A Note on Hopfield Neural Network Stability

Jiemin Zhao

***Abstract*—Without assuming the symmetry of the connected matrix and the boundedness or monotonicity of the nonlinear activation functions, some new sufficient criterions are obtained for the asymptotic stability in the large for a class of neural networks. These criterions are applied easily and widely too.**

***Index Terms*—Asymptotic stability in the large, Boundedness, Monotonicity, Hopfield Neural network**

I. Introduction

The Hopfield neural network (HNN [1]–[3]) theory and its application are extensively discussed in literature, for example, [3]–[17], [19], [21]–[29] and the references cited therein. M. Ghatee and M. Niksirat [7] dealt with maximum cut problem on a graph with fuzzy edges. An adaptive Hopfield neural network with modern simulated annealing cooling schedule was proposed to solve the problem. Its efficiency was tested for some simulated benchmark examples. M. Sheikhan and E. Hemmati [13] explored the use of Hopfield neural network as a path set selection algorithm. The authors used link expiration time between two nodes to estimate link reliability. In this method, node-disjoint and link-disjoint path sets can be found simultaneously with route discovery algorithm. Simulation results showed that the proposed protocol can find path sets with higher reliability comparing to other recently proposed algorithms. In monograph [19] some sufficient conditions were obtained for the stability of the HNN. In [29] a new 5 × 7 optimized FLC-coupled HNN maximum tracking technique was designed. A HNN was used to tune the fuzzy membership function routinely. Entire components of a PV array, a DC–DC buck-boost zeta converter and a designed MPP tracking controller were implemented in a Matlab–Simulink tool to validate the HNN. The results validated the effectiveness and execution of the HNN using the optimized fuzzy system. The designed system was successfully tested on an experimental prototype.

As we know, determining the stability of the equilibrium points is essential for the HNN and its application [4], [5], [12], [17], [19]. It is commonly assumed that the connected matrix is symmetrical or the nonlinear activation functions are monotonously non-decreasing or bounded ([3]–[16] etc.) in the stability criterions for HNN.

In the paper, we prove some new sufficient criterions of asymptotic stability in the large for the HNN by means of analysis and calculation method without assuming the symmetry of the connected matrix, and the boundedness [20], or the monotonicity of the nonlinear activation functions.

This paper is arranged as follow. In section 2, we state the main results. In section 3, we give the proofs of theorems. In section 4, we briefly discuss our conclusion.

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II. Main Results

Consider the following Hopfield neural network

$$C_i \frac{d\,u_i(t)}{d\,t} = \sum_{j=1}^{n} w_{ij} V_j - \frac{u_i(t)}{R_i} + I_i\,, \qquad u_i = g_i^{-1}(V_i), \quad i = 1, 2, \cdots, n, \tag{1}$$

where $w_{ij}V_i$ represents the electrical current input to cell i due to the present potential of cell j, and w_{ij} is thus the synapse efficacy. Linear summing of inputs is assumed. I_i is any other (fixed) input current to neuron i. u_i is the neural voltages. R_i is the resistances. C_i is the capacitance [1]–[4].

We know that HNN there exists equilibrium point [19]. Suppose $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T$ is an equilibrium point of system (1). Let $x = u - u^* = (x_1, x_2, \cdots, x_n)^T$, then system (1) can be rewritten as following equivalent system [19]

$$C_i \frac{dx_i}{dt} = -\frac{x_i}{R_i} + \sum_{j=1}^{n} w_{ij}\, f_j(x_j),\ i = 1, 2, \cdots, n, \qquad f_j(0) = 0,\ j = 1, 2, \cdots, n, \tag{2}$$

where $f_j(x_j) = g_j(x_j + u_j^*) - g_j(u_j^*)$.

Define $nsgn\, x_i$, a_{ij} as follows:

$$nsgn\, x_i = \begin{cases} 1, & x_i \ge 0, \\ -1, & x_i < 0, \end{cases} \quad (i = 1, 2, \cdots, n),$$

$$a_{ij} = R_i w_{ij}\, nsgn\, x_i \ \ (i, j = 1, 2, \cdots, n).$$

The following Theorem 1 through 5 do not need the symmetry of the connected matrix or the boundedness and the monotonicity of the nonlinear activation functions.

Theorem 1. *If*

$$f_j(x_j) \sum_{i=1}^{n} a_{ij} < |\, x_j\,| \quad for\ \ x_j \ne 0,\ \ j = 1, 2, \cdots, n, \tag{3}$$

then system (2) is asymptotically stable in the large.

Theorem 2. *If*

$$|\, f_j(x_j)\,| \ \sum_{i=1}^{n} R_i \,|\, w_{ij}\,| < |\, x_j| \ \text{ for } x_j \ne 0,\ \ j = 1, 2, \cdots, n,$$

then system (2) is asymptotically stable in the large.

Theorem 3. *If* $|\, f_j'(x_j)\,| \le \omega_j$ $(\omega_j = const. \ge 0)$ *and*

$$\omega_j \sum_{i=1}^{n} R_i \, | w_{ij} | < 1, \quad j = 1, 2, \cdots, n,$$

then system (2) is asymptotically stable in the large.

Theorem 4. *If* $| f_j(x_j) | \le K_j \, | x_j |$ ($K_j = const. \ge 0$) *and*

$$K_j \sum_{i=1}^{n} R_i \, | w_{ij} | < 1, \quad j = 1, 2, \cdots, n,$$

then System (2) is asymptotically stable in the large.

Theorem 5. *If*

$$| f_j(u) - f_j(v) | \le L_j \, | u - v | \ \ (u, v \in R, \ L_j = const. \ge 0)$$

and

$$L_j \sum_{i=1}^{n} R_i \, | w_{ij} | < 1, \quad j = 1, 2, \cdots, n,$$

then system (2) is asymptotically stable in the large .

For the HNN, it is commonly assumed that the connected matrix is symmetrical and the hyperbolic tangent function $\tanh cx$ is taken as activation function ([7], [16] etc.). We have no requirement for the symmetry of the connected matrix in the following Theorem 6 and Theorem 7.

Theorem 6. *Suppose*

$$\sum_{i=1}^{n} R_i \, | w_{ij} | < 1, \quad j = 1, 2, \cdots, n \ .$$

1) *If*

$$f_j(x_j) = b_j \tan^{-1} c_j x_j, \ b_j, \ c_j = const, \ 0 < b_j c_j \le 1,$$

then system (2) is asymptotically stable in the large.

2) *If* $f_j(x_j) = b_j \tanh c_j x_j$, *then system (2) is asymptotically stable in the large.*

3) *If* $f_j(x_j) = b_j \sinh^{-1} c_j x_j$, *then system (2) is asymptotically stable in the large.*

Theorem 7. *If* $f_j(x_j) = \dfrac{| x_j + 1 | - | x_j - 1 |}{2}$ *and*

$$\sum_{i=1}^{n} R_i \, | w_{ij} | < 1, \quad j = 1, 2, \cdots, n,$$

then system (2) is asymptotically stable in the large.

III. Proof of the Theorems

Proof of the Theorem 1

We need following Lemma in the proof of Theorem 1.

Lemma. *Suppose that the functions* $f_{ij}(x_j)$ *are continuous and satisfy the conditions such that the uniqueness theorem of solutions holds for the system*

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} f_{ij}(x_j), \ i = 1, 2, \cdots, n, \ f_{ij}(0) = 0 \ . \tag{4}$$

We also suppose, that there are numbers

$$a_i > 0, \ b_i > 0 \ \ (1 \le i \le n)$$

such that

$$\sum_{i=1}^{n} \Phi_i(x_i) f_{ij}(x_j) < 0 \ \ \text{for} \ x_j \ne 0 \ \ (1 \le j \le n) \tag{5}$$

where

$$\Phi_i(x_i) = \begin{cases} a_i, & x_i \ge 0, \\ -b_i, & x_i < 0, \end{cases} \tag{6}$$

then the zero solution of (4) is asymptotically stable in the large [18].

Proof of the Lemma. To prove the Lemma we use the comparison function

$$V(x) = \sum_{i=1}^{n} \Phi_i(x_i) x_i \quad [x = (x_1, \ \cdots, x_n)^T], \tag{7}$$

which satisfies a Lipschitz condition in R^n . We have

$$\begin{aligned} | V(\bar{x}) - V(x) | &\le \sum_{i=1}^{n} | \Phi_i(\bar{x}_i) \bar{x}_i - \Phi_i(x_i) x_i | \\ &\le \sum_{i=1}^{n} \text{Max} \ (a_i, b_i) \, | \bar{x}_i - x_i | \\ &\le \text{Max} \ (a_1, \cdots, b_n) \sum_{i=1}^{n} | \bar{x}_i - x_i | . \end{aligned}$$

Let

$$\lambda' = \operatorname*{Min}_i(a_i, b_i), \ \lambda'' = \operatorname*{Max}_i(a_i, b_i)$$

and $| x | = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$, then

$$| V(\bar{x}) - V(x) | \le \sqrt{n} \, \lambda'' \, | \bar{x} - x |$$

Specially $V(x) \le \sqrt{n} \, \lambda'' \, | x |$. In addition,

$$V(x) = \sum_{i=1}^{n} | \Phi_i(x_i) | \, | x_i | \ge \lambda' \sum_{i=1}^{n} | x_i | \ge \lambda' \, | x | \to \infty$$

as $| x | \to \infty$. That implies that V is positive-definite and "infinitely large".

Now we consider an arbitrary (non-trivial) solution of (4)

$$x(t; \tau, \xi) \ \ [x(\tau; \tau, \xi) = \xi \ne 0] .$$

Firstly, we prove that the comparison function V , i.e., $\varphi(t) = V(x(t; \tau, \xi))$ is strict decreasing along this solution. It is enough to show

$$\varphi(t) < \varphi(\tau) = V(\xi) \quad \text{for} \ t > \tau \tag{8}$$

as long as the solution exists. The existence of solution for $\tau \le t < \infty$ then follows from the estimate.

By

$$\left. \frac{dx_i}{dt} \right|_{t=\tau} = \sum_{j=1}^{n} f_{ij}(\xi_j) = f_i(\xi), \ i = 1, 2, \cdots, n$$

we have

$$\lim_{t \to \tau} \frac{x_i(t; \tau, \xi) - \xi_i}{t - \tau} = f_i(\xi),$$

So, for any given positive number $\varepsilon > 0$, we can assign a positive number $\delta = \delta(\varepsilon) > 0$ such that for $\tau \le t \le \tau + \delta$, the following inequalities hold

$$\xi_i + (f_i(\xi) - \varepsilon)(t - \tau) \le x_i(t; \tau, \xi) \le \xi_i + (f_i(\xi) + \varepsilon)(t - \tau).$$

In case of $x_i(t; \tau, \xi) \ge 0$, we have

$$\Phi_i(x_i) x_i \le \Phi_i(x_i) \xi_i + \Phi_i(x_i)(f_i(\xi) + \varepsilon)(t - \tau) .$$

And in case of $x_i(t; \tau, \xi) < 0$, we have

$$\Phi_i(x_i) x_i \le \Phi_i(x_i) \xi_i + \Phi_i(x_i)(f_i(\xi) - \varepsilon)(t - \tau) ,$$

Taking $\varepsilon_i = \pm \varepsilon$ in two cases, we obtain

$$\Phi_i(x_i)x_i \le \Phi_i(x_i)\xi_i + \Phi_i(x_i)(f_i(\xi)+\varepsilon_i)(t-\tau),$$

Therefore

$$\varphi(t) \le \sum_{i=1}^{n} \Phi_i(x_i)\xi_i + (t-\tau)\sum_{i,j=1}^{n} \Phi_i(x_i) f_{ij}(\xi_j) +$$
$$n\varepsilon\lambda''(t-\tau),\ t\in[\tau, \tau+\delta].$$

Choosing δ so that the signs of every non-zero component ξ_i and the corresponding component $x_i(t;\tau,\xi)$ coincide (that is, $\Phi_i(x_i)=\Phi_i(\xi_i)$ for $\xi_i \ne 0$), we obtain

$$\varphi(t) \le \varphi(\tau) + (t-\tau)\sum_j \{\Phi_j(\xi_j) f_{jj}(\xi_j) +$$
$$\sum_{i\ne j} \Phi_i(x_i) f_{ij}(\xi_j)\} + n\varepsilon\lambda''(t-\tau),$$

$$\varphi(t) \le \varphi(\tau) - (t-\tau)\sum_j \mu_j(\xi_j) + n\varepsilon\lambda''(t-\tau) < \varphi(\tau)$$

for all j with $\xi_j \ne 0$ and ε sufficiently small.

If there is a time $\tau_2 > \tau$ such that

$$\varphi(\tau_2) = \varphi(\tau), \quad \varphi(t) < \varphi(\tau) \quad \text{for } \tau < t < \tau_2,$$

then the function $\varphi(t)$ attains its minimum in (τ, τ_2), at τ_1, say.

Denoting $\xi_1 = x(\tau_1;\tau,\xi)$, $x(t;\tau_1,\xi_1) \equiv x(t;\tau,\xi)$, we can prove similarly

$$\varphi(t) = V(x(t;\tau_1,\xi_1)) < \varphi(\tau_1) = V(\xi_1) \quad \text{for } \tau_1 < t \le \tau_1 + \delta_1.$$

But this is a contradiction, so (8) is true.

Because the function

$$\varphi(t) = V(x(t;\tau,\xi)) \quad \text{for } t \ge \tau$$

is decreasing, it follows immediately that $x=0$ is a stable solution, and

$$|x(t;\tau,\xi)| \le \frac{1}{\lambda'}\varphi(t) \le \frac{1}{\lambda'}\varphi(\tau) \le \sqrt{n}\frac{\lambda''}{\lambda'}|\xi| \le \varepsilon \quad (t \ge \tau),$$

for

$$|\xi| \le \delta(\varepsilon) = \frac{\lambda'\varepsilon}{\lambda''\sqrt{n}}.$$

Hence every solution $x(t;\tau,\xi)$ is bounded for $t \ge \tau$, and the following limit exists

$$\lim_{t\to\infty} V(x(t;\tau,\xi)) = V_0 \ge 0.$$

Since the solution has a non-empty ω - limit set, there also exists an ω - limit trajectory $x(t_0)$, and for every fixed t we can find a monotone increasing divergent sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} x(t_k;\tau,\xi) = x_0(t)$$
$$[\text{and } \lim_{k\to\infty} V(x(t_k;\tau,\xi)) = V(x_0(t))].$$

Consequently, we have for all t

$$V(x_0(t)) = V_0,$$

which shows that $x_0(t) \equiv 0$ and $V_0 = 0$, because the comparison function decreases strictly along every non-trivial solution.

It follows immediately that

$$\lim_{t\to\infty} x(t;\tau,\xi) = 0$$

This establishes the Lemma [18].

Using this Lemma, we can prove the Theorem 1. In fact, let $\varphi_i(x_i)$ be the piecewise continuous functions defined as following

$$\varphi_i(x_i) = \begin{cases} C_i R_i, & x_i \ge 0, \\ -C_i R_i, & x_i < 0, \end{cases} \quad i = 1, 2, \cdots, n$$

and the functions $f_{ij}(x_j)$ defined by

$$f_{i1}(x_1) = \frac{w_{i1}}{C_i} f_1(x_1),\ f_{i2}(x_2) = \frac{w_{i2}}{C_i} f_2(x_2), \cdots\cdots,$$

$$f_{ii-1}(x_{i-1}) = \frac{w_{ii-1}}{C_i} f_{i-1}(x_{i-1}),\ f_{ii}(x_i) = -\frac{x_i}{C_i R_i} + \frac{w_{ii}}{C_i} f_i(x_i),$$

$$f_{ii+1}(x_{i+1}) = \frac{w_{ii+1}}{C_i} f_{i+1}(x_{i+1}), \cdots\cdots,\ f_{in}(x_n) = \frac{w_{in}}{C_i} f_n(x_n),$$

then

$$\varphi_1(x_1) f_{1j}(x_j) + \varphi_2(x_2) f_{2j}(x_j) + \cdots + \varphi_{j-1}(x_{j-1}) f_{j-1j}(x_j) +$$
$$\varphi_j(x_j) f_{jj}(x_j) + \varphi_{j+1}(x_{j+1}) f_{j+1j}(x_j) + \cdots + \varphi_n(x_n) f_{nj}(x_j) =$$
$$\frac{w_{1j}}{C_1} f_j(x_j)\varphi_1(x_1) + \frac{w_{2j}}{C_2} f_j(x_j)\varphi_2(x_2) + \cdots +$$
$$\frac{w_{j-1j}}{C_{j-1}} f_j(x_j)\varphi_{j-1}(x_{j-1}) + \left(\frac{w_{jj}}{C_j} f_j(x_j) - \frac{x_j}{C_j R_j}\right)\varphi_j(x_j) +$$
$$\frac{w_{j+1j}}{C_{j+1}} f_j(x_j)\varphi_{j+1}(x_{j+1}) + \cdots + \frac{w_{nj}}{C_n} f_j(x_j)\varphi_n(x_n) =$$

$$\frac{w_{1j}}{C_1} f_j(x_j)\cdot C_1 R_1\, nsgn\, x_1 + \frac{w_{2j}}{C_2} f_j(x_j)\cdot C_2 R_2\, nsgn\, x_2 + \cdots +$$
$$\frac{w_{j-1j}}{C_{j-1}} f_j(x_j)\cdot C_{j-1} R_{j-1}\, nsgn\, x_{j-1} + \left(\frac{w_{jj}}{C_j} f_j(x_j) - \frac{x_j}{C_j R_j}\right)\times$$
$$C_j R_j\, nsgn\, x_j + \frac{w_{j+1j}}{C_{j+1}} f_j(x_j)\cdot C_{j+1} R_{j+1}\, nsgn\, x_{j+1} + \cdots + \frac{w_{nj}}{C_n}\times$$
$$f_j(x_j)\cdot C_n R_n\, nsgn\, x_n = (R_1 w_{1j}\, nsgn\, x_1 + R_2 w_{2j}\, nsgn\, x_2 + \cdots +$$
$$R_{j-1} w_{j-1j}\, nsgn\, x_{j-1} + R_j w_{jj}\, nsgn\, x_j + R_{j+1} w_{j+1j}\, nsgn\, x_{j+1} + \cdots +$$
$$R_n w_{nj}\, nsgn\, x_n) f_j(x_j) - x_j\, nsgn\, x_j = (R_1 w_{1j}\, nsgn\, x_1 +$$
$$R_2 w_{2j}\, nsgn\, x_2 + \cdots + R_{j-1} w_{j-1j}\, nsgn\, x_{j-1} + R_j w_{jj}\, nsgn\, x_j +$$
$$R_{j+1} w_{j+1j}\, nsgn\, x_{j+1} + \cdots + R_n w_{nj}\, nsgn\, x_n) f_j(x_j) - |x_j| =$$
$$(a_{1j} + a_{2j} + \cdots + a_{j-1j} + a_{jj} + a_{j+1j} + \cdots + a_{nj}) f_j(x_j) - |x_j|.$$

If

$$(a_{1j} + a_{2j} + \cdots + a_{j-1j} + a_{jj} + a_{j+1j} + \cdots + a_{nj}) f_j(x_j) - |x_j|$$
$$< 0 \text{ for } x_j \ne 0 \ (j = 1, 2, \cdots, n),$$

then the Hopfield neural network (2) is asymptotically stable in the large. This ends the proof of the Theorem 1.

Proof of the Theorem 2

If $|f_j(x_j)|\sum_{i=1}^{n} R_i |w_{ij}| < |x_j|$ for $x_j \ne 0$, $j = 1, 2, \cdots, n$, then

$$f_j(x_j)\sum_{i=1}^{n} a_{ij} \le$$
$$|f_j(x_j)| \sum_{i=1}^{n} R_i |w_{ij}| < |x_j| \text{ for } x_j \ne 0,\ j = 1, 2, \cdots, n.$$

Therefore, the proof of the theorem 2 is completed.

Proof of the Theorem 3

If $|f_j'(x_j)| \le \omega_j$ ($\omega_j = \text{const.} \ge 0$), then

$$|f_j(x_j)| = |f_j(x_j) - f_j(0)| =$$
$$|f_j'(\xi)(x_j - 0)| = |x_j\, f_j'(\xi)| \le \omega_j |x_j|,$$

where ξ belongs to $(0, x_j)$ for $x_j \ge 0$ or ξ belongs to $(x_j, 0)$ for $x_j \le 0$. So,

$$f_j(x_j)\sum_{i=1}^{n} a_{ij} \le \omega_j |x_j| \sum_{i=1}^{n} R_i |w_{ij}| <$$
$$|x_j| \text{ for } x_j \ne 0,\ \ j = 1, 2, \cdots, n.$$

The system (2) is asymptotically stable in the large by the Theorem 1. This ends the proof of theorem 3.

Proof of the Theorem 4

If $|f_j(x_j)| \le K_j |x_j|$ ($K_j = \text{const.} \ge 0$) and $K_j \sum_{i=1}^{n} R_i |w_{ij}| < 1$, $j = 1, 2, \cdots, n$, then

$$f_j(x_j)\sum_{i=1}^{n} a_{ij} \le K_j |x_j| \sum_{i=1}^{n} R_i |w_{ij}| <$$
$$|x_j| \text{ for } x_j \ne 0,\ \ j = 1, 2, \cdots, n.$$

This establishes the theorem 4 by the theorem 1.

Proof of the Theorem 5

If the function $f_j(x_j)$ satisfies the Lipschitz condition, and

$$L_j \sum_{i=1}^{n} R_i |w_{ij}| < 1,\ \ j = 1, 2, \cdots, n,$$

then all of the conditions of theorem 4 are satisfied. Therefore, the system (2) is asymptotically stable in the large. The proof of the Theorem 5 is completed.

Proof of the Theorem 6

1) Let

$$g_j(x_j) = x_j - f_j(x_j) = x_j - b_j \tan^{-1} c_j x_j$$
$$\text{for } x_j \ge 0,\ \ j = 1, 2, \cdots, n.$$

If $b_j c_j \in (0, 1]$, then

$$\frac{d}{dx_j} g_j(x_j) = 1 - \frac{b_j c_j}{1 + (c_j x_j)^2} =$$
$$\frac{(c_j x_j)^2 + (1 - b_j c_j)}{1 + (c_j x_j)^2} > 0 \text{ for } x_j > 0,\ \ j = 1, 2, \cdots, n,$$

and $g_j(0) = 0$. Consequently we have

$$x_j - f_j(x_j) > 0 \text{ for } x_j > 0,\ \ j = 1, 2, \cdots, n.$$

Therefore,

$$f_j(x_j)\sum_{i=1}^{n} a_{ij} \le |b_j \tan^{-1} c_j x_j| \sum_{i=1}^{n} |a_{ij}| \le |x_j| \sum_{i=1}^{n} R_i |w_{ij}|$$
$$< |x_j| \text{ for } x_j \ne 0,\ \ j = 1, 2, \cdots, n.$$

By the theorem 1, the system (2) is asymptotically stable in the large.

2) Let

$$h_j(x_j) = x_j - f_j(x_j) = x_j - b_j \tanh c_j x_j$$
$$\text{for } x_j \ge 0,\ \ j = 1, 2, \cdots, n.$$

We have

$$\frac{d}{dx_j} h_j(x_j) = 1 - b_j c_j \operatorname{sech}^2 c_j x_j \ge 1 - b_j c_j \operatorname{sech}^2 0 = 1 - b_j c_j \ge 0,$$

and $h_j(0) = 0$. Therefore,

$$x_j - b_j \tanh c_j x_j > 0 \text{ for } x_j > 0,\ \ j = 1, 2, \cdots, n.$$

and

$$f_j(x_j)\sum_{i=1}^{n} a_{ij} \le |b_j \tanh c_j x_j| \sum_{i=1}^{n} |a_{ij}| \le = |x_j| \sum_{i=1}^{n} R_i |w_{ij}|$$
$$< |x_j| \text{ for } x_j \ne 0,\ \ j = 1, 2, \cdots, n.$$

Hence, the system (2) is asymptotically stable in the large.

3) Let

$$u_j(x_j) = x_j - f_j(x_j) = x_j - b_j \sinh^{-1} c_j x_j$$
$$\text{for } x_j \ge 0,\ \ j = 1, 2, \cdots, n.$$

then

$$\frac{d}{dx_j} u_j(x_j) = 1 - \frac{b_j c_j}{\sqrt{1 + (c_j x_j)^2}} = \frac{\sqrt{1 + (c_j x_j)^2} - b_j c_j}{\sqrt{1 + (c_j x_j)^2}}$$
$$\text{for } x_j > 0,\ \ j = 1, 2, \cdots, n,$$

and $u_j(0) = 0$. Consequently we have

$$x_j - f_j(x_j) > 0 \text{ for } x_j > 0,\ \ j = 1, 2, \cdots, n.$$

Therefore,

$$f_j(x_j)\sum_{i=1}^{n} a_{ij} \le |x_j| \sum_{i=1}^{n} R_i |w_{ij}| < |x_j|$$
$$\text{for } x_j \ne 0,\ \ j = 1, 2, \cdots, n.$$

It follows that the system (2) is asymptotically stable in the large. The proof of the theorem 6 is completed.

Proof of the Theorem 7

The function $f_j(x_j) = \dfrac{|x_j + 1| - |x_j - 1|}{2}$ is sketched in Figure 1.

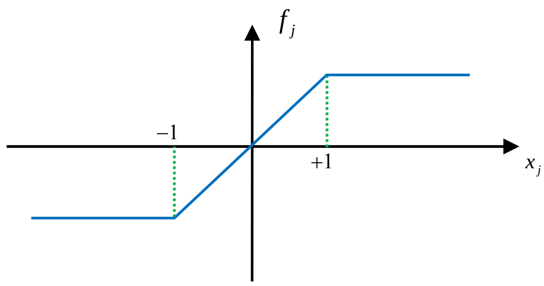

Fig. 1. The sketch of the function $f_j(x_j)$

If $\sum_{i=1}^{n} R_i |w_{ij}| < 1$, $j = 1, 2, \cdots, n$, then

$$f_j(x_j)\sum_{i=1}^{n} a_{ij} \le |x_j| \sum_{i=1}^{n} |a_{ij}| = |x_j| \sum_{i=1}^{n} R_i |w_{ij}|$$
$$< |x_j| \text{ for } x_j \ne 0,\ \ j = 1, 2, \cdots, n.$$

By the theorem 1, the system (2) is asymptotically stable in the large. The proof of the Theorem 7 is completed.

IV. Conclusion

This paper mainly focuses on the stability analysis of Hopfield neural network. Seven new sufficient criterions are given to guarantee the stability of Hopfield neural network. These conditions can be satisfied easily. And these results can be applied widely in more areas. The Hopfield model based neural network is used frenquently in practical applications.

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