 1 \end{pmatrix}$$

satisfy the intrinsic order criterion, since obviously they have no $\binom{1}{0}$ columns!
Thus, for all $u \in \{0,1\}^n$ and for all $\{p_i\}_{i=1}^n$ s.t. (2)

$$\Pr\left\{\left(1, \overset{n}{\ldots}, 1\right)\right\} \leq \Pr\{(u_1, \ldots, u_n)\} \leq \Pr\left\{\left(0, \overset{n}{\ldots}, 0\right)\right\}.$$

Many different properties of the intrinsic order relation can be derived from its simple matrix description IOC (see, e.g., [2], [3], [4], [5]).

B. A Graph for the Intrinsic Order

Now, we present the most common graphical representation of our poset $I_n = (\{0,1\}^n, \preceq)$. The usual representation of a poset is its Hasse diagram (see [12] for more details about these diagrams). Specifically, for our poset I_n, its Hasse diagram is a directed graph (digraph, for short) whose vertices are the 2^n binary n-tuples of 0s and 1s, and whose edges go upward from v to u whenever u covers v, denoted by $u \rhd v$. This means that u is intrinsically greater than v with no other elements between them, i.e.,

$$u \rhd v \quad \Leftrightarrow \quad u \succ v \text{ and } \nexists \ w \in \{0,1\}^n \quad \text{s.t.} \quad u \succ w \succ v.$$

A simple matrix characterization of the covering relation for the intrinsic order is given in the next theorem; see [4] for the proof.

Theorem 2.2 (Covering relation in I_n): Let $n \geq 1$ and let $u, v \in \{0,1\}^n$. Then $u \rhd v$ if and only if the only columns of matrix M_v^u different from $\binom{0}{0}$ and $\binom{1}{1}$ are either its last column $\binom{0}{1}$ or just two columns, namely one $\binom{1}{0}$ column immediately preceded by one $\binom{0}{1}$ column, i.e., either

$$M_v^u = \begin{pmatrix} u_1 & \dots & u_{n-1} & 0 \\ u_1 & \dots & u_{n-1} & 1 \end{pmatrix} \tag{3}$$

or there exists i $(2 \leq i \leq n)$ s.t.

$$M_v^u = \begin{pmatrix} u_1 & \dots & u_{i-2} & 0 & 1 & u_{i+1} & \dots & u_n \\ u_1 & \dots & u_{i-2} & 1 & 0 & u_{i+1} & \dots & u_n \end{pmatrix}. \tag{4}$$

Example 2.5: For $n = 4$, we have

$$6 \rhd 7 \text{ since } M_7^6 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \text{ has the pattern (3),}$$

$$10 \rhd 12 \text{ since } M_{12}^{10} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ has the pattern (4).}$$

The Hasse diagram of the poset I_n will be also called the *intrinsic order graph* for n variables, denoted as well by I_n.

For small values of n, the intrinsic order graph I_n can be directly constructed by using either Theorem 2.1 (matrix description of the intrinsic order) or Theorem 2.2 (matrix description of the covering relation for the intrinsic order). For instance, for $n = 1$: $I_1 = (\{0,1\}, \preceq)$, and its Hasse diagram is shown in Fig. 1. Indeed I_1 contains a downward

Fig. 1. The Intrinsic Order Graph for $n = 1$.

edge from 0 to 1 because (see Theorem 2.1) $0 \succ 1$, since matrix $\binom{0}{1}$ has no $\binom{1}{0}$ columns! Alternatively, using Theorem 2.2, we have that $0 \rhd 1$, since matrix $\binom{0}{1}$ has the pattern (3)! Moreover, this is in accordance with the obvious fact that

$$\Pr\{0\} = 1 - p_1 \geq p_1 = \Pr\{1\},$$

since $p_1 \leq 1/2$, due to (2)!

However, for large values of n, a more efficient method is needed. For this purpose, in [4] the following algorithm for iteratively building up I_n (for all $n \geq 2$) from I_1 (depicted in Fig. 1), has been developed.

Theorem 2.3 (Building up I_n from I_1): Let $n \geq 2$. The graph of the poset $I_n = \{0, \dots, 2^n - 1\}$ (on 2^n nodes) can be drawn simply by adding to the graph of the poset $I_{n-1} = \{0, \dots, 2^{n-1} - 1\}$ (on 2^{n-1} nodes) its isomorphic copy $2^{n-1} + I_{n-1} = \{2^{n-1}, \dots, 2^n - 1\}$ (on 2^{n-1} nodes). This addition must be performed placing the powers of 2 at consecutive levels of the Hasse diagram of I_n. Finally, the edges connecting one vertex of I_{n-1} with the other vertex of $2^{n-1} + I_{n-1}$ are given by the set of 2^{n-2} vertex pairs

$$\left\{ \left(u_{(10}, 2^{n-2} + u_{(10} \right) \mid 2^{n-2} \leq u_{(10} \leq 2^{n-1} - 1 \right\}.$$

In Fig. 2, we illustrate the above iterative process for the first few values of n, denoting all the binary n-tuples by their decimal equivalents. Basically, we first add to I_{n-1} its isomorphic copy $2^{n-1} + I_{n-1}$. This addition must be performed by placing the powers of two, 2^{n-2} and 2^{n-1}, at consecutive levels in the intrinsic order graph. The reason is simply that

$$2^{n-2} \rhd 2^{n-1} \quad \text{since matrix } M_{2^{n-1}}^{2^{n-2}} \quad \text{has the pattern (4).}$$

Then, we connect one-to-one the nodes of "the second half of the first half" to the nodes of "the first half of the second half": A nice fractal property of I_n!

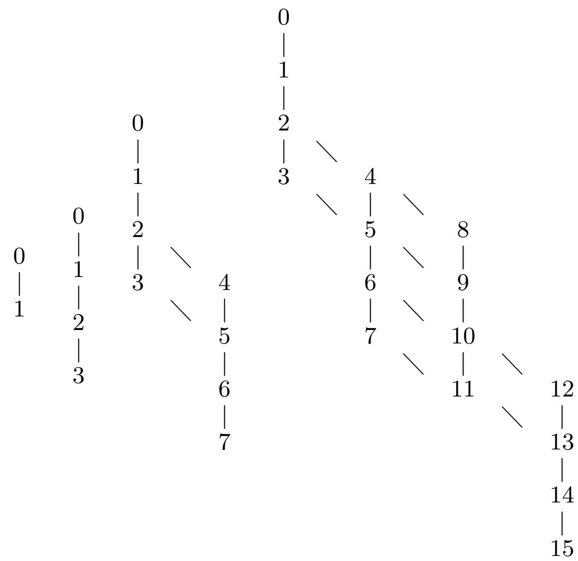

Fig. 2. The Intrinsic Order Graphs for $n = 1, 2, 3, 4$.

Each pair (u, v) of vertices connected in I_n either by one edge or by a longer path, descending from u to v, means that u is intrinsically greater than v, i.e., $u \succ v$. On the contrary, each pair (u, v) of non-connected vertices in I_n either by one edge or by a longer descending path, means that u and v are incomparable by intrinsic order, i.e., $u \nsucc v$ and $v \nsucc u$.

Looking at any of the four graphs in Fig. 2, we can confirm the fact that 0 and $2^n - 1$ are the maximum and minimum elements, respectively, in the poset I_n (see Example 2.4). Also, Theorems 2.1 and 2.2 are illustrated by Fig. 2.

The edgeless graph for a given graph is obtained by removing all its edges, keeping its nodes or vertices at the same positions [1]. In Fig. 3, the edgeless intrinsic order graph of I_5 is depicted.

For further theoretical properties and practical applications of the intrinsic order and the intrinsic order graph, we refer the reader to e.g., [2], [3], [4], [5], [6], [7], [8], [9], [10], [11].

0							
1							
2							
3	4						
	5	8					
	6	9		16			
	7	10		17			
		11	12	18			
			13	19	20		
			14		21	24	
			15		22	25	
					23	26	
						27	28
							29
							30
							31

Fig. 3. The Edgeless Intrinsic Order Graph for $n = 5$.

III. Intrinsic Order and Lexicographic Order

The lexicographic order between binary n-tuples is the usual truth-table order on $\{0,1\}^n$ (denoted here by the symbol "$\leq_l$") beginning with the n-tuple $\left(0, \overset{n}{\ldots}, 0\right)$ and finishing with the n-tuple $\left(1, \overset{n}{\ldots}, 1\right)$. As is well-known, this ordering coincides with the natural ordering between the decimal equivalents of the rows (binary n-tuples) of the truth-table. That is,

$$u \leq_l v \Leftrightarrow u_{(10} \leq v_{(10}\,. \tag{5}$$

Example 3.1: Let $n = 4$, $u = (1, 0, 1, 1)$, $v = (1, 1, 0, 1)$. Then

$$u = (1, 0, 1, 1) <_l (1, 1, 0, 1) = v$$

since u precedes v in the truth-table, or since

$$u_{(10} = 11 < 13 = v_{(10}\,.$$

The lexicographic order is a necessary condition for the intrinsic order. More precisely, we have the following corollary of Theorem 2.1; see [3] for the proof.

Corollary 3.1: For all $n \geq 1$ and for all $u, v \in \{0,1\}^n$

$$u \succeq v \Rightarrow u \leq_l v, \tag{6}$$

i.e.,

$$u \succeq v \Rightarrow u_{(10} \leq v_{(10}\,. \tag{7}$$

However, the necessary condition for intrinsic order stated by Corollary 3.1 is not sufficient. That is,

$$u \leq_l v \nRightarrow u \succeq v,$$

as the following simple counter-example (indeed, the simplest one that one can find!) shows.

Example 3.2: For

$$n = 3, \quad u = 3 \equiv (0, 1, 1)\,, \quad v = 4 \equiv (1, 0, 0)\,,$$

we have (see the digraph of I_3, the third one from left to right in Fig. 2)

$$u \leq_l v.$$

However $3 \not\succeq 4$, since matrix

$$M_4^3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

does not satisfy IOC.

However, for some special binary n-tuples $u \in \{0,1\}^n$ the necessary condition stated by Corollary 3.1 is also sufficient. Let us characterize in a very simple way such binary strings. First, we must set the following notation.

Definition 3.1: For all $n \geq 1$ and for any given binary n-tuple u, C^u (C_u, respectively) is the set of binary n-tuples v intrinsically less (greater, respectively) than or equal to v, i.e.,

$$C^u = \{v \in \{0,1\}^n \mid u \succeq v\}\,,$$
$$C_u = \{v \in \{0,1\}^n \mid u \preceq v\}\,.$$

Equivalently, according to Definition 2.1, C^u and C_u can be defined as

$$C^u = \{v \in \{0,1\}^n \,|\Pr\{u\} \geq \Pr\{v\}\,,\ \forall \{p_i\}_{i=1}^n \text{ s.t. } (2)\}\,,$$
$$C_u = \{v \in \{0,1\}^n \,|\Pr\{u\} \leq \Pr\{v\}\,,\ \forall \{p_i\}_{i=1}^n \text{ s.t. } (2)\}\,.$$

Definition 3.2: For all $n \geq 1$ and for any given binary n-tuple u, L_u (L^u, respectively) is the set of binary n-tuples v whose decimal equivalents are greater (less, respectively) than or equal to the decimal equivalent of u, i.e.,

$$L_u = \left\{v \in \{0,1\}^n \;\middle|\; u_{(10} \leq v_{(10}\right\}\,,$$
$$L^u = \left\{v \in \{0,1\}^n \;\middle|\; u_{(10} \geq v_{(10}\right\}\,.$$

Equivalently, according to (5), L_u and L^u can be defined as

$$L_u = \{v \in \{0,1\}^n \mid u \leq_l v\}\,,$$
$$L^u = \{v \in \{0,1\}^n \mid u \geq_l v\}\,.$$

With this notation, the implications (6) or (7) can be simply rewritten as

$$C^u \subseteq L_u \tag{8}$$

and the question of characterizing the binary n-tuples u for which

$$u \succeq v \Leftrightarrow u_{(10} \leq v_{(10}$$

is equivalent to characterize the binary n-tuples u for which

$$C^u = L_u$$

The following theorem provides the answer to this question (see [5] for the proof)

Theorem 3.1: Let $n \geq 1$ and $u = (u_1, \ldots, u_n) \in \{0,1\}^n$. Then

$$C^u = L_u$$

if and only if u does not contain any 0 bit followed by two (or more) 1 bits, placed at consecutive or non consecutive positions, i.e., u has the general pattern

$$u = (\underbrace{1, \ldots, 1}_{p}, \underbrace{0, \ldots, 0}_{q}, \underbrace{1}_{1}, \underbrace{0, \ldots, 0}_{r}),\ p + q + r + 1 = n, \tag{9}$$

where any (but not all!) of the above four subsets of bits grouped together can be omitted.

Example 3.3: For $n = 4$ and $u = 10 \equiv (1, 0, 1, 0)$, we have (see the digraph of I_4, the right-most one in Fig. 2)

$$C^{10} = \{10, 11, 12, 13, 14, 15\} = L_{10}$$

since $u = 10 \equiv (1, 0, 1, 0)$ has the pattern (9).

To establish the dual result of Theorem 3.1, we need the following definition.

Definition 3.3: Let $n \geq 1$. Then

(i) The complementary n-tuple of a given binary n-tuple $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ is obtained by changing its 0s by 1s and its 1s by 0s

$$u^c = (u_1, \ldots, u_n)^c = (1 - u_1, \ldots, 1 - u_n).$$

Obviously, two binary n-tuples are complementary if and only if their decimal equivalents sum up to

$$\left(1, \overset{n}{\ldots}, 1\right)_{(10} = 2^n - 1.$$

(ii) The complementary set of a given subset $S \subseteq \{0,1\}^n$ of binary n-tuples is the set of the complementary n-tuples of all the n-tuples of S

$$S^c = \{u^c \mid u \in S\}.$$

Now, interchanging the roles of u and v in (6), we get

$$v \succeq u \Rightarrow v \leq_l u, \tag{10}$$

Using the notation introduced in Definitions 3.1 and 3.2, the implication (10) can be rewritten as the following set inclusion, dual of the inclusion (8)

$$C_u \subseteq L^u. \tag{11}$$

Sometimes, for some binary n-tuples u, the inclusion (11) becomes the set identity $C_u = L^u$. These binary strings u satisfying this nice property are characterized by the following theorem, which is the dual of Theorem 3.1 because the 0s are changed by 1s and the 1s are changed by 0s in the corresponding positional criteria. The proof is straightforward, using Theorem 3.1 and the fact that (see [5] for more details)

$$(C_u)^c = C^{u^c}, \qquad (L^u)^c = L_{u^c}.$$

Theorem 3.2: Let $n \geq 1$ and $u = (u_1, \ldots, u_n) \in \{0,1\}^n$. Then

$$C_u = L^u$$

if and only if u does not contain any 1 bit followed by two (or more) 0 bits, placed at consecutive or non consecutive positions, i.e., u has the general pattern

$$u = (\underbrace{0, \ldots, 0}_{p}, \underbrace{1, \ldots, 1}_{q}, \underbrace{0}_{1}, \underbrace{1, \ldots, 1}_{r}), \; p + q + r + 1 = n, \tag{12}$$

where any (but not all!) of the above four subsets of bits grouped together can be omitted.

Example 3.4: For $n = 4$ and $u = 5 \equiv (0,1,0,1)$, we have (see the digraph of I_4, the right-most one in Fig. 2)

$$C_5 = \{0,1,2,3,4,5\} = L^5$$

since $u = 5 \equiv (0,1,0,1)$ has the pattern (12).

IV. Intrinsic Order and Hamming Weight

The Hamming weight –or simply the weight– of u is the sum of all its n bits. In other words, the Hamming weight of a binary n-tuple is the number of its 1-bits, and it will be denoted by

$$w_H(u) = \sum_{i=1}^{n} u_i.$$

Example 4.1: For $n = 7$, we have

$$w_H(1,0,1,0,1,1,0) = 4.$$

The intrinsic order respects the Hamming weight. More precisely, we have the following corollary of Theorem 2.1 (see, e.g., [3], [9] for the proof).

Corollary 4.1: For all $n \geq 1$ and for all $u, v \in \{0,1\}^n$

$$u \succeq v \;\; \Rightarrow \;\; w_H(u) \leq w_H(v).$$

Now, we present some relations between the intrinsic ordering and the Hamming weight. Our starting point is Corollary 4.1. This corollary has stated that a necessary condition for u being intrinsically greater than or equal to v is that the weight of u must be less than or equal to the weight of v. That is, let u be an arbitrary, but fixed, binary n-tuple. Then

$$u \succeq v \;\; \Rightarrow \;\; w_H(u) \leq w_H(v) \quad \text{for all } v \in \{0,1\}^n \tag{13}$$

or, equivalently,

$$w_H(u) > w_H(v) \;\; \Rightarrow \;\; u \not\succeq v.$$

For instance, looking at the digraph I_4, the right-most one in Fig. 2, we can confirm that

$$4 \equiv (0,1,0,0) \succ 13 \equiv (1,1,0,1),$$
$$w_H(4) = 1 < 3 = w_H(13)$$

and that

$$3 \equiv (0,0,1,1) \succ 12 \equiv (1,1,0,0),$$
$$w_H(3) = 2 = w_H(12).$$

However, the necessary condition for intrinsic order stated by Corollary 4.1 is not sufficient. That is,

$$w_H(u) \leq w_H(v) \nRightarrow u \succeq v,$$

as the following simple counter-example (indeed, the simplest one that one can find!) shows.

Example 4.2: For

$$n = 3, \;\; u = 4 \equiv (1,0,0), \;\; v = 3 \equiv (0,1,1),$$

we have (see the digraph of I_3, the third one from left to right in Fig. 2)

$$w_H(4) = 1 < 2 = w_H(3).$$

However $4 \not\succeq 3$, since matrix

$$M_3^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

does not satisfy IOC (or, more easily, since $4 > 3$; see Corollary 3.1).

Moreover, even assuming that the two necessary conditions stated by Corollaries 3.1 & 4.1 simultaneously hold, this does not imply intrinsic order. That is,

$$u_{(10} < v_{(10} \text{ and } w_H(u) \leq w_H(v) \nRightarrow u \succeq v,$$

as the following simple counter-example (indeed, the simplest one that one can find!) shows.

Example 4.3: For

$$n = 4, \;\; u = 6 \equiv (0,1,1,0), \;\; v = 9 \equiv (1,0,0,1),$$

we have (see the digraph of I_4, the right-most one in Fig. 2)

$$u_{(10} = 6 < v_{(10} = 9 \text{ and } w_H(6) = 2 = w_H(9).$$

However $6 \not\succeq 9$, since matrix

$$M_9^6 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

does not satisfy IOC.

Moreover, even though assuming that the Hamming weight of u is strictly less than the Hamming weight of v, the two necessary conditions stated by Corollaries 3.1 & 4.1 do not imply intrinsic order. That is,

$$u_{(10} < v_{(10} \text{ and } w_H(u) < w_H(v) \nRightarrow u \succeq v,$$

as the following simple counter-example (indeed, the simplest one that one can find!) shows.

Example 4.4: For

$$n = 5, \quad u = 12 \equiv (0,1,1,0,0), \quad v = 19 \equiv (1,0,0,1,1),$$

we have

$$u_{(10} = 12 < v_{(10} = 19 \text{ and } w_H(12) = 2 < 3 = w_H(19).$$

However $12 \not\succeq 19$, since matrix

$$M_{19}^{12} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

does not satisfy IOC.

In this context, two dual questions naturally arise. They are posed in the two subsections of this section. First, we need to set the following notation.

Definition 4.1: For every binary n-tuple $u \in \{0,1\}^n$, H^u (H_u, respectively) is the set of all binary n-tuples v whose Hamming weights are less (greater, respectively) than or equal to the Hamming weight of u, i.e.,

$$H^u = \{v \in \{0,1\}^n \mid w_H(u) \geq w_H(v)\},$$
$$H_u = \{v \in \{0,1\}^n \mid w_H(u) \leq w_H(v)\}.$$

A. Greater Weight and Less Probability

Looking at the implication (13), the following question immediately arises.

Can we characterize the binary n-tuples u for which the necessary condition (13) is also sufficient? That is, we try to identify those bitstrings $u \in \{0,1\}^n$ for which the set of binary n-tuples v with weights greater than or equal to the one of u coincides with the set of binary n-tuples v with occurrence probabilities less than or equal to the one of u, i.e.,

$$u \succeq v \quad \Leftrightarrow \quad w_H(u) \leq w_H(v), \text{ i.e., } C^u = H_u.$$

The following theorem provides the answer to this question, in a very simple way.

Theorem 4.1: Let $n \geq 1$ and $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ with Hamming weight $w_H(u) = m$ $(0 \leq m \leq n)$. Then

$$C^u = H_u$$

if and only if either u is the zero n-tuple ($m = 0$) or the m 1-bits of u ($m > 0$) are placed at the m right-most positions, i.e., if and only if u has the general pattern

$$u = \left(0, \overset{n-m}{\smile\!\!\!\ldots}, 0, 1, \overset{m}{\smile\!\!\!\ldots}, 1\right) \equiv 2^m - 1, \quad 0 \leq m \leq n, \tag{14}$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted.

Proof: Sufficient condition. We distinguish two cases:

(i) If u is the zero n-tuple $0 \equiv \left(0, \overset{n}{\ldots}, 0\right)$, then u is the maximum element for the intrinsic order (as we have proved in Example 2.4). Then

$$C^0 = \{v \in \{0,1\}^n \mid 0 \succeq v\} = \{0,1\}^n$$
$$= \{v \in \{0,1\}^n \mid w_H(0) = 0 \leq w_H(v)\} = H_0.$$

(ii) If u is not the zero n-tuple, then u has the pattern (14) with $m > 0$. Let $v \in H_u$, i.e., let v let a binary n-tuple with Hamming weight greater than or equal to m (the Hamming weight of u). We distinguish two subcases:

(ii)-(a) Suppose that the weight of v is

$$w_H(v) = m = w_H(u).$$

Then v has exactly m 1-bits and $n - m$ 0-bits. Call r the number of 1-bits of v placed among the m right-most positions ($\max\{0, 2m-n\} \leq r \leq m$). Obviously, v has r 1-bits and $m - r$ 0-bits placed among the m right-most positions, and also it has $m - r$ 1-bits and $n - 2m + r$ 0-bits placed among the $n - m$ left-most positions. These are the positions of the

$$r + (m-r) + (m-r) + (n-2m+r) = m + (n-m) = n$$

bits of the binary n-tuple v.

Hence, matrix M_v^u has exactly $m - r$ $\binom{1}{0}$ columns (all placed among the m right-most positions) and exactly $m - r$ $\binom{0}{1}$ columns (all placed among the $n - m$ left-most positions). Thus, M_v^u satisfies IOC and then $u \succeq v$, i.e., $v \in C^u$.
So, for this case (ii)-(a), we have proved that

$$\{v \in \{0,1\}^n \mid w_H(v) = w_H(u) = m\} \subseteq C^u \tag{15}$$

(ii)-(b) Suppose that the weight of v is

$$w_H(v) = m + p > m = w_H(u) \quad (0 < p \leq n - m).$$

Then define a new binary n-tuple s as follows. First, select any p 1-bits in v (say, for instance, $v_{i_1} = \cdots = v_{i_p} = 1$). Second, s is constructed by changing these p 1-bits of v by 0-bits, assigning to the remainder $n - p$ bits of s the same values as the ones of v. Formally, $s = (s_1, \ldots, s_n)$ is defined by

$$s_i = \begin{cases} 0 & \text{if } i \in \{i_1, \ldots, i_p\}, \\ v_i & \text{if } i \notin \{i_1, \ldots, i_p\}. \end{cases}$$

On one hand, $u \succeq s$ since

$$w_H(s) = w_H(v) - p = m = w_H(u)$$

and then we can apply case (ii)-(a) to s.

On the other hand, $s \succeq v$ since matrix M_v^s has p $\binom{0}{1}$ columns (placed at positions $i_1, \ldots, i_p$), while its $n - p$ reminder columns are either $\binom{0}{0}$ or $\binom{1}{1}$. Hence M_v^s has no $\binom{1}{0}$ columns, so that it satisfies IOC.

Finally, from the transitive property of the intrinsic order, we derive

$$u \succeq s \text{ and } s \succeq v \quad \Rightarrow \quad u \succeq v, \text{ i.e., } v \in C^u.$$

So, for this case (ii)-(b), we have proved that

$$\{v \in \{0,1\}^n \mid w_H(v) > w_H(u) = m\} \subseteq C^u \tag{16}$$

From (15) & (16), we get

$$\{v \in \{0,1\}^n \mid w_H(v) \geq w_H(u) = m\} \subseteq C^u,$$

i.e., $H_u \subseteq C^u$, and this set inclusion together with the converse inclusion $C^u \subseteq H_u$ (which is always satisfied for every binary n-tuple u; see Corollary 4.1) leads to the set equality $C^u = H_u$. This proves the sufficient condition.

Necessary condition. Conversely, suppose that not all the m 1-bits of u are placed at the m right-most positions. In other words, suppose that

$$u \neq \left(0, \overset{n-m}{\smile\!\!\!\ldots}, 0, 1, \overset{m}{\smile\!\!\!\ldots}, 1\right).$$

Since, by assumption, $w_H(u) = m$ then simply using the necessary condition we derive that

$$\left(0, \overset{n-m}{\smile\!\!\!\ldots}, 0, 1, \overset{m}{\smile\!\!\!\ldots}, 1\right) \succ u,$$

and then

$$\left(0, \overset{n-m}{\smile\!\!\!\ldots}, 0, 1, \overset{m}{\smile\!\!\!\ldots}, 1\right) \in H_u - C^u$$

so that,

$$H_u \nsubseteq C^u.$$

This proves the necessary condition. ■

Corollary 4.2: Let $n \geq 1$ and let

$$u = \left(0, \overset{n-m}{\smile\!\!\!\ldots}, 0, 1, \overset{m}{\smile\!\!\!\ldots}, 1\right) \equiv 2^m - 1, \quad 0 \leq m \leq n,$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted. Then the number of binary n-tuples intrinsically less than or equal to u is

$$|C^u| = \binom{n}{m} + \binom{n}{m+1} + \cdots + \binom{n}{n}.$$

Proof: Using Theorem 4.1, we have

$$\begin{aligned} C^u = H_u \Rightarrow |C^u| &= |H_u| \\ &= |\{v \in \{0,1\}^n \mid w_H(u) = m \leq w_H(v)\}| \\ &= |\{v \in \{0,1\}^n \mid w_H(v) = m, m+1, \ldots, n\}| \\ &= \binom{n}{m} + \binom{n}{m+1} + \cdots + \binom{n}{n}, \end{aligned}$$

as was to be shown. ■

B. Less Weight and Greater Probability

Interchanging the roles of u & v, (13) can be rewritten as follows. Let u be an arbitrary, but fixed, binary n-tuple. Then

$$v \succeq u \quad \Rightarrow \quad w_H(v) \leq w_H(u) \quad \text{for all } v \in \{0,1\}^n. \quad (17)$$

Looking at the implication (17), the following dual question of the one posed in Section IV-A, immediately arises.

Can we characterize the binary n-tuples u for which the necessary condition (17) is also sufficient? That is, we try to identify those bitstrings $u \in \{0,1\}^n$ for which the set of binary n-tuples v with weights less than or equal to the one of u coincides with the set of binary n-tuples v with occurrence probabilities greater than or equal to the one of u, i.e.,

$$v \succeq u \quad \Leftrightarrow \quad w_H(v) \leq w_H(u), \text{ i.e., } C_u = H^u.$$

The following theorem provides the answer to this question, in a very simple way. For a very short proof of this theorem, we use Definition 3.3.

Theorem 4.2: Let $n \geq 1$ and $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ with Hamming weight $w_H(u) = m$ $(0 \leq m \leq n)$. Then

$$C_u = H^u$$

if and only if either u is the zero n-tuple ($m = 0$) or the m 1-bits of u ($m > 0$) are placed at the m left-most positions, i.e., if and only if u has the general pattern

$$u = \left(1, \overset{m}{\smile\!\!\!\ldots}, 1, 0, \overset{n-m}{\smile\!\!\!\ldots}, 0\right) \equiv 2^n - 2^{n-m}, \quad 0 \leq m \leq n, \quad (18)$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted.

Proof: Using Theorem 4.1 and the facts that (see, e.g., [5], [7])

$$(C_u)^c = C^{u^c}, \qquad (H^u)^c = H_{u^c},$$

we get

$$\begin{aligned} &C_u = H^u \Leftrightarrow (C_u)^c = (H^u)^c \Leftrightarrow C^{u^c} = H_{u^c} \\ &\quad \Leftrightarrow u^c \text{ has the pattern (14)} \ \Leftrightarrow u \text{ has the pattern (18)}, \end{aligned}$$

as was to be shown. ■

Corollary 4.3: Let $n \geq 1$ and let

$$u = \left(1, \overset{m}{\smile\!\!\!\ldots}, 1, 0, \overset{n-m}{\smile\!\!\!\ldots}, 0\right) \equiv 2^n - 2^{n-m}, \quad 0 \leq m \leq n,$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted. Then the number of binary n-tuples intrinsically greater than or equal to u is

$$|C_u| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}.$$

Proof: Using Corollary 4.2, we get

$$\begin{aligned} |C_u| = |(C_u)^c| &= \left|C^{u^c}\right| \\ &= \binom{n}{n-m} + \binom{n}{n-m+1} + \cdots + \binom{n}{n} \\ &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}, \end{aligned}$$

as was to be shown. ■

Example 4.5: Let $n = 5$.
According to Theorem 4.1, the 6 binary 5-tuples $u = 2^m - 1$ $(0 \leq m \leq 5)$, for which $C^u = H_u$ are:

$$(0,0,0,0,0) \equiv 0, \ (0,0,0,0,1) \equiv 1, \ (0,0,0,1,1) \equiv 3,$$
$$(0,0,1,1,1) \equiv 7, \ (0,1,1,1,1) \equiv 15, \ (1,1,1,1,1) \equiv 31.$$

Note that obviously $\{2^n - 2^{n-m}\}_{m=0}^n = \{2^n - 2^m\}_{m=0}^n$. Then, according to Theorem 4.2, the 6 binary 5-tuples $u = 2^5 - 2^m$ $(0 \leq m \leq 5)$, for which $C_u = H^u$ are the complementary ones of the above 5-tuples :

$$(1,1,1,1,1) \equiv 31, \ (1,1,1,1,0) \equiv 30, \ (1,1,1,0,0) \equiv 28,$$
$$(1,1,0,0,0) \equiv 24, \ (1,0,0,0,0) \equiv 16, \ (0,0,0,0,0) \equiv 0.$$

V. Intrinsic Order and Vector Order

Let $n \geq 1$ and let u be a nonzero binary n-tuple, with Hamming weight $w_H(u) = m > 0$. Then the vector of positions of 1s of u is defined as the vector of positions of its m 1-bits, with the convention that these positions are arranged in increasing order from the right-most possible position 1 to the left-most possible position n. This vector will be denoted by

$$[i_1, i_2, \ldots, i_m]_n \quad (1 \leq i_1 < i_2 < \cdots < i_m \leq n),$$

so that

$$\left\{ \begin{array}{lll} i \in \{i_1, i_2, \ldots, i_m\} & \text{iff} & u_{n+1-i} = 1, \\ i \notin \{i_1, i_2, \ldots, i_m\} & \text{iff} & u_{n+1-i} = 0. \end{array} \right. \tag{19}$$

We use again the symbol "$\equiv$" to denote the conversion between the new vector notation and the binary and decimal representations of the bitstrings, e.g.,

$$\left(\overset{\smile}{\underset{}{0}}{}^{5}, \overset{4}{1}, \overset{3}{0}, \overset{2}{0}, \overset{1}{1} \right) \equiv [i_1, i_2]_5 = [1, 4]_5 \equiv 9.$$

A new order relation between binary n-tuples with the same weight m, the so-called *vector order*, is introduced in the following definition.

Definition 5.1: Let $n \geq 2$ and let u, v be two binary n-tuples with the same Hamming weight

$$w_H(u) = w_H(v) = m \quad (0 < m < n)$$

and with vectors of positions of 1s

$$u = [i_1, i_2, \ldots, i_m]_n, \quad v = [j_1, j_2, \ldots, j_m]_n.$$

Then we say that u precedes or is equal to v in the vector order, denoted by $u \leq_v v$, if and only if either

$$i_p = j_p \ (1 \leq p \leq m),$$

or

$$q = \min\{p \in \{1, 2, \ldots, m\} \mid i_p \neq j_p\} \Rightarrow i_q < j_q.$$

Example 5.1: For $n = 5$ and $m = 3$, we have

$$[1, 2, 3]_5 <_v [1, 3, 5]_5 <_v [3, 4, 5]_5 \text{ i.e.,}$$

$$(0, 0, 1, 1, 1) <_v (1, 0, 1, 0, 1) <_v (1, 1, 1, 0, 0), \text{ i.e.,}$$

$$7 <_v 21 <_v <_v 28.$$

The following theorem provides us with a simple characterization of the intrinsic order between two binary n-tuples with the same weight, using their vectors of positions of 1s, instead of their binary representations used in Theorem 2.1 (IOC).

Theorem 5.1: Let $n \geq 2$ and let u, v be two binary n-tuples, with the same Hamming weight

$$w_H(u) = w_H(v) = m \quad (0 < m < n)$$

and with vectors of positions of 1s

$$u = [i_1, i_2, \ldots, i_m]_n, \quad v = [j_1, j_2, \ldots, j_m]_n.$$

Then

$$u \succeq v \quad \Leftrightarrow \quad j_p \geq i_p \text{ for all } p = 1, 2, \ldots, m.$$

Proof: Using Definition 2.1 and Remark 2.4, we have that $u \succeq v$ iff matrix M_v^u satisfies IOC iff either u has no 1-bits or for each 1-bit in u there exists (at least) one corresponding 1-bit in v placed at the same or at a previous position. Now, according to (19), sweeping the m 1-bits of u from left to right, the last assertion is equivalent to saying that

$$j_m \geq i_m, \ j_{m-1} \geq i_{m-1}, \ \ldots \ , j_1 \geq i_1,$$

i.e., $j_p \geq i_p$ for all $p = 1, 2, \ldots, m$. ■

The following corollary establishes the relationship between the intrinsic order and the vector order.

Corollary 5.1: Let $n \geq 2$ and let u, v be two binary n-tuples, with the same Hamming weight

$$w_H(u) = w_H(v) = m \quad (0 < m < n).$$

Then

$$u \succeq v \quad \Rightarrow \quad u \leq_v v.$$

Proof: Let

$$u = [i_1, i_2, \ldots, i_m]_n, \quad v = [j_1, j_2, \ldots, j_m]_n$$

be the vectors of positions of 1s of u and v, respectively. Then, using Theorem 5.1, we have

$$u \succeq v \quad \Rightarrow \quad j_p \geq i_p \text{ for all } p = 1, 2, \ldots, m. \tag{20}$$

We distinguish the following two cases.
(i) If $j_p = i_p$ for all $p = 1, 2, \ldots, m$, then $u = v$, so that, clearly, $u \leq_v v$.
(ii) If $j_p \neq i_p$ for some $p = 1, 2, \ldots, m$, then the least index q for which $i_q \neq j_q$ necessarily satisfies $i_q < j_q$, due to (20). Hence, using Definition 5.1, we get $u \leq_v v$. ■

The converse of Corollary 5.1 does not hold, as the following simple counter-example (indeed, the simplest one that one can find!) shows.

Example 5.2: For $n = 4$, $m = 2$ and for

$$\begin{array}{lllllll} u = (1,0,0,1) & \equiv & [i_1, i_2]_4 & = & [1, 4]_4 & \equiv & 9, \\ v = (0,1,1,0) & \equiv & [j_1, j_2]_4 & = & [2, 3]_4 & \equiv & 6, \end{array}$$

using Definition 5.1, we have

$$u \leq_v v \quad \text{since} \quad i_1 = 1 < 2 = j_1$$

and using Corollary 3.1, we have (see the digraph of I_4, the right-most one in Fig. 2)

$$u \not\succeq v \quad \text{since} \quad u_{(10} = 9 > 6 = v_{(10}.$$

VI. Conclusion

In this paper, we have established three different necessary conditions for the intrinsic order. The first one has involved the lexicographic (truth-table) order in the set $\{0,1\}^n$ of all binary n-tuples. The second one deals with the Hamming weight of the binary strings. The third one has been expressed in terms of the vector order, defined between binary n-tuples with the same weight. We have provided different simple counter-examples for proving that the above necessary conditions for the intrinsic ordering are not sufficient, in general. Moreover, for the first two cases we have also established the general patterns of the special binary n-tuples u for which the corresponding necessary conditions are also sufficient, i.e., they become equivalences. These patterns have been expressed by simple positional criteria of the $0s$ & $1s$ in the corresponding binary n-tuples.

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