Constructive Proof of the Fan-Glicksberg Fixed Point Theorem for Sequentially Locally Non-constant Multi-functions in a Locally Convex Space

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***Abstract*—In this paper we constructively prove the Fan-Glicksberg fixed point theorem for sequentially locally non-constant multi-functions (multi-valued functions or correspondences) with uniformly closed graph in a locally convex space, and apply it to the proof of the existence of a social equilibrium in an abstract economy where each payoff function has sequentially locally at most one maximum.**

***Index Terms*—**

the Fan-Glicksberg fixed point theorem; sequentially locally non-constant multi-functions; constructive mathematics; abstract economy; social equilibrium

I. Introduction

It is well known that Brouwer's fixed point theorem can not be constructively proved.

> [6] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics à la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive (See [4] or [12]).

Thus, Kakutani's fixed point theorem and the Fan-Glicksberg fixed point theorem in a locally convex space for multi-functions (multi-valued functions or correspondences) also can not be constructively proved. On the other hand, Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma (See [12] and [13]). Also Dalen in [12] states a conjecture that a function f from a simplex to itself, with property that each open set contains a point x such that x is not equal to $f(x)$ ($x \neq f(x)$) and on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. Recently Berger and Ishihara[2] showed that the following theorem is equivalent to Brouwer's fan theorem, and so it is non-constructive.

> Each uniformly continuous function from a compact metric space into itself with at most one fixed point and approximate fixed points has a fixed point.

By reference to the notion of *sequentially at most one maximum* in Berger, Bridges and Schuster[1] we require a more general and somewhat stronger condition of *sequential local non-constancy* to functions, and in [7], [9] and [11] we showed the following result.

> If each uniformly continuous function from a compact metric space into itself is *sequentially locally non-constant*, then it has a fixed point,

without the fan theorem. It is a partial answer to Dalen's conjecture.

In this paper we extend the sequential local non-constancy to multi-functions in a locally convex space, and constructively prove the Fan-Glicksberg fixed point theorem for sequentially locally non-constant multi-functions with uniformly closed graph. The uniformly closed graph property of multi-functions is a stronger version of the closed graph property. Also we apply this theorem to prove the existence of a social equilibrium in an abstract economy where each payoff function has sequentially locally at most one maximum. In a previous paper [8] we constructively proved the approximate version of the Fan-Glicksberg fixed point theorem for multi-functions with uniformly closed graph in a locally convex space. In this paper we use this result and prove the exact version of the Fan-Glicksberg fixed point theorem for sequentially locally non-constant multi-functions with uniformly closed graph. We follow the Bishop style constructive mathematics according to [3], [4] and [5].

II. The Fan-Glicksberg fixed point theorem

In constructive mathematics a nonempty set is called an *inhabited* set. A set S is inhabited if there exists an element of S.

> Note that in order to show that S is inhabited, we cannot just prove that it is impossible for S to be empty: we must actually construct an element of S (see page 12 of [5]).

Also in constructive mathematics compactness of a set means *total boundedness with completeness*. A set S is *finitely enumerable* if there exist a natural number N and a mapping of the set $\{1, 2, \dots, N\}$ onto S. An ε-approximation to S, a set in a metric space, is a subset of S such that for each $x \in S$ there exists y in that ε-approximation with $|x-y| < \varepsilon$($|x-y|$ is the distance between x and y). S is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to S. Completeness of

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a set, of course, means that every Cauchy sequence in the set converges.

We consider an n-dimensional simplex Δ as a compact metric space. According to Corollary 2.2.12 of [5], we have the following result.

Lemma 1: For each $\varepsilon > 0$ there exist totally bounded sets $H_1, H_2, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^n H_i$.

The notion that a function f from Δ to itself has at most one fixed point by [2] is defined as follows.

Definition 1 (At most one fixed point): For all $x, y \in \Delta$, if $x \neq y$, then $f(x) \neq x$ or $f(y) \neq y$.

By reference to the notion that a function has *sequentially at most one maximum* in [1], we define the property of *sequential local non-constancy* as follows.

Definition 2 (Sequential local non-constancy of functions): There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^m H_i$, and if for all sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ in each H_i, $|f(x_n) - x_n| \longrightarrow 0$ and $|f(y_n) - y_n| \longrightarrow 0$, then $|x_n - y_n| \longrightarrow 0$.

Let Φ be a compact and convex valued multi-function from Δ to the collection of its inhabited subsets. Since Δ and $\Phi(x)$ for $x \in \Delta$ are compact, $\Phi(x)$ is located (see Proposition 2.2.9 in [5]), that is, $|\Phi(x) - y| = \inf_{z\in\Phi(x)} |z - y|$ for $y \in \Delta$ exists.

The definition of sequential local non-constancy for multi-functions is as follows.

Definition 3: (Sequential local non-constancy of multi-functions): There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^m H_i$, and if for all sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ in each H_i $|\Phi(x_n) - x_n| \longrightarrow 0$ and $|\Phi(y_n) - y_n| \longrightarrow 0$, then $|x_n - y_n| \longrightarrow 0$.

Now we consider a locally convex space. A locally convex space consists of a vector space E and a family $(p_i)_{i\in I}$ of semi-norms on X, where I is an index set, for example, the set of positive integers. X is a subset of E. For each finitely enumerable subset F of I we define a basic neighborhood of a set S as follows.

$$V(S, F, \varepsilon) = \{y \in X | \sum_{i\in F} p_i(y - z) < \varepsilon \text{ for some } z \in S\}.$$

The closure of $V(S, F, \varepsilon)$ is denoted by $\overline{V}(S, F, \varepsilon)$, and it is represented as follows.

$$\overline{V}(S, F, \varepsilon) = \{y \in X | \sum_{i\in F} p_i(y - z) \leq \varepsilon \text{ for some } z \in S\}.$$

We call it a closed basic neighborhood of S. Compactness of a set in constructive mathematics means total boundedness with completeness also in a locally convex space. According to [5] we define total boundedness of a set in a locally convex space as follows.

Definition 4: (Total boundedness of a set in a locally convex space): Let X be a subset of E, F be a finitely enumerable subset of I, and $\varepsilon > 0$. By an ε-approximation to X relative to F we mean a subset T of X such that for each $x \in X$ there exists $y \in T$ with $\sum_{i\in F} p_i(x - y) < \varepsilon$.

X is totally bounded relative to F if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to X relative to F. It is totally bounded if it is totally bounded relative to each finitely enumerable subset of I.

Let X be a compact subset of a locally convex space E and Φ be a compact and convex valued multi-function from X to the collection of its inhabited subsets.

An approximate fixed point of a multi-function Φ is defined as follows.

Definition 5: (Approximate fixed point of a multi-function): For each $\varepsilon > 0$ x^* is an approximate fixed point of a multi-function Φ from X to the collection of its inhabited subsets if

$$\sum_{i\in F} p_i(x^* - \Phi(x^*)) < \varepsilon,$$

for each finitely enumerable $F \subset I$, where $p_i(x^* - \Phi(x^*)) = \inf_{y\in\Phi(x^*)} p_i(x^* - y)$. This infimum exists because a totally bounded set in a locally convex space is located (Proposition 5.4.4 in [5]).

On the other hand, a fixed point of Φ is defined as follows.

Definition 6 (Fixed point of a multi-function): x^* is a fixed point of Φ if

$$\sum_{i\in F} p_i(x^* - \Phi(x^*)) = 0,$$

for each finitely enumerable $F \subset I$.

A graph of a multi-function Φ from X to the collection of its inhabited subsets is

$$G(\Phi) = \cup_{x\in X}\{x\} \times \Phi(x).$$

If $G(\Phi)$ is a closed set, we say that Φ has a closed graph. It implies the following fact.

> For sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ such that $y_n \in \Phi(x_n)$, If $x_n \longrightarrow x$, then for some $y \in \Phi(x)$ we have $y_n \longrightarrow y$.

On the other hand, if the following condition is satisfied, we say that Φ has a uniformly closed graph.

> For sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$, $(x'_n)_{n\geq 1}$, $(y'_n)_{n\geq 1}$ such that $y_n \in \Phi(x_n)$, $y'_n \in \Phi(x'_n)$, if $\sum_{i\in F} p_i(x_n - x'_n) \longrightarrow 0$, then for any y_n and some y'_n, we have $\sum_{i\in F} p_i(y_n - y'_n) \longrightarrow 0$, and for any y'_n and some y_n, we have $\sum_{i\in F} p_i(y_n - y'_n) \longrightarrow 0$ for each finitely enumerable $F \subset I$.
>
> Let $y \in \Phi(x)$, $(x'_n)_{n\geq 1} = \{x, x, \dots\}$ and $(y'_n)_{n\geq 1} = \{y, y, \dots\}$ be sequences with constant points x and y. If $\sum_{i\in F} p_i(x_n - x'_n) = \sum_{i\in F} p_i(x_n - x) \longrightarrow 0$, then $\sum_{i\in F} p_i(y_n - y'_n) = \sum_{i\in F} p_i(y_n - y) \longrightarrow 0$ for each finitely enumerable $F \subset I$, that is, if $x_n \longrightarrow x$, then $y_n \longrightarrow y$, and so uniformly closed graph property implies closed graph property.

In this definition

> $\sum_{i\in F} p_i(x_n - x'_n) \longrightarrow 0$ means that for any $\delta > 0$ there exists n_0 such that when $n \geq n_0$ we have $\sum_{i\in F} p_i(x_n - x'_n) < \delta$, and $\sum_{i\in F} p_i(y_n - y'_n) \longrightarrow 0$ means that for any $\varepsilon > 0$ there exists n'_0 such that when $n \geq n'_0$, we have $\sum_{i\in F} p_i(y_n - y'_n) < \varepsilon$.

If X is totally bounded relative to each finitely enumerable subset of I, there exists a finitely enumerable τ-approximation $\{x^0, x^1, \dots, x^m\}$ to X relative to each finitely enumerable $F \subset I$, that is, for each $x \in X$ we have

$\sum_{i\in F} p_i(x - x^i) < \tau$ for at least one $x^i,\ i = 0, 1, \dots, m$ for each F. Let

$$\Phi_\tau(x) = \overline{V}(\Phi(x), F, \tau),$$

where $\overline{V}(\Phi(x), F, \tau)$ is a closed basic neighborhood of $\Phi(x)$. If Φ has a uniformly closed graph, Φ_τ also has a uniformly closed graph. Now let

$$X_V = \left\{ \sum_{i=0}^{n} \alpha_i x_i | x_i \in X,\ \sum_{i=0}^{n} \alpha_i = 1,\ \alpha_i \geq 0 \right\}. \quad (1)$$

This is the convex-hull of $\{x^0, x^1, \dots, x^m\}$. If X is convex and compact, we have $X_V \subset X$ and for $x \in X$,

$$\Phi(x) \subset X \subset \overline{V}(X_V, F, \tau).$$

Thus,

$$\Phi(x) \cap \overline{V}(X_V, F, \tau) \neq \emptyset,$$

and so

$$\Phi_\tau(x) \cap X_V \neq \emptyset.$$

Let

$$\Phi_{X_V}(x) = \Phi_\tau(x) \cap X_V \text{ for } x \in X_V.$$

Then, it is a compact and convex valued multi-function with uniformly closed graph from X_V to the collection of its inhabited subsets. If the dimension of X_V is n, X_V is homeomorphic to an n-dimensional simplex $\Delta = \{(\alpha^0, \alpha^1, \dots, \alpha^n)|\ \sum_{i=0}^{n} \alpha_i = 1\}$. Therefore, a multi-function with uniformly closed graph from Δ to the collection of its inhabited subsets corresponds one to one to a multi-function with uniformly closed graph from X_V to the collection of its inhabited subsets.

The definition of sequential local non-constancy of multi-functions from X_V to the collection of its inhabited subsets is as follows.

Definition 7: (Sequential local non-constancy of multi-functions in a locally convex space): There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_n$, each of diameter less than or equal to ε, such that $X_V = \cup_{i=1}^{m} H_i$, and if for all sequences $(x_n)_{n\geq1}$, $(y_n)_{n\geq1}$ in each H_i $\sum_{i\in F} p_i(\Phi(x_n) - x_n) \longrightarrow 0$ and $\sum_{i\in F} p_i(\Phi(y_n) - y_n) \longrightarrow 0$, then $\sum_{i\in F} p_i(x_n - y_n) \longrightarrow 0$ for each finitely enumerable $F \subset I$.

Now we show the following lemma.

Lemma 2: Let Φ be a multi-function with uniformly closed graph from a compact and convex set X to the collection of its inhabited subsets in a locally convex space, and assume that $\inf_{x\in H_j} \sum_{i\in F} p_i(\Phi(x) - x) = 0$ in some H_j such that $\cup_{j=1}^{m} H_j = X$. If the following condition holds:

> For each $\varepsilon > 0$ there exists $\eta > 0$ such that if $x, y \in H_i$, $\sum_{i\in F} p_i(\Phi(x) - x) < \eta$ and $\sum_{i\in F} p_i(\Phi(y) - y) < \eta$, then $\sum_{i\in F} p_i(x - y) \leq \varepsilon$.

Then, there exists a point $z \in X$ such that $\Phi(z) = z$, that is, a fixed point of Φ.

Proof: Choose a sequence $(x_n)_{n\geq1}$ in H_i such that $\sum_{i\in F} p_i(\Phi(x_n) - x_n) \longrightarrow 0$. Compute N such that $\sum_{i\in F} p_i(\Phi(x_n) - x_n) < \eta$ for all $n \geq N$. Then, for $m, n \geq N$ we have $\sum_{i\in F} p_i(x_m - x_n) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $(x_n)_{n\geq1}$ is a Cauchy sequence in S, and converges to a limit $z \in H_i$. The uniformly closed graph property of ϕ yields $\sum_{i\in F} p_i(\Phi(z) - z) = 0$, that is, $\Phi(z) = z$. ■

Next we show the following theorem.

Theorem 1: (The Fan-Glicksberg fixed point theorem for sequentially locally non-constant multi-functions): Let X be a compact (totally bounded and complete) and convex subset of a locally convex space E, and Φ be a convex and compact valued, sequentially locally non-constant multi-function with uniformly closed graph from X to the collection of inhabited subsets of X. Then, Φ has a fixed point.

Proof:

1) According to [8] Φ_{X_V} has an approximate fixed point, that is, for any $\eta > 0$ there exists x^* such that

$$\sum_{i\in F} p_i(x^* - \Phi_{X_V}(x^*)) < \eta,$$

for each finitely enumerable $F \subset I$. Then,

$$\sum_{i\in F} p_i(x^* - \Phi(x^*)) < \eta + \tau$$

for $\tau > 0$. Let $\varepsilon = \eta + \tau$. We have

$$\sum_{i\in F} p_i(x^* - \Phi(x^*)) < \varepsilon.$$

Since η and τ are arbitrary, and so ε is arbitrary, we obtain

$$\inf_{x\in H_j} \sum_{i\in F} p_i(x^* - \Phi(x^*)) = 0$$

for some H_j such that $X = \cup_{j=1}^{m} H_j$.

2) Choose a sequence $(z_n)_{n\geq1}$ in some H_i such that $\sum_{i\in F} p_i(\Phi(z_n) - z_n) \longrightarrow 0$. In view of Lemma 2 it is enough to prove that the following condition holds for each finitely enumerable $F \subset I$.

For each $\varepsilon > 0$ there exists $\eta > 0$ such that if $x, y \in H_i$, $\sum_{i\in F} p_i(\Phi(x) - x) < \eta$ and $\sum_{i\in F} p_i(\Phi(y) - y) < \eta$, then $\sum_{i\in F} p_i(x - y) \leq \varepsilon$.

Assume that the set

$$K = \{(x, y) \in H_i \times H_i : \sum_{i\in F} p_i((x - y) \geq \varepsilon\}$$

is nonempty and compact (Theorem 2.2.13 of [5]). Since the mapping $(x, y) \longrightarrow \max(\sum_{i\in F} p_i(\Phi(x) - x), \sum_{i\in F} p_i(\Phi(y) - y))$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n\geq1}$ such that

$$\lambda_n = 0$$
$$\Rightarrow \inf_{(x,y)\in K} \max(\sum_{i\in F} p_i(\Phi(x) - x), \sum_{i\in F} p_i(\Phi(y) - y)) < 2^{-n},$$

$$\lambda_n = 1$$
$$\Rightarrow \inf_{(x,y)\in K} \max(\sum_{i\in F} p_i(\Phi(x) - x), \sum_{i\in F} p_i(\Phi(y) - y)) > 2^{-n-1}.$$

It suffices to find n such that $\lambda_n = 1$. In that case, if $\sum_{i\in F} p_i(\Phi(x) - x) < 2^{-n-1}$, $\sum_{i\in F} p_i(\Phi(y) - y) < 2^{-n-1}$, we have $(x, y) \notin K$ and $\sum_{i\in F} p_i(x - y) \leq \varepsilon$.

Assume $\lambda_1 = 0$. If $\lambda_n = 0$, choose $(x_n, y_n) \in K$ such that $\max(\sum_{i\in F} p_i(\Phi(x_n) - x_n), \sum_{i\in F} p_i(\Phi(y_n) - y_n)) < 2^{-n}$, and if $\lambda_n = 1$, set $x_n = y_n = z_n$. Then, $\sum_{i\in F} p_i(\Phi(x_n) - x_n) \longrightarrow 0$ and $\sum_{i\in F} p_i(\Phi(y_n) - y_n) \longrightarrow 0$, so $\sum_{i\in F} p_i(x_n - y_n) \longrightarrow 0$. Computing N such that $\sum_{i\in F} p_i(x_N - y_N) < \varepsilon$, we must have $\lambda_N = 1$. We have completed the proof.

■

III. Social equilibrium in an abstract economy

In this section using the Fan-Glicksberg fixed point theorem for sequentially locally non-constant multi-functions we prove the existence of a social equilibrium in an abstract economy. It is a generalization of Nash equilibrium in a strategic game

First we present a constructive version of the Maximum Theorem. We define continuity of multi-functions as follows.

Definition 8 (Continuity of multi-functions): Let X, Y be locally convex spaces. A multi-function Φ from X to the collection of subsets of Y is continuous if it satisfies the following conditions.

1) It has a uniformly closed graph.
2) For a sequence $(x_n)_{n\geq 1}$ and $y \in Y$ such that $x_n \longrightarrow x$ and $y \in \Phi(x)$ there exist a sequence $(y_n)_{n\geq 1}$ such that $y_n \in \Phi(x_n)$ and $y_n \longrightarrow y$.

Let X, Y be locally convex spaces, f be a uniformly continuous function from $X \times Y$ to the set of real numbers $\mathbb{R}$, and let Φ be a compact valued continuous multi-function from X to the set of inhabited subsets of Y. Consider a maximization problem.

$$\text{maximize } f(x, y) \text{ subject to } y \in \Phi(x).$$

In constructive mathematics we can not generally prove the existence of the maximum of f even if $\Phi(x)$ is compact and f is uniformly continuous, but we can prove the existence of the supremum (see Proposition 5.4.3 in [5]). It is represented as

$$\sup_{y\in\Phi(x)} f(x, y). \tag{2}$$

We will show that we can find the maximum of a real valued function in a locally convex space if it has sequentially locally at most one maximum. The definition of the notion of *sequentially locally at most one maximum* is as follows.

Definition 9: (Sequentially locally at most one maximum): Let $M = \sup f_{y\in\Phi(x)} f(x, y)$. There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $\Phi(x) = \cup_{i=1}^m H_i$, and if for all sequences $(y_n)_{n\geq 1}$, $(y'_n)_{n\geq 1}$ in each H_i, $|f(x, y_n) - M| \longrightarrow 0$ and $|f(x, y'_n) - M| \longrightarrow 0$, then $\sum_{i\in F} p_i(y_n - y'_n) \longrightarrow 0$.

Now we show the following lemma, which is based on Lemma 2 of [1].

Lemma 3: Let $f(x, y)$ be a uniformly continuous function from a compact set $\Phi(x)$ to $\mathbb{R}$. Let $\sup_{y\in\Phi(x)} f(x, y) = M$. then, $\sup_{y\in H_i} f(x, y) = M$ for some $H_j \subset \Phi(x)$ such that $\Phi(x) = \cup_{j=1}^m H_j$. If the following condition holds:

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $y, y' \in H_i$, $f(x, y) \geq M - \varepsilon$ and $f(x, y') \geq M - \varepsilon$, then $\sum_{i\in F} p_i(y - y') \leq \delta$.

Then, there exists a point $z \in H_i$ such that $f(x, z) = M$, that is, $f(x, y)$ has the maximum in $\Phi(x)$.

Proof: Choose a sequence $(y_n)_{n\geq 1}$ in H_i such that $f(x, y_n) \longrightarrow M$. Compute N such that $f(x, y_n) \geq M - \varepsilon$ for all $n \geq N$. Then, for $m, n \geq N$ we have $\sum_{i\in F} p_i(y_m - y_n) \leq \delta$. Since $\delta > 0$ is arbitrary, $(y_n)_{n\geq 1}$ is a Cauchy sequence in H_i, and converges to a limit $z \in H_i$. The continuity of $f(x, y)$ yields $f(x, z) = M$. ■

Next we show the following lemma, which is based on Proposition 3 of [1].

Lemma 4: Each uniformly continuous function $f(x, y)$ from $\Phi(x)$ to $\mathbb{R}$, which has sequentially locally at most one maximum, has the maximum.

Proof: Choose a sequence $(z_n)_{n\geq 1}$ in H_i such that $f(x, z_n) \longrightarrow M$. In view of Lemma 3 it is enough to prove that the following condition holds.

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $y, y' \in H_i$, $f(x, y) \geq M - \varepsilon$ and $f(x, y') \geq M - \varepsilon$, then $\sum_{i\in F} p_i(y - y') \leq \delta$.

Assume that the set

$$K = \{(y, y') \in H_i \times H_i : \sum_{i\in F} p_i(y - y') \geq \delta\}$$

is nonempty and compact (for a metric space see Theorem 2.2.13 of [5]. It similarly holds for a locally convex space). Since the mapping $(y, y') \longrightarrow \min(f(x, y), f(x, y'))$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n\geq 1}$ such that

$$\lambda_n = 0 \Rightarrow \sup_{(y,y')\in K} \min(f(x, y), f(x, y')) > M - 2^{-n},$$

$$\lambda_n = 1 \Rightarrow \sup_{(y,y')\in K} \min(f(x, y), f(x, y')) < M - 2^{-n-1}.$$

It suffices to find n such that $\lambda_n = 1$. In that case, if $f(x, y) > M - 2^{-n-1}$, $f(x, y') > M - 2^{-n-1}$, we have $(y, y') \notin K$ and $\sum_{i\in F} p_i(y - y') \leq \delta$. Assume $\lambda_1 = 0$. If $\lambda_n = 0$, choose $(y_n, y'_n) \in K$ such that $\min(f(x, y_n), f(x, y'_n)) > M - 2^{-n}$, and if $\lambda_n = 1$, set $y_n = y'_n = z_n$. Then, $f(x, y_n) \longrightarrow M$ and $f(x, y'_n) \longrightarrow M$, so $\sum_{i\in F} p_i(y_n - y'_n) \longrightarrow 0$. Computing N such that $\sum_{i\in F} p_i(y_N - y'_N) < \delta$, we must have $\lambda_N = 1$. We have completed the proof. ■

This lemma means that $f(x, y)$ has the maximum in $\Phi(x)$, that is, $\max_{y\in\Phi(x)} f(x, y)$ exists. We define

$$\psi(x) = \max_{x\in X, y\in\Phi(x)} f(x, y).$$

It is a function from X to $\mathbf{R}$, and define

$$\Psi(x) = \{y \in \Phi(x) | f(x, y) = \psi(x)\}.$$

It is a multi-function from X to the set of inhabited subsets of Y.

Now we show the following theorem which is the maximum theorem for functions with sequentially locally at most one maximum. It is based on [10].

Theorem 2: Let X, Y be locally convex spaces, let f be a uniformly continuous function with sequentially locally at most one maximum from $X \times Y$ to $\mathbf{R}$, and let Φ be a compact valued continuous multi-function from X to the set of inhabited subsets of Y. Then,

1) ψ defined above is uniformly continuous in X, and

2) Ψ defined above has a uniformly closed graph.

Proof: Consider sequences $(x_n)_{n\geq 1}$ in X and $(y_n)_{n\geq 1}$ in Y such that $y_n \in \Psi(x_n)$, $x_n \longrightarrow x$ and $y_n \longrightarrow y$. $y_n \in \Psi(x_n)$ means $y_n \in \Phi(x_n)$ and $f(x_n, y_n) = \psi(x_n)$. Since F is a continuous multi-function, we have $y \in \Phi(x)$, and for every $y' \in \Phi(x)$ there exist sequences $(x_n)_{n\geq 1}$ and $(y'_n)_{n\geq 1}$ such that $y'_n \in \Psi(x_n)$, $x_n \longrightarrow x$ and $y'_n \longrightarrow y'$. Assume $f(x, y') > f(x, y)$. Then, $f(x_n, y'_n) > f(x_n, y_n)$ for sufficiently large n, But it contradicts $y_n \in \Psi(x_n)$, and so $f(x, y) = \psi(x)$ and $y \in \Psi(x)$. Since Φ has a uniformly closed graph, Ψ also has a uniformly closed graph.

Since $\psi(x_n) = f(x_n, y_n) \longrightarrow f(x, y) = \psi(x)$, ψ is uniformly continuous because f is uniformly continuous. ■

An abstract economy is described as follows. There are n players. n is a finite positive integer. Let X^i be the set of strategies of player i, and denote his each strategy by x^i. X^i is a finite or infinite set. The set of strategies of all players is denoted by $X = \prod_{i=1}^n X^i$, and a combination of strategies of all players, which is called a profile, is denoted by x. The set of available strategies of each player is not fixed, but depends on strategies chosen by players other than him. Define a multi-function $\Phi^i : X \to X^i$. $\Phi^i(x) \subset X^i$ denotes a set of available strategies for player i when a profile of strategies of all players is x. It is a continuous multi-function in the sense defined above, and we assume that $\Phi^i(x)$ is a compact and convex set in a locally convex space. We denote a set of available strategies for players other than i at a profile x by $\Phi^{-i}(x)$. The payoff of player i is represented by a function u^i. u^i depends on strategies chosen by all players including player i himself, and it is uniformly continuous, quasi-concave and has sequentially locally at most one maximum.

Quasi-concavity of payoff functions in our constructive situation is defined as follows:

> Let $x^i \in X^i$ be a strategy of player i and x^{-i} be a profile of strategies of players other than i. If, for each pair $x^i, x^{i'} \in X^i$ and $\delta > 0$, u^i satisfies the condition
>
> $$u^i(\lambda x^i + (1-\lambda)x^{i'}, x^{-i}) > \min_{x^i\in\Phi^i(x), x^{-i}\in\Phi^{-i}(x)} (u^i(x^i, x^{-i}), u^i(x^{i'}, x^{-i})) - \delta,$$
>
> then it is quasi-concave.

A Nash equilibrium of the abstract economy is called a *social equilibrium*. It is defined as a profile of strategies of players which maximize the payoff of each player under the constraint expressed by $\Phi^i(x)$.

Theorem 3: Under these assumptions there exists a social equilibrium of the abstract economy.

Proof: Define a function ψ^i by

$$\psi^i(x) = \max_{x^i\in\Phi^i(x), x^{-i}\in\Phi^{-i}(x)} u^i(x),\ i = 1, 2, \dots, n,$$

and define a multi-function $\Psi = (\Psi^1, \Psi^2, \dots, \Psi^n)$ by

$$\Psi^i(x) = \{x^i \in \Phi^i(x) | u^i(x^i, x^{-i}) = \psi^i(x)\},\quad i = 1, 2, \dots, n,$$

where $x = (x^1, x^2, \dots, x^n)$. It is the set of best response strategies of player i. By Theorem 2 ψ^i is uniformly continuous, and Ψ^i has a uniformly closed graph. Since $u^i(x)$ is uniformly continuous, $\Psi^i(x)$ is totally bounded (Theorem 5.4.6 of [5]). Let us check convexity of $\Psi^i(x)$. Let $x^i,\ x^{i'} \in \Phi^i(x)$ and for $0 \leq \lambda \leq 1$ define $x^{i\lambda} = \lambda x^i + (1-\lambda)x^{i'}$. Since $\Phi^i(x)$ is convex, $x^{i\lambda} \in \Phi^i(x)$. Quasi-concavity of $u^i(x)$ implies that for each $x^i, x^{i'} \in \Psi^i(x)$ and $\delta > 0$ we have

$$u^i(\lambda x^i + (1-\lambda)x^{i'}, x^{-i}) > \min_{x^i\in\Phi^i(x)} (u^i(x^i, x^{-i}), u^i(x^{i'}, x^{-i})) - \delta.$$

Since δ is arbitrary, we have

$$u^i(\lambda x^i + (1-\lambda)x^{i'}, x^{-i}) \geq \min_{x^i\in\Phi^i(x)} (u^i(x^i, x^{-i}), u^i(x^{i'}, x^{-i}))$$

Thus, $x^{i\lambda}$ is a best response strategy of player i, and $\Psi^i(x)$ is convex.

Finally we check that Ψ is sequentially locally non-constant. Let $(x_n)_{n\geq 1}$, $(x'_n)_{n\geq 1}$ be sequences in H_j such that $\sum_{i\in F} p_i(\Psi^i(x_n) - x_n) \longrightarrow 0$ and $\sum_{i\in F} p_i(\Psi^i(x'_n) - x'_n) \longrightarrow 0$, where $\cup_{j=1}^m H_j = \Phi(x) = (\Phi^1(x), \Phi^2(x), \dots, \Phi^n(x))$ and $H_j, j = 1, \dots, m$, are totally bounded sets whose diameters are smaller than ε for some $\varepsilon > 0$. Since u^i is uniformly continuous, $|u^i(x_n) - M| \longrightarrow 0$ where M is $\max u^i$ in a domain including x_n for all n, and similarly $|u^i(x'_n) - M'| \longrightarrow 0$ where M' is $\max u^i$ in a domain including x_n for all n'. Since u^i has sequentially locally at most one maximum, $M = M'$ with sufficiently small ε. Thus, we have $|u^i(x'_n) - M| \longrightarrow 0$. Then,

$$\sum_{i\in F} p_i(x_n - x'_n) \longrightarrow 0.$$

Therefore, Ψ is uniformly sequentially non-constant, and so by the Fan-Glicksberg fixed point theorem for sequentially locally non-constant multi-functions there exists x^* such that $x^* \in \Psi(x^*)$. ■

At x^* all players maximize their payoffs under the constraints expressed by $\Phi(x^*)$, and so x^* is a social equilibrium in an abstract economy.

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