Machine Learning and Statistical Analysis for BRDF Data from Computer Graphics and Multidimensional Reflectometry

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***Abstract*—Characterizing the appearance of real-world surfaces is a fundamental problem in multidimensional reflectometry, computer vision and computer graphics. For many applications, appearance is sufficiently well characterized by the bidirectional reflectance distribution function. BRDF is one of the fundamental concepts in such diverse fields as multidimensional reflectometry, computer graphics and computer vision.**

In this paper, we treat BRDF measurements as samples of points from high-dimensional non-linear non-convex manifolds. We argue that any realistic statistical analysis of BRDF measurements, or any parameter or manifold learning procedure applied to BRDF measurements has to account both for nonlinear structure of the data as well as for a very ill-behaved noise. Standard statistical and machine learning methods can not be safely directly applied to BRDF data.

We discuss the differences and the common points of data analysis and modelling for BRDFs in both physical as well as in virtual application domains. We outline a mathematical framework that captures some important problems in both types of application domains, and allows for application and performance comparisons of statistical and machine learning methods. For comparisons between the methods, we use criteria that are relevant to both statistics and machine learning, as well as to both virtual and physical application domains. This outlines a possible unified approach to BRDF data analysis and modelling relevant for the whole generality of application domains.

Specifically, we apply the notion of Pitman closeness to compare different estimators and learning procedures for BRDF models. This criterion for comparison is loss function-free and seems to be especially appropriate for applications in metrology and in comparing different types of learning methods.

Additionally, we propose a class of multiple testing procedures to test a hypothesis that a material has diffuse reflection in a generalized sense. We treat a general case where the number of hypotheses can potentially grow with the number of measurements. Our approach leads to tests that are more powerful than the generic multiple testing procedures.

***Index Terms*—BRDF, computer graphics, metrology, data analysis, statistics of manifolds, learning manifolds.**

I. Introduction

CHARACTERIZING the appearance of real-world surfaces is a fundamental problem in multidimensional reflectometry, computer vision and computer graphics. For many applications, appearance is sufficiently well characterized by the bidirectional reflectance distribution function (BRDF).

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In the case of a fixed wavelength, BRDF describes reflected light as a four-dimensional function of incoming and outgoing light directions. In a special case of rotational symmetry, isotropic BRDFs are are used. Isotropic BRDFs are functions of only three angles. On the other hand, for modelling or describing complicated visual effects such as goniochromism or irradiance, an extra dimension accounting for the wave length has to be added. The BRDF is applied under the assumption that all light falls at a single surface point. The classical device for measuring BRDFs is the gonio-reflectometer, which is composed of a photometer and light source that are moved relative to a surface sample under computer control.

In computer graphics and computer vision, usually either physically inspired analytic reflectance models [1], [2], [3], or parametric reflectance models chosen via qualitative criteria [4], [5], [6], [7] are used to model BRDFs. These BRDF models are only crude approximations of the reflectance of real materials. Moreover, analytic reflectance models are limited to describing only special subclasses of materials.

In multidimensional reflectometry, an alternative approach is usually taken. One directly measures values of the BRDF for different combinations of the incoming and outgoing angles and then fits the measured data to a selected analytic model using optimization techniques. There are several shortcomings to this approach as well.

An alternative approach to fitting parametric models is in constructing more realistic BRDFs on the basis of actual BRDF measurements. This approach bridges the gap between computer graphics and industrial reflectometry. For example, [8] and [9] modelled reflectance of materials in nature as a linear combination of a small set of basis functions derived from analyzing a large number of densely sampled BRDFs of different materials.

There were numerous efforts to use modern machine learning techniques to construct data-driven BRDF models as well. [10] proposed a method to generate new analytical BRDFs using a heuristic distance-based search procedure called Genetic Programming. In [11], an active learning algorithm using discrete perceptional data was developed and applied to learning parameters of BRDF models such as the Ashikhmin - Shirley model [12].

In computer graphics, it is important that BRDF models should be processed in real-time. Computer-modelled materials have to remind real materials qualitatively, but quantitative accuracy is not as important. The picture in reflectometry and metrology is almost the opposite: there is typically no need in real-time processing of BRDFs, but quantitative accuracy is the paramount. In view of this,

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some of the breakthrough results from computer vision and animation would not fit applications in reflectometry and in many industries.

Another difference with virtual reality models is that in computer graphics measurement uncertainties are essentially never present. This is not the case in metrology, reflectometry and in any real-world based industry [13]. Since measurement errors can greatly influence shape and properties of BRDF manifolds, there is a clear need to develop new methods for handling BRDFs with measurement uncertainties.

In this paper, we treat BRDF measurements as samples of points from a high-dimensional and highly non-linear non-convex manifold. We argue that any realistic statistical analysis of BRDF measurements, or any parameter or manifold learning procedure applied to BRDF measurements has to account both for nonlinear structure of the data as well as for a very ill-behaved noise. Standard statistical and machine learning methods can not be safely directly applied to BRDF data. Our study of parameters for generalized Lambertian models in Sections IV and V clarifies certain pitfalls in analysis of BRDF data, and helps to understand and develop more refined estimates for generalized Lambertian models in Section VI.

We introduce and apply in Section V the notion of Pitman closeness to compare different estimators and parameter learning methods that could be applied to BRDF models. To the best of our knowledge, the present paper together with [14] are the first works where the Pitman closeness criterion was introduced to either fields of computer graphics as well as metrology. This criterion for comparison of estimators appeals to the actually observable precision of estimators and is assumption-free and loss function-independent, and thus seems to be especially appropriate for applications in metrology, as well as for comparative studies of parameter learning procedures derived for different types of loss functions. Based on this and other criteria, we show that, in the context of the BRDF model parameter estimation and parameter learning, estimators based on either median or trimmed mean are safer to use and are often more accurate than estimators based on sample means.

We use the generalized Lambertian model parameter estimators from Section VI to build statistical tests to test a hypothesis whether any particular material is diffuse, even if in a weak sense, or not. Testing validity of BRDF models is important for computer graphics, even though rarely done in a rigorous way, with [15] being a notable exception dealing with several types of tests for parametric models. Perhaps surprisingly, rigorous hypothesis testing for BRDF data is rarely studied in metrology and reflectometry as well. Recent works [16] and [17] deals with hypothesis testing for diffuse reflection standards. In this paper, we treat a more general case of generalized Lambertian BRDFs, which demands simultaneous testing for a set of stochastically ordered hypotheses, where the number of those hypothesis is the number of measured BRDF layers and so can potentially grow with the number of measurements available. We build a class of tests for this complicated set of hypotheses, and show that our approach leads to tests that are more powerful than the generic multiple testing procedures often recommended by default in the literature.

II. Main definition

The bidirectional reflectance distribution function (BRDF), $f_r(\omega_i,\, \omega_r))$ is a four-dimensional function that defines how light is reflected at an opaque surface. The function takes a negative incoming light direction, ω_i, and outgoing direction, ω_r, both defined with respect to the surface normal $\mathbf{n}$, and returns the ratio of reflected radiance exiting along ω_r to the irradiance incident on the surface from direction ω_i. Each direction ω is itself parameterized by azimuth angle ϕ and zenith angle θ, therefore the BRDF as a whole is 4-dimensional. The BRDF has units sr^{-1}, with steradians (sr) being a unit of solid angle.

The BRDF was first defined by Nicodemus in [18]. The defining equation is:

$$f_r(\omega_i,\, \omega_r) \,=\, \frac{d\, L_r(\omega_r)}{d\, E_i(\omega_i)} \,=\, \frac{d\, L_r(\omega_r)}{L_i(\omega_i)\cos\theta_i \, d\, \omega_i}\,. \qquad (1)$$

where L is radiance, or power per unit solid-angle-in-the-direction-of-a-ray per unit projected-area-perpendicular-to-the-ray, E is irradiance, or power per unit surface area, and θ_i is the angle between ω_i and the surface normal, $\mathbf{n}$. The index i indicates incident light, whereas the index r indicates reflected light.

In the basic definition it is assumed that the wavelength λ is fixed and is the same for both the incoming and the reflected light. In order to model complicated visual effects such as iridescence, luminescence and structural coloration, or to model materials such as pearls, crystals or minerals, as well as to analyze the related data, it is necessary to have an extended, wavelength-dependent definition of BRDFs. Fortunately, formally this new definition is relatively straightforward and is obtained by rewriting equation (1) for $f_r(\lambda_i,\, \omega_i,\, \lambda_r,\, \omega_r)$, where λ_i and λ_r are the wavelengths of the incoming and the reflected light respectively.

III. Important models of diffuse reflection

A. Lambertian model

Lambertian model [4] represents reflection of perfectly diffuse surfaces by a constant BRDF. Because of its simplicity, Lambertian model is extensively used as one of the building blocks for models in computer graphics. Most of the recent studies of light reflection by means of advanced machine learning methods still rely on the Lambertian model. Examples include color studies [19], [20], analytic inference [21], perception studies [22], and face detection [23].

It was believed for a long time that the so-called standard diffuse reflection materials exhibit Lambertian reflectance, but recent studies with actual BRDF measurements convincingly reject this hypothesis [24], [17], [16].

B. Oren-Nayar model

Oren-Nayar model [1] is a ”directed–diffuse” microfacet model, with perfectly diffuse (rather than specular) microfacets. It can be viewed as a generalization of the Lambertian model. This is a reflectance model for diffuse reflection from rough surfaces. Oren-Nayar model is typically used to predict the appearance of rough surfaces, such as concrete or sand.

Resently, a more sophisticated model was proposed by [25]. This new model includes as special cases both Lambertian model and the Oren–Nayar model, as well as the Torrance–Sparrow model with specular microfacets.

C. Lommel-Seeliger

Lommel-Seeliger model [26] is used to simulate the brightening of a rough surface when illuminated from directly behind the observer. This is a physically inspired model of a special class of reflections from diffuse surfaces. This model is typically applied to model astronomical data, such as lunar and Martian reflection.

IV. Statistical analysis of BRDF models

In this section, we treat parameter estimation for BRDF models of standard diffuse reference materials. These materials are supposed to have ideal diffuse reflection with constant BRDFs. Graphically, for each incoming anlge θ_i, φ_i, the resulting BRDF $f_r(\omega_i, \omega_r)$ is a (subset of) two-dimensional upper hemisphere. The radius ρ of this hemisphere is the parameter that we aim to estimate in this paper.

As we mentioned before, the Lambertian model has been shown to be inaccurate even for those materials that were designed to be as close to perfectly diffuse as possible. Therefore, parameter estimates determined for the Lambertian model can hardly be used in practice. However, there are two methodological reasons that make these estimators worth a separate study.

First, BRDF measurements represent a sample of points from a high-dimensional and highly non-linear non-convex manifold. Moreover, these measurements are collected via a nontrivial process, possibly involving random or systematic measurement errors of digital or geometric nature. These two observations suggest that any realistic statistical analysis of BRDF measurements has to account both for nonlinear structure of the data as well as for a very ill-behaved noise and heavy-tailed noise. Any type of statistical inference is more complicated in these conditions, see, e.g., [27]. Standard statistical methods typically assume nice situations like i.i.d. normal errors, and can not be safely directly applied to BRDF data. The same applies to statistical analysis of image data in general [28]. Our study of parameters for Lambertian models clarifies certain pitfalls in analysis of BRDF data, and helps to understand and develop more refined estimates for more realistic BRDF models that will be studied in subsequent papers.

Second, we would use the Lambertian model parameter estimators to build statistical tests to test a hypothesis whether any particular material is perfect diffuse or not. This will be studied in a separate paper.

Suppose we have measurements of a BRDF available for the *set of incoming angles*

$$\Omega_{inc} = \left\{\omega_i^{(p)}\right\}_{p=1}^{P_{inc}} = \left\{\left(\theta_i^{(p)}, \varphi_i^{(p)}\right)\right\}_{p=1}^{P_{inc}} . \tag{2}$$

Here $P_{inc} \geq 1$ is the total number of incoming angles where the measurements were taken. Say that for an incoming angle $\left\{\omega_i^{(p)}\right\}$ we have measurements available for angles from the *set of reflection angles*

$$\Omega_{refl} = \bigcup_{p=1}^{P_{inc}} \Omega_{refl}(p) , \tag{3}$$

where

$$\Omega_{refl}(p) = \left\{\omega_r^{(q)}\right\}_{q=1}^{P_{refl}(p)} = \left\{\left(\theta_r^{(q)}, \varphi_r^{(q)}\right)\right\}_{q=1}^{P_{refl}(p)} ,$$

where $\left\{P_{refl}(p)\right\}_{p=1}^{P_{inc}}$ are (possibly different) numbers of measurements taken for corresponding incoming angles.

Overall, we have the set of random observations

$$\Big\{ f\left(\theta_i^{(p)}, \varphi_i^{(p)}, \theta_r^{(q)}, \varphi_r^{(q)}\right) \mid \quad \left(\theta_i^{(p)}, \varphi_i^{(p)}\right) \in \Omega_{inc} , \quad \left(\theta_r^{(q)}, \varphi_r^{(q)}\right) \in \Omega_{refl}(p) \Big\} .$$

Our aim is to infer properties of the BRDF funtion (1) from the set of observations (4). In general, the connection between the true BRDF and its measurements is described via a stochastic transformation T, i.e.

$$f(\omega_i, \omega_r) = T\big(f_r(\omega_i, \omega_r)\big) , \tag{4}$$

where

$$T : \mathcal{M} \times \mathcal{P} \times \mathcal{F}_4 \to F_4 , \tag{5}$$

with $\mathcal{M} = (M, \mathfrak{A}, \mu)$ is an (unknown) measurable space, $\mathcal{P} = (\Pi, \mathfrak{P}, \mathbb{P})$ is an unknown probability space, $\mathcal{F}_4$ is the space of all Helmholtz-invariant energy preserving 4-dimensional BRDFs, and F_4 is the set of all functions of 4 arguments on the 3-dimensional unit sphere S^3 in $\mathbb{R}^4$.

Equations (4) and (5) mean that there could be errors of both stochastic or non-stochastic origin. In this setting, the problem of inferring the BRDF can be seen as a statistical inverse problem. However, contrary to much literature on this subject, we do not assume linearity of the transformation T, we do not assume that T is purely stochastic, and we do not assume an additive model with zero-mean parametric errors, as these assumptions do not seem realistic for BRDF measurements.

Of course, this setup is intractable in full generality, but for special cases such as inference for Lambertian model, we would be able to obtain quite general solutions (see also [29]).

It is also easily observable (see, e.g., [24]) that for all materials their sub-BRDFs, consisting of measurements for different incoming angles, look substantially different (no matter if we believe in the underlying Lambertian model or not). This suggests that different sub-BRDFs of the same material still have different parameter values, and this in turn calls for applying statistical procedures separately for different sub-BRDFs and for combining the results via model selection, multitesting and related techniques.

V. Means, medians and robust estimators

A. Basic properties of distributions in BRDF data

In our choice of estimators for parameters in BRDF models, we have to take into account specific properties of BRDF data. It is important to notice that, due to the complicated structure of measurement devices, outliers are possible in the data. Additionally, due to technical difficulties in measuring peak values of BRDFs (see [30], [31]), we have to count on the fact that certain (even though small) parts of the data contain observations with big errors. This also leads us to conclusion that, even for simplest additive error models, we cannot blindly assume that random errors are identically distributed throughout the whole manifold. Additionally, missing data are possible and even inevitable for certain angles. Measurement angles themselves can be also arbitrary and non-uniformly distributed.

In view of the above arguments, a useful estimator for any BRDF model has to exhibit certain robustness against outliers and dependent or mixed errors.

An estimator has to be universal enough in the sense that it has to be applicable to BRDF samples without requiring extra regularity in the data set, such as uniformly distributed (or other pre-specified) design points, pre-specified large number of measurements, or absence of missing values. This observation suggests that simpler estimators are more practical for BRDF data than complicated (even if possibly asymptotically optimal) estimators, as the later class of estimators has to rely on rather strict regularity assumptions about the underlying model.

B. Pitman closeness of estimators

Let Ω be a probability space and let $\widehat{\theta}_1 : \Omega \to \mathbb{R}$ and $\widehat{\theta}_2 : \Omega \to \mathbb{R}$ be estimators of a parameter $\theta \in \mathbb{R}$. Then the *Pitman relative closeness* of these two estimators at the point θ is defined as

$$\mathcal{P}\big(\widehat{\theta}_1, \widehat{\theta}_2; \theta\big) \,=\, \mathbb{P}\Big(\big|\widehat{\theta}_1 - \theta\big| < \big|\widehat{\theta}_2 - \theta\big|\Big). \tag{6}$$

The estimator $\widehat{\theta}_1$ is *Pitman closer* to θ than $\widehat{\theta}_2$, if

$$\mathcal{P}\big(\widehat{\theta}_1, \widehat{\theta}_2; \theta\big) \,>\, 1/2\,.$$

While this criterion for comparison of estimators is much less known as, say, unbiasedness or asymptotic variance, it appeals to the actually observable precision of estimators, and thus seems of much interest for applications in metrology. To the best of our knowledge, the present paper together with [14] are the first works where the Pitman closeness criterion was introduced to either fields of computer graphics as well as metrology.

The closeness criterion appeals to the actually observed precision of estimators and is assumption-free and loss function-independent, and thus seems to be especially appropriate for comparative studies of parameter learning procedures derived for different types of loss functions. As a drawback, the Pitman closeness has some nontrivial properties such as non-transitivity [32], which leads to counterintuitive results in several examples [33]. On the other hand, these nontrivial properties help to clarify some classic statistical paradoxes such as the Stein paradox [34], [35].

We refer to [36] for an extensive discussion of the relative closeness of estimators and other related notions and their properties. Besides unbiasedness, asymptotic variance and relative closeness, there are many other criteria for comparing quality of statistical estimators. At least 7 of them can be found in [37].

We apply the notion of Pitman closeness to compare different estimators that could be used in BRDF models. Based on this and other criteria, we show that, in the context of the BRDF model parameter estimation and parameter learning, procedures based on either median or trimmed mean are safer to use and are often more accurate than procedures based on sample means.

C. Definitions of basic estimators

To exemplify these arguments, we consider the following basic estimators for the radius parameter of the Lambertian model: sample mean; sample median; trimmed (truncated) mean.

Let $X_1, X_2, \ldots, X_n$ be a sample from probability distribution F. Then the *sample mean* is defined as

$$sm(X) \,=\, \frac{1}{n}\sum_{i=1}^{n} X_i\,. \tag{7}$$

Let $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ be the order statistics of the sample $X_1, X_2, \ldots, X_n$. The *sample median* is defined as

$$smed(X) \,=\, \begin{cases} X_{((n+1)/2)}, & \text{n is odd;} \\ \frac{1}{2}\big(X_{(n/2)} + X_{(n/2+1)}\big), & \text{n is even.} \end{cases}$$

Let $0 \le \alpha < 1/2$ be a number, and let $[\cdot]$ denote the integer part of a real number. Then the *sample trimmed mean* is defined as

$$\begin{aligned} tm_\alpha(X) \;&=\; \frac{1}{n(1-2\alpha)}\Bigg\{\sum_{i=[n\alpha]+2}^{n-[n\alpha]-1} X_{(i)} \,+ \\ &\qquad \big([n\alpha]+1-n\alpha\big)\big(X_{([n\alpha]+1)} + X_{(n-[n\alpha])}\big)\Bigg\}. \end{aligned}$$

If F_n denotes the empirical distribution function of the sample $X_1, X_2, \ldots, X_n$, then we can write, equivalently,

$$tm_\alpha(X) \,=\, \frac{1}{1-2\alpha}\int_\alpha^{1-\alpha} F_n^{-1}(t)\,dt\,. \tag{8}$$

D. Mean and median

Sample mean is known to be an asymptotically efficient estimator, as well as a uniformly minimum-variance unbiased estimator, for the expected value of the random variable. However, it is important to note that these nice properties are guaranteed only for sufficiently "nice" distributions (see [38] or [39]), while sometimes even marginal deviations from these nice models seriously spoil performance of the sample mean estimator. Even if the regularity conditions are only slightly violated, the sample mean estimator can lose its efficiency or can become inconsistent.

In view of the above discussion of properties of BRDF data, we conclude that it is not advisable to apply the sample mean directly as an estimator of the Lambertian radius.

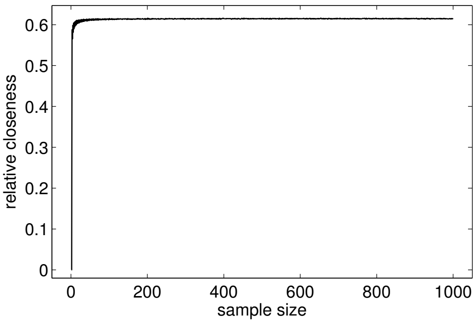

Fig. 1. Mean beats median for the standard normal distribution

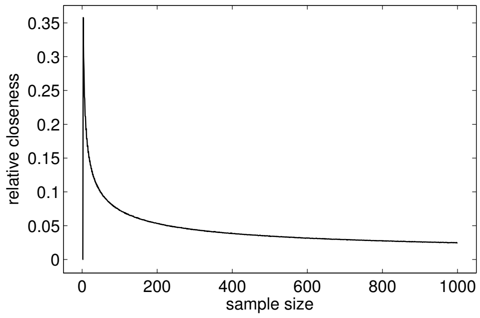

Fig. 2. Median grossly outperforms mean for heavy-tailed distributions

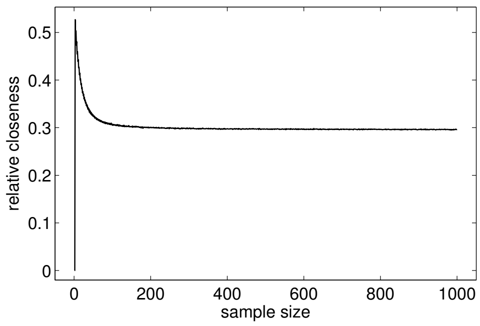

Fig. 3. Median can outperform mean for mixtures of normal distributions

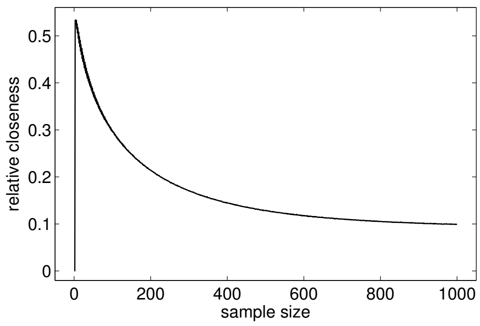

Fig. 4. Median can outperform mean for mixtures of normal distributions with small errors

Here we bring some examples to illustrate our points. An example from pp. 2 - 5 of [40] shows that, while the sample mean is even finite sample efficient for estimating parameters of normal distribution, this estimator loses its nice performance properties already for mixtures of two normal distributions. Moreover, even a mixture with 0.2 percent of a different normal distribution can cause the sample mean to lose its efficiency.

In this and in the next subsection, we present some results of an extensive Monte Carlo experiment comparing relative closeness of different types of basic nonparametric estimators. Each of the graphs contains values of relative closeness obtained for samples of all sizes ranging from 1 to 1000 observations. We performed 1000000 comparisons for each sample size.

Figure 1 shows that for a sample of i.i.d. normal random variables mean has better relative closeness than the median, with the actual value being above 0.6. We notice that the situation remains essentially the same regardless of the variance of the underlying normal distribution.

However, if we are dealing with a heavy-tailed distribution, the picture changes. Suppose we are presented with a Cauchy distribution, and our goal is to estimate the mode (the mean does not exist in this case). Then Figure 2 shows that the relative closeness of the mean tends to 0 when compared with the median.

Mean surprisingly loses its efficiency even in rather smooth toy situations. Suppose that a sample from i.i.d. standard normal distribution is contaminated with 5% of i.i.d. normals with mean 0 and variance 10. The result is shown on Figure 3. Mean's closeness compared to median drops to 0.3. Even more surprisingly, if we start with a sample of i.i.d. normals with mean 0 and variance 100 and contaminate this sample with just 5% of i.i.d. normals with mean 0 and small variance 1, the drop in mean's closeness compared to median is even worse. Figure 4 shows that the relative closeness of mean drops to 0.1.

E. Truncated mean and mean

If our data are generated by sufficiently nice distribution such as, say, a normal distribution, then the sample mean possesses is an efficient estimator. In those cases, it can be rigorously proven that Mean is better than Trimmed Mean in the sense of both Pitman closeness, as well as asymptotic relative efficiency.

The picture can be reversed when our data are allowed to contain outliers or when the data can be, at least partially, generated by a heavy-tailed distribution (which is the case when large values of measurement errors are possible, as is the case for BRDF measurements of specular peaks). We give here a toy example with a Cauchy distribution. Figure 5 illustrates the relative efficiency of mean compared to the trimmed mean with 10% of the extremes in data being discarded. The unusual shape of the relative closeness curve has no explanation at the moment.

Here the mean is an inconsistent estimator of the median of the distribution, while the truncated mean is not only a consistent estimator of the median, but, with a proper choice of the truncation point, is capable of outperforming the sample median in estimating the median [41]! One needs to drop out about 76% of the data, though. In fact, even more efficient estimators exist [42], but they require to drop out almost all of the data, and we would not advise to use them for estimation in BRDF models or for any work with moderate sample sizes.

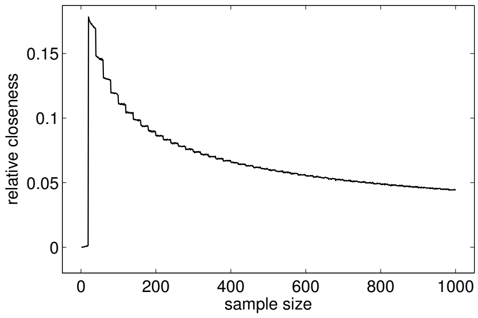

Fig. 5. Trimmed mean totally dominates mean for Cauchy distributions

VI. Parameter estimation for generalized Lambertian models

For each $\omega_i^{(p)}$ from the set of incoming angles Ω_{inc}, let $\rho^{(p)}$ denote the Lambertian radius of the BRDF's layer

$$\left\{ f\big(\theta_i^{(p)}, \varphi_i^{(p)}, \theta_r^{(q)}, \varphi_r^{(q)}\big) \,\big|\, \big(\theta_r^{(q)}, \varphi_r^{(q)}\big) \in \Omega_{refl}(p) \right\}, \quad (9)$$

where $\Omega_{refl}(p)$ is defined by (3). Thus, we are estimating the P_{inc}-dimensional parameter vector

$$\left\{ \rho^{(p)} \right\}_{p=1}^{P_{inc}}. \quad (10)$$

For $1 \le p \le P_{inc}$, let

$$\left\{ f_{(i)}^{(p)} \right\}_{i=1}^{P_{refl}(p)} \quad (11)$$

be the non-decreasing sequence of order statistics of the subsample (9). Then the *sample median estimator* of the parameter vector (10) is defined as

$$\left\{ \widehat{smed^{(p)}} \right\}_{p=1}^{P_{inc}}, \quad (12)$$

where

$$\widehat{smed^{(p)}}(f) = \begin{cases} f^{(p)}_{((P_{refl}(p)+1)/2)}, & \text{if } P_{refl}(p) \text{ is odd;} \\ \frac{1}{2}\big(f^{(p)}_{(P_{refl}(p)/2)} + f^{(p)}_{(P_{refl}(p)/2+1)}\big), & \\ & \text{if } P_{refl}(p) \text{ is even.} \end{cases}$$

Let $0 \le \alpha < 1/2$ be a number, and let $[\cdot]$ denote the integer part of a real number. Then the *sample trimmed mean* estimator of the parameter vector (10) is defined as

$$\left\{ \widehat{tm_\alpha^{(p)}} \right\}_{p=1}^{P_{inc}}, \quad (13)$$

where

$$\widehat{tm_\alpha^{(p)}}(f) = \frac{1}{P_{refl}(p)(1-2\alpha)} \times$$

$$\Bigg\{ \Big([P_{refl}(p)\alpha] + 1 - P_{refl}(p)\alpha\Big)\Big(f^{(p)}_{([P_{refl}(p)\alpha]+1)} +$$

$$f^{(p)}_{(P_{refl}(p)-[P_{refl}(p)\alpha])}\Big) + \sum_{i=[P_{refl}(p)\alpha]+2}^{P_{refl}(p)-[P_{refl}(p)\alpha]-1} f^{(p)}_{(i)} \Bigg\}.$$

VII. Hypothesis testing for generalized diffuse reflection models

It is rather straightforward to build a test for checking whether any particular material is perfectly diffuse. Indeed, the corresponding null hypothesis can be tested via a *t*-statistic on the basis of the observed set of BRDF values. However, as we noted above, testing this hypothesis is not very informative as this null hypothesis will be rejected even for those materials that serve as diffuse reflectance standards.

Therefore, it makes more sense to test a hypothesis that a material has diffuse reflection in general, even though not perfectly diffuse with the same level of reflection for each incoming angle. This amounts to building a multiple testing procedure for testing the joint hypothesis $H_0 = \bigcap_{1\le p\le P_{inc}} H_p$, where H_p is the p-th null hypothesis stating that the p-th layer (9) is laying on a sphere.

As an application of the above estimators, we propose now a class of tests for the compound hypothesis H_0. Consider any sequence of test statistics $\{MT_p\}_{1\le p\le P_{inc}}$, where MT_p is used for testing the corresponding hypothesis H_p. For a given sample of points from the BRDF, let us apply the test based on MT_p for testing the hypothesis H_p for all p. Denote the corresponding resulting p-values by $PV_1, \ldots, PV_{P_{inc}}$, and let $PV_{(1)} \le \ldots \le PV_{(P_{inc})}$ be the ordered set of these p-values. Then one could suggest to reject H_0 if $PV_{(p)} \le p\alpha/P_{inc}$ for at least one p.

Under certain conditions, this multiple testing procedure is asymptotically consistent and more powerful than the procedure based on the Bonferroni principle applied to the same sequence of test statistics $\{MT_p\}_{1\le p\le P_{inc}}$, which is often assumed to be the default way of testing several hypothesis simultaneously. Our procedure capitalizes on the physical fact that, as the incoming light angle grows, deviations from diffuse reflection can only grow as well. Therefore, in mathematical terms, the test statistics $\{MT_p\}_{1\le p\le P_{inc}}$ would be highly positively correlated for any reasonable choice of these statistics. See [43] for details related to rigorous analysis of this type of multiple testing methods.

To give a specific example, for simplicity of notation, we use the median-based estimator. Consider the sequence of test statistics $\{MT_p\}_{1\le p\le P_{inc}}$, where

$$MT_p = \frac{\sqrt{n}}{SSD^{(p)}} \times \quad (14)$$

$$\Bigg(\frac{1}{\big|\Omega_{refl}(p)\big|} \sum_{\left(\theta_r^{(q)}, \varphi_r^{(q)}\right) \in \Omega_{refl}(p)} f\big(\theta_i^{(p)}, \varphi_i^{(p)}, \theta_r^{(q)}, \varphi_r^{(q)}\big)$$

$$- \min\big\{\widehat{smed^{(p)}},\, 1/\pi\big\} \Bigg)$$

and $SSD^{(p)}$ denotes the sample standard deviation of the $p-$th BRDF subsample (9). We pick this particularly simple form of a test statistic only for illustrative purposes. In fact, MT_p is best suited for testing a hypothesis that the points of the true $p-$th layer of the BRDF belong to a sphere, while they were measured with normally distributed independent errors, versus the alternative that the points of the $p-$th layer are not symmetric about the median and tend to have bigger deviations from the median. We also introduced the $1/\pi$-correction in (14) in order to account for the energy conservation law, as we are only interested in testing against physically plausible alternatives. Depending on the assumptions that we make about measurement errors, it is possible to use any other appropriate test statistics instead of $\{MT_p\}_{1\leq p\leq P_{inc}}$. The principle of constructing the multiple test remains the same.

Note that it is crucial to take into account the multiplicity of tests. Otherwise, irrespectively of what kind of test statistics we use, if the decisions about each of the basic hypothesis $H_0, \ldots, H_{P_{inc}}$ are made on the basis of the unadjusted marginal $p-$values, then the probability to reject some true null hypothesis will be too large and the test will not be reliable. Unfortunately, this mistake is commonly made in applications of multiple testing.

VIII. Conclusion

BRDF is one of the fundamental concepts in such diverse fields as multidimensional reflectometry, computer graphics and computer vision. Most of BRDF models are only crude approximations of reflectance of real materials. In view of this, some of the breakthrough results from computer vision and animation would not fit applications in reflectometry and in many industries.

Since measurement errors can greatly influence shape and properties of BRDF manifolds, there is a clear need to develop new methods for handling BRDFs with measurement uncertainties. Moreover, analytic reflectance models are limited to describing only special subclasses of materials. In computer graphics and vision, it is important that BRDF models should be processed in real-time, but quantitative accuracy is not as important. In reflectometry and metrology, it is the opposite: there is typically no need in real-time processing of BRDFs, but quantitative accuracy is the paramount.

In this paper, we treated BRDF measurements as samples of points from high-dimensional non-linear non-convex manifolds. We have shown that statistical analysis of BRDF measurements has to account both for nonlinear structure of the data as well as for ill-behaved noise. Standard statistical methods can not be safely directly applied to BRDF data. Our study of parameters for generalized Lambertian models clarified certain pitfalls in the analysis of BRDF data. We developed more refined estimators for BRDF models of standard diffuse reference materials.

We also introduced the notion of Pitman closeness to metrology and to computer graphics and applied this closeness criterion to compare different estimators for BRDF models. This criterion for comparison of estimators seems to be especially appropriate for applications in metrology, and for comparing between statistical and machine learning procedures that can potentially rely on different loss functions or different model assumptions. Based on this and other criteria, we have shown that, in the context of BRDF model parameter estimation, estimators based on either median or trimmed mean are safer to use and are often more precise than estimators based on sample means. Additionally, we outlined an efficient approach to build multiple testing procedures for testing composite hypotheses about BRDFs and their layers.

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