C^1 Positivity-preserving Interpolation Schemes with Local Free Parameters

Xiangbin Qin, Lei Qin, and Qingsong Xu

***Abstract*—A kind of rational cubic/quadratic interpolation spline with three local free parameters is presented. Simple sufficient conditions for constructing positivity-preserving interpolation curves are given. By using the boolean sum of cubic interpolating operators to blend together the constructed rational cubic/quadratic interpolation splines as four boundary functions, a class of C^1 bi-cubic partially blended rational cubic/quadratic interpolation surface with six families of local free parameters is constructed. By developing new constrains on the boundary functions, simple sufficient data dependent conditions are theoretically deduced on the local free parameters for generating C^1 positivity-preserving interpolation surfaces on rectangular domain.**

***Index Terms*—Data visualization, Interpolation surface, Positivity-preserving, Local free parameter.**

I. Introduction

Spline is relevant to a wide range of applications, such as data fitting [1], principal components analysis [2], signal restoration [3] and so on. In scientific data visualization, when the data are arising from some complex function or from some scientific phenomena, it becomes crucial that the resulting spline can preserve the shape features of the data. Positivity is one of the essential shape features of data. Many physical situations have entities that gain meaning only when their values are positive, such as a probability distribution function, monthly rainfall amounts, speed of winds at different intervals of time, and half-life of a radioactive substance and so on. For given positive data, ordinary interpolation spline methods such as the classical cubic interpolation spline usually ignores positive characteristic. Thus constructing positivity-preserving interpolation spline is an essential problem and has been attracted widespread interests. Various positivity-preserving interpolation spline methods have been proposed, such as the cubic interpolation spline methods [4], the rational trigonometric interpolation spline methods [5], [6] and the rational polynomial interpolation spline methods [7], [8], [9], [10], [11], [12], [13].

Some C^1 positivity-preserving bivariate rational interpolation splines with local free parameters have been proposed by using Coons surface technique developed in [14], see for example [15], [16], [17], [18], [19] and the references quoted therein. In [15], [16], by exchanging the cubic Hermite blending functions with two different kinds of rational cubic blending functions, two classes of C^1 rational bi-cubic interpolation splines were presented. And constrains concerning the local free parameters were given for visualizing 3D positive data on rectangular grids. Like the classical bi-cubic Coons surface technique, these schemes need to provide the twists on the grid lines in advance for generating interpolation surfaces. In [17], [18], [19], based upon the boolean sum of cubic interpolating operators, by blending different rational cubic interpolation splines as the boundary curves, simpler schemes without making use of twists for constructing C^1 positive interpolation of gridded data were given. These rational bi-cubic partially blended interpolation spline methods are convenience since they are possible to control the shape of the interpolation surfaces by using the boundary curves, though they have to pay the price that the generated surfaces have zero twist vectors at the data points.

However, the sufficient conditions for generating positivity-preserving interpolation surfaces developed in [17], [18], [19] were based on the claim given in [20]: bi-cubic partially blended interpolation surface patch inherits all the properties of network of boundary curves. Thus, these methods have a common point that the positivity of the global interpolation surfaces are determined by the positivity of the four boundary curves of each local interpolation surface patch respectively. As it was pointed out in [16] that, these methods did not depict the positive surfaces due to the coon patches because they conserved the shape of data only on the boundaries of patch not inside the patch. In fact, in order to preserve the positivity of data inside the patch, more constrains are needed on the boundaries of patch. In this paper, new constrain conditions on the boundary curves of each local interpolation surface patch are developed, which is different from those used in [17], [18], [19] and sufficient to preserve the positivity of the resulting interpolation surfaces everywhere in the domain.

In this paper, a C^1 bi-cubic partially blended rational cubic/quadratic interpolation scheme with six local free parameters is developed to handle the problem of constructing a positivity-preserving surface through 3D positive data on rectangular grid. It improves on the existing schemes in some ways:

(1) The smoothness of bi-quadratic interpolant given in [10] is C^0 while in this paper it is C^1.

(2) The interpolation scheme developed in [8] does not allow the designer to refine the positive surface as per consumers demand. Whereas, the given method is done by introducing free parameters which they are used in the description of bi-cubic partially blended rational cubic/quadratic interpolation surface.

(3) In [17], a kind of positivity-preserving interpolation surfaces was developed based on a class of rational cubic/quadratic interpolation spline with two local free parameters. However, the conditions for the interpolation spline

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preserving positivity given in [17] are not sufficient. In fact, one purpose of this paper is to overcome these shortcomings. As we will see later, by introducing a new local tension parameter τ_i into the rational cubic/quadratic interpolation spline given in [17], we can achieve this goal.

(4) In [17], [18], [19], the authors claimed that the rational bi-cubic partially blended functions (coon patches) generated a positive surface but they conserved the shape of data only on the boundaries of patch not inside the patch and they did not provide proof that the conditions given [17], [18], [19] will be always sufficient to generate positivity-preserving interpolation surfaces everywhere in the domain. In contrast, we develop new constrain conditions on the boundary curves of each local interpolation surface patch and the given conditions are sufficient to generate positivity-preserving interpolation surfaces everywhere in the domain with theory proving.

The rest of this paper is organized as follows. Section II recalls the rational cubic/quadratic interpolation spline with two local free parameters given in [17] and analyzes the problem of the positivity-preserving conditions given there. By introducing a new local tension parameter τ_i into the rational cubic/quadratic interpolation spline given in [17], we construct a kind of rational cubic/quadratic interpolation spline with three local free parameters and develop simple sufficient conditions for constructing positivity-preserving interpolation curves. In section III, a kind of C^1 bi-cubic partially blended rational cubic/quadratic interpolation surface with six families of local free parameters is described. Simple sufficient data dependent constraints are derived on the local free parameters to preserve the shape of 3D positive data on rectangular grids. Several numerical examples are also given to prove the worth of the new developed schemes. Conclusion is given in the section IV.

II. New C^1 rational cubic/quadratic interpolation spline

In this section, we firstly recall the rational cubic/quadratic interpolation spline with two local free parameters given in [17] and analyze the problem of the positivity-preserving conditions given there. Then by introducing a new local tension parameter τ_i into the rational cubic/quadratic interpolation spline given in [17], we construct a new C^1 rational cubic/quadratic interpolation spline with three local free parameters and develop sufficient conditions for constructing positivity-preserving interpolation curves.

A. Rational cubic/quadratic interpolation spline with two local free parameters

Let $f_i \in \mathcal{R}$, $i = 1, ..., n$, be data given at the distinct knots $x_i \in \mathcal{R}$, $i = 1, ..., n$, with interval spacing $h_i = x_{i+1} - x_i > 0$, and let $d_i \in \mathcal{R}$ be denoted the first derivative values defined at the knots. In [17], for $x \in [x_i, x_{i+1}]$, a piecewise C^1 rational cubic/quadratic interpolation spline with two local free parameters u_i and v_i is defined over each subinterval $I_i = [x_i, x_{i+1}]$ as follows

$$R(x) = \frac{\sum_{j=0}^{3} A_{ij}(1-t)^{3-j}t^j}{u_i(1-t)^2 + 2(1-t)t + v_i t^2}, \tag{1}$$

where $t = (x - x_i)/h_i$, $u_i, v_i \in (0, +\infty)$, $i = 1, 2, ..., n-1$, and

$$A_{i0} = u_i f_i,$$
$$A_{i1} = 2f_i + u_i(f_i + d_i h_i),$$
$$A_{i2} = 2f_{i+1} + v_i(f_{i+1} - d_{i+1} h_i),$$
$$A_{i3} = v_i f_{i+1}.$$

In applications, the first derivative values d_i, $i = 1, 2, \ldots, n$ are not known and should be specified in advance. In this paper, they are computed by using the following Arithmetic mean method

$$\begin{cases} d_1 = \Delta_1 - \frac{h_1}{h_1+h_2}(\Delta_2 - \Delta_1), \\ d_i = \frac{\Delta_{i-1}+\Delta_i}{2}, i = 2, 3, \ldots, n-1, \\ d_n = \Delta_{n-1} + \frac{h_{n-1}}{h_{n-2}+h_{n-1}}(\Delta_{n-1} - \Delta_{n-2}), \end{cases} \tag{2}$$

where $\Delta_i := (f_{i+1} - f_i)/h_i$. This Arithmetic mean method is the three-point difference approximation based on arithmetic calculation, which is computationally economical and suitable for visualization of shaped data, see for example [17].

For given positivity data set $\{(x_i, f_i), i = 1, 2, \ldots, n\}$, in [17], based on the following constrains for the cubic polynomial numerator $P_i(x) := A_{i0}(1-t)^3 + A_{i1}(1-t)^2 t + A_{i2}(1-t)t^2 + A_{i3}t^3$ preserving positivity

$$(P_i'(0), P_i'(1)) \in \left\{(a, b) : a > \frac{-3P_i(0)}{h_i}, b < \frac{3P_{i+1}(1)}{h_i}\right\},$$

or

$$\begin{cases} \frac{-2u_i f_i + u_i d_i h_i + 2f_i}{h_i} > \frac{-3u_i f_i}{h_i}, \\ \frac{2v_i f_{i+1} + v_i d_{i+1} h_i - 2f_{i+1}}{h_i} < \frac{3v_i f_{i+1}}{h_i}, \end{cases} \tag{3}$$

the following sufficient conditions for the interpolation spline given in (1) preserving positivity were established in Theorem 4.1 of [17]

$$\begin{cases} u_i = w_i + \text{Max}\left\{0, \frac{-2f_i}{f_i + h_i d_i}\right\}, \ w_i > 0, \\ v_i = r_i + \text{Max}\left\{0, \frac{-2f_i}{f_{i+1} - h_i d_{i+1}}\right\}, \ r_i > 0. \end{cases} \tag{4}$$

The constrains given in (3) are equivalent to the following constrains

$$\begin{cases} u_i(f_i + d_i h_i) + 2f_i > 0, \\ v_i(f_{i+1} - d_{i+1} h_i) + 2f_{i+1} > 0, \end{cases} \tag{5}$$

or

$$A_{i1} > 0, \quad A_{i2} > 0. \tag{6}$$

Now let us consider a possible case $f_i + d_i h_i < 0$ and $f_{i+1} - d_{i+1} h_i < 0$. Then in order to ensure the constrains (5) hold, the free parameters must satisfy the following conditions

$$\begin{cases} u_i < \frac{-2f_i}{f_i + d_i h_i}, \\ v_i < \frac{-2f_{i+1}}{f_{i+1} - d_{i+1} h_i}, \end{cases} \tag{7}$$

which is contradict to the sufficient conditions given (4). This also imply that the conditions (4) developed in [17] are not sufficient to generate positivity-preserving interpolation curves for every possible data set.

Fig. 1 shows the interpolation curves generated by using the conditions (4) with different free parameters w_i and r_i. Figs. 1(A–C) shows the interpolation curves for the positivity date sets given in Tab. I and Figs. 1(D–F) shows

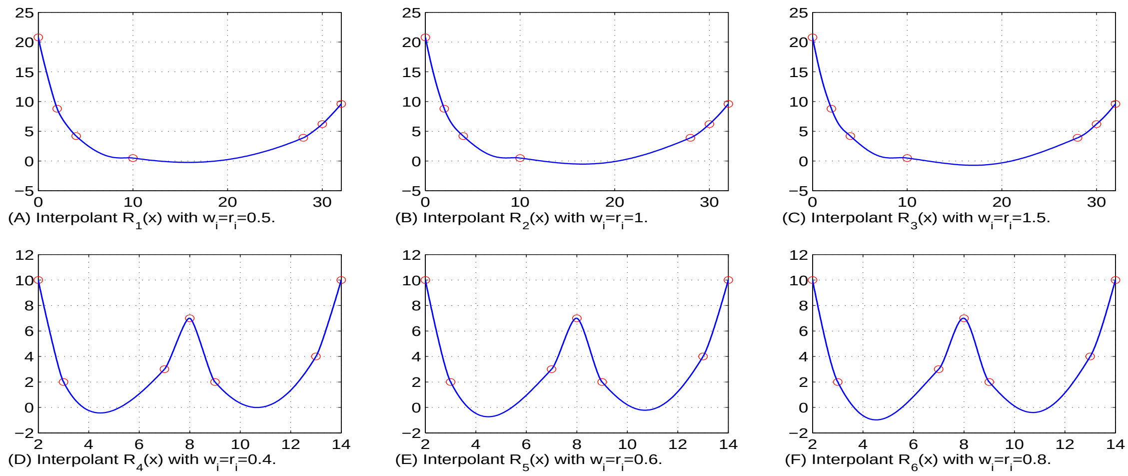

Fig. 1. Interpolation curves generated by using the conditions (4) with different free parameters w_i and r_i for the positive date sets given in Tab. I and Tab. II.

TABLE I
THE POSITIVE DATA SET GIVEN IN [17].

i	1	2	3	4	5	6	7
x_i	0	2	4	10	28	30	32
f_i	20.8	8.8	4.2	0.5	3.9	6.2	9.6

TABLE II
THE POSITIVE DATA SET GIVEN IN [17].

i	1	2	3	4	5	6	7
x_i	2	3	7	8	9	13	14
f_i	10	2	3	7	2	4	10

the interpolation curves for the positivity date sets given in Tab. II, respectively. From the results, we can observe that the conditions (4) given in [17] fail to generate positivity-preserving interpolation curves.

B. New rational cubic/quadratic interpolation spline with three local free parameters

Tension parameter plays a intuitive adjusting role on generating interpolation curves and it is useful for generating positivity-preserving interpolation curves. In this subsection, in order to overcome the drawback of the positivity-preserving sufficient conditions given in (1), we want to introduce a new local tension parameter τ_i into the interpolation spline given in (1) and develop simple sufficient conditions for generating positivity-preserving interpolation curves.

For $x \in [x_i, x_{i+1}]$, a new piecewise rational cubic/quadratic interpolation spline with three local free parameters u_i, v_i and τ_i is defined over each subinterval $I_i = [x_i, x_{i+1}]$ as follows

$$R(x) = \frac{\sum\limits_{j=0}^{3} C_{ij}(1-t)^{3-j}t^j}{u_i(1-t)^2 + \tau_i\,(1-t)\,t + v_i t^2}, \tag{8}$$

where $t = (x - x_i)/h_i$, $u_i, v_i \in (0, +\infty), \tau_i \in [0, +\infty), i = 1, 2, ..., n-1$, and

$$C_{i0} = u_i f_i,$$

$$C_{i1} = \tau_i f_i + u_i\,(f_i + d_i h_i)\,,$$

$$C_{i2} = \tau_i f_{i+1} + v_i\,(f_{i+1} - d_{i+1} h_i)\,,$$

$$C_{i3} = v_i f_{i+1}.$$

It is worth noting that, in [12], [16], one can also find a kind of rational cubic/quadratic interpolation spline with three local free parameters. However, as compared the new constructed rational cubic/quadratic interpolation spline given in (8) with that given in [12], [16], one can find that the new constructed rational cubic/quadratic interpolation spline has a more concise expression than that given [12], [16]. It is obvious that, for all $\tau_i = 2$, the new constructed interpolant (8) will return to the rational cubic/quadratic interpolation spline with two local free parameters given in formula (1).

The spline given in (8) is a C^1 interpolation spline as it satisfies the interpolation properties: $R(x_i) = f_i, R(x_{i+1}) = f_{i+1}, R'(x_i) = d_i, R'(x_{i+1}) = d_{i+1}$. Here $R'(x)$ denotes the derivative with respect to the variable x. It is interesting to note that for all $u_i = v_i = 1$ and $\tau_i = 2$, the piecewise rational cubic/quadratic interpolation spline will reduce to the standard cubic Hermite interpolation spline.

From the expression of the interpolation spline $R(x)$ given in (8), it is easy to check that

$$\lim_{\tau_i \to +\infty} R\,(x) = (1-t)\,f_i + t f_{i+1},$$

which implies that the new introduced parameter τ_i serves as tension parameter.

For given positive data set $\{(x_i, f_i), i = 1, 2, \ldots, n\}$, it is obvious that the interpolation spline $R(x)$ given in (8) is positive in each subinterval $I_i = [x_i, x_{i+1}]$ on condition that $u_i > 0$, $\tau_i > 0$, $v_i > 0$, $C_{i1} \geq 0$ and $C_{i2} \geq 0$. From these, we can immediately obtain the following sufficient conditions for $R(x)$ $(x \in [x_i, x_{i+1}])$ preserving positivity

$$\begin{cases} u_i > 0, \quad v_i > 0, \\ \tau_i = \max\left\{-\frac{u_i(f_i + d_i h_i)}{f_i}, -\frac{v_i(f_{i+1} - d_{i+1} h_i)}{f_{i+1}}, 0\right\} + w_i, \end{cases} \tag{9}$$

where $w_i \geq 0$.

Fig. 2 shows the positivity-preserving interpolation curves generated by using the conditions (9) with different free

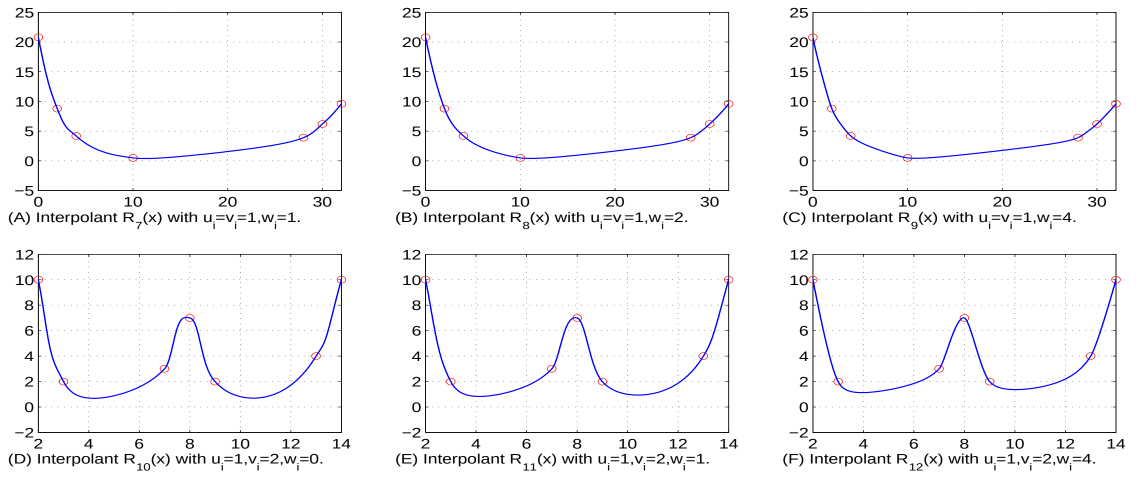

Fig. 2. Positivity-preserving interpolation curves generated by using the conditions (9) with different free parameters w_i for the positive date sets given in Tab. I and Tab. II.

parameters u_i, v_i and w_i. Figs. 2(A–C) shows the positivity-preserving interpolation curves for the positive date set given in Tab. I and Figs. 2(D–F) shows the positivity-preserving interpolation curves for the positive date set given in Tab. II, respectively. From the results, we can see that the interpolation curves preserve the shape of the positive data sets given in Tabs. I and II genially.

For convenience, in the following discussion, we will denote the interpolation spline $R(x)$ given in (8) as $R(t; f_i, f_{i+1}; d_i, d_{i+1}; u_i, \tau_i, v_i)$ for $x \in [x_i, x_{i+1}]$.

III. C^1 POSITIVITY-PRESERVING INTERPOLATION SURFACES

In this section, by using the boolean sum of cubic interpolating operators to blend together the proposed rational cubic/quadratic interpolation splines given in (8) as four boundary functions, we shall construct a class of C^1 bi-cubic partially blended rational cubic/quadratic interpolation surface with six families of local free parameters. By developing new constrains on the boundary functions, we will also theoretically deduce simple sufficient data dependent conditions on the local free parameters to generate C^1 positivity-preserving interpolation surfaces for positive data on rectangular grids.

A. *Bi-cubic partially blended rational cubic/quadratic interpolation surfaces*

Let $\{(x_i, y_i, F_{ij}), i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$ be a given set of data points defined over the rectangular domain $D = [x_1, x_n] \times [y_1, y_m]$, where $\pi_x : x_1 < x_2 < \ldots < x_n$ is the partition of $[x_1, x_n]$ and $\pi_y : y_1 < y_2 < \ldots < y_m$ is the partition of $[y_1, y_m]$. For $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, by using the boolean sum of cubic interpolating operators to blend together the rational cubic interpolation splines (8) as four boundary functions, a new bi-cubic partially blended rational cubic/quadratic interpolation surface is given as follows

$$\begin{aligned} F(x,y) = {} & b_0(s)F(x, y_j) + b_1(s)F(x, y_{j+1}) \\ & + b_0(t)F(x_i, y) + b_1(t)F(x_{i+1}, y) \\ & - b_0(t)b_0(s)F_{i,j} - b_0(t)b_1(s)F_{i,j+1} \\ & - b_1(t)b_0(s)F_{i+1,j} - b_1(t)b_1(s)F_{i+1,j+1}, \end{aligned} \tag{10}$$

where $h_i^x = x_{i+1} - x_i, h_j^y = y_{j+1} - y_j, t = (x - x_i)/h_i^x, s = (y - y_j)/h_j^y$ and

$$b_0(z) := (1 - z)^2 [1 + 2z], \qquad b_1(z) := z^2 [1 + 2(1 - z)],$$

$$F(x, y_j) := R(t; F_{i,j}, F_{i+1,j}; D^x_{i,j}, D^x_{i+1,j}; u^x_{i,j}, \tau^x_{i,j}, v^x_{i,j}),$$

$$F(x_i, y) := R(s; F_{i,j}, F_{i,j+1}; D^y_{i,j}, D^y_{i,j+1}; u^y_{i,j}, \tau^y_{i,j}, v^y_{i,j}).$$

Here, $D^x_{i,j}, D^y_{i,j}$ are known as the first partial derivatives at the grid point (x_i, y_j) and $(u^x_{i,j})_{(n-1)\times m}$, $(\tau^x_{i,j})_{(n-1)\times m}$, $(v^x_{i,j})_{(n-1)\times m}$, $(u^y_{i,j})_{n\times(m-1)}$, $(\tau^y_{i,j})_{n\times(m-1)}$, $(v^y_{i,j})_{n\times(m-1)}$ are called six families of local free parameters. From the interpolation surface $F(x, y)$ given in (10), we can see that the changes of a local free parameter $u^x_{i,j}$ will affect the shape of two neighboring patches $F(x, y)$ defined in the domain $(x, y) \in (x_i, x_{i+1}) \times (y_{i-1}, y_{i+1})$. And the changes of a local shape parameter $u^y_{i,j}$ will affect the shape of two neighboring patches $F(x, y)$ defined in the domain $(x, y) \in (x_{i-1}, x_{i+1}) \times (y_i, y_{i+1})$. And the local free parameters $\tau^x_{i,j}$, $v^x_{i,j}$ and $\tau^y_{i,j}$, $v^y_{i,j}$ have the same effect region on the shape of the generated interpolation surface $F(x, y)$ as that of the local parameters $u^x_{i,j}$ and $u^y_{i,j}$, respectively. Since the four boundary functions $F(x, y_j)$, $F(x, y_{j+1})$, $F(x_i, y)$ and $F(x_{i+1}, y)$ are all C^1 continuous, we can easily conclude that the given bi-cubic partially blended rational cubic interpolation surface $F(x, y)$ is global C^1 continuous over the rectangular domain $[x_1, x_n] \times [y_1, y_m]$.

In most applications, the first partial derivatives $D^x_{i,j}$ and $D^y_{i,j}$ are not given and hence must be determined either from given data or by some other means. In this paper, we use the following arithmetic mean method to compute the first partial derivatives

$$\begin{cases} D^x_{1,j} = \Delta^x_{1,j} + \left(\Delta^x_{1,j} - \Delta^x_{2,j}\right) \frac{h^x_1}{h^x_1 + h^x_2}, \\ D^x_{n,j} = \Delta^x_{n-1,j} + \left(\Delta^x_{n-1,j} - \Delta^x_{n-2,j}\right) \frac{h^x_{n-1}}{h^x_{n-2} + h^x_{n-1}}, \\ D^x_{i,j} = \frac{\Delta^x_{i-1,j} + \Delta^x_{i,j}}{2}, \quad i = 2,3,\ldots,n-1; j = 1,2,\ldots,m, \\ D^y_{i,1} = \Delta^y_{i,1} + \left(\Delta^y_{i,1} - \Delta^y_{i,2}\right) \frac{h^y_1}{h^y_1 + h^y_2}, \\ D^y_{i,m} = \Delta^y_{i,m-1} + \left(\Delta^y_{i,m-1} - \Delta^y_{i,m-2}\right) \frac{h^y_{m-1}}{h^y_{m-2} + h^y_{m-1}}, \\ D^y_{i,j} = \frac{\Delta^y_{i,j-1} + \Delta^y_{i,j}}{2}, \quad i = 1,2,\ldots,n; j = 2,3,\ldots,n-1, \end{cases} \tag{11}$$

where $\Delta^x_{i,j} = (S_{i+1,j} - S_{i,j})/h^x_i$ and $\Delta^y_{i,j} = (S_{i,j+1} - S_{i,j})/h^y_j$. This arithmetic mean method is computationally economical and suitable for visualization of shaped data [17].

B. Positivity-preserving conditions

In this subsection, we want to develop simply schemes so that the C^1 interpolation surface $F(x,y)$ given in (10) can preserve the shape of 3D positive data on rectangular grides.

Let $\{(x_i, y_i, F_{i,j})\}$ be a positive data set defined over the rectangular grid $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 1,2,\cdots,n-1$, $j = 1,2,\cdots,m-1$ such that $F_{i,j} > 0, \forall i,j$. The interpolation surface $F(x,y)$ given in (10) preserves the shape of positive data if

$$F(x,y) > 0, \quad \forall (x,y) \in [x_1, x_n] \times [y_1, y_m].$$

For $(x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we want to rewrite the expression of the interpolation surface $F(x,y)$ given in (10) as the following form

$$\begin{aligned} F(x,y) = {} & b_0(s)\left[F(x,y_j) - \tfrac{1}{2}b_0(t)F_{i,j} - \tfrac{1}{2}b_1(t)F_{i+1,j}\right] \\ & + b_1(s)\left[F(x,y_{j+1}) - \tfrac{1}{2}b_0(t)F_{i,j+1} - \tfrac{1}{2}b_1(t)F_{i+1,j+1}\right] \\ & + b_0(t)\left[F(x_i,y) - \tfrac{1}{2}b_0(s)F_{i,j} - \tfrac{1}{2}b_1(s)F_{i,j+1}\right] \\ & + b_1(t)\left[F(x_{i+1},y) - \tfrac{1}{2}b_0(s)F_{i+1,j} - \tfrac{1}{2}b_1(s)F_{i+1,j+1}\right]. \end{aligned} \tag{12}$$

Without loss of generality, for any $(x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, from formula (10), since $b_0(z)$ and $b_1(z)$ are strict positive for any $z \in (0,1)$, we can see that the interpolation surface $F(x,y)$ is positive everywhere in the domain $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ if the following constrains hold

$$\begin{cases} F(x,y_j) - \frac{1}{2}b_0(t)F_{i,j} - \frac{1}{2}b_1(t)F_{i+1,j} > 0, \\ F(x,y_{j+1}) - \frac{1}{2}b_0(t)F_{i,j+1} - \frac{1}{2}b_1(t)F_{i+1,j+1} > 0, \\ F(x_i,y) - \frac{1}{2}b_0(s)F_{i,j} - \frac{1}{2}b_1(s)F_{i,j+1} > 0, \\ F(x_{i+1},y) - \frac{1}{2}b_0(s)F_{i+1,j} - \frac{1}{2}b_1(s)F_{i+1,j+1} > 0. \end{cases} \tag{13}$$

For $F(x,y_j) - \frac{1}{2}b_0(t)F_{i,j} - \frac{1}{2}b_1(t)F_{i+1,j}$, from (8), we have

$$F(x,y_j) - \tfrac{1}{2}b_0(t)F_{i,j} - \tfrac{1}{2}b_1(t)F_{i+1,j} = \frac{\sum_{k=0}^{5} C^x_{ijk}(1-t)^{5-k}t^k}{u^x_{i,j}(1-t)^2 + \tau^x_{i,j}(1-t)t + v^x_{i,j}t^2},$$

where

$$C^x_{ij0} = \frac{1}{2}u^x_{i,j}F_{i,j},$$

$$C^x_{ij1} = \frac{1}{2}\left[\tau^x_{i,j}F_{i,j} + u^x_{i,j}\left(3F_{i,j} + 2D^x_{i,j}h^x_i\right)\right],$$

$$\begin{aligned} C^x_{ij2} = {} & \tfrac{1}{2}\left[\tau^x_{i,j}F_{i,j} + 2u^x_{i,j}\left(3F_{i,j} + 2D^x_{i,j}h^x_i\right) - v^x_{i,j}F_{i,j}\right] \\ & + \left[\tau^x_{i,j}F_{i+1,j} + v^x_{i,j}\left(F_{i+1,j} - D^x_{i+1,j}h^x_i\right) - \tfrac{3}{2}u^x_{i,j}F_{i+1,j}\right], \end{aligned}$$

$$\begin{aligned} C^x_{ij3} = {} & \left[\tau^x_{i,j}F_{i,j} + u^x_{i,j}\left(F_{i,j} + D^x_{i,j}h^x_i\right) - \tfrac{3}{2}v^x_{i,j}F_{i,j}\right] \\ & + \tfrac{1}{2}\left[\tau^x_{i,j}F_{i+1,j} + 2v^x_{i,j}\left(3F_{i+1,j} - 2D^x_{i+1,j}h^x_i\right) - u^x_{i,j}F_{i+1,j}\right], \end{aligned}$$

$$C^x_{ij4} = \frac{1}{2}\left[\tau^x_{i,j}F_{i+1,j} + v^x_{i,j}\left(3F_{i+1,j} - 2D^x_{i+1,j}h^x_i\right)\right],$$

$$C^x_{ij5} = \frac{1}{2}v^x_{i,j}F_{i+1,j}.$$

Thus we can see that the conditions $u^x_{i,j} > 0, v^x_{i,j} > 0$ together with $C^x_{ijk} \geq 0, k = 1,2,3,4$ are sufficient to ensure $F(x,y_j) - \frac{1}{2}b_0(t)F_{i,j} - \frac{1}{2}b_1(t)F_{i+1,j} > 0$.

For $C^x_{ijk} \geq 0, k = 1,2,3,4$, we have the following sufficient conditions

$$\begin{cases} \tau^x_{i,j}F_{i,j} + u^x_{i,j}\left(3F_{i,j} + 2D^x_{i,j}h^x_i\right) \geq 0, \\ \tau^x_{i,j}F_{i,j} + 2u^x_{i,j}\left(3F_{i,j} + 2D^x_{i,j}h^x_i\right) - v^x_{i,j}F_{i,j} \geq 0, \\ \tau^x_{i,j}F_{i,j} + u^x_{i,j}\left(F_{i,j} + D^x_{i,j}h^x_i\right) - \frac{3}{2}v^x_{i,j}F_{i,j} \geq 0, \\ \tau^x_{i,j}F_{i+1,j} + v^x_{i,j}\left(F_{i+1,j} - D^x_{i+1,j}h^x_i\right) - \frac{3}{2}u^x_{i,j}F_{i+1,j} \geq 0, \\ \tau^x_{i,j}F_{i+1,j} + 2v^x_{i,j}\left(3F_{i+1,j} - 2D^x_{i+1,j}h^x_i\right) - u^x_{i,j}F_{i+1,j} \geq 0, \\ \tau^x_{i,j}F_{i+1,j} + v^x_{i,j}\left(3F_{i+1,j} - 2D^x_{i+1,j}h^x_i\right) \geq 0. \end{cases} \tag{14}$$

Let us consider a tough case $D^x_{i,j} < 0$ and $D^x_{i+1,j} > 0$. For this case, we have

$$\begin{aligned} & \min\left\{\tau^x_{i,j}F_{i,j} + u^x_{i,j}\left(3F_{i,j} + 2D^x_{i,j}h^x_i\right),\right. \\ & \tau^x_{i,j}F_{i,j} + 2u^x_{i,j}\left(3F_{i,j} + 2D^x_{i,j}h^x_i\right) - v^x_{i,j}F_{i,j}, \\ & \left.\tau^x_{i,j}F_{i,j} + u^x_{i,j}\left(F_{i,j} + D^x_{i,j}h^x_i\right) - \tfrac{3}{2}v^x_{i,j}F_{i,j}\right\} \\ & \geq \tau^x_{i,j}F_{i,j} + u^x_{i,j}\left(F_{i,j} + 2D^x_{i,j}h^x_i\right) - \tfrac{3}{2}v^x_{i,j}F_{i,j}, \\ & \min\left\{\tau^x_{i,j}F_{i+1,j} + v^x_{i,j}\left(F_{i+1,j} - D^x_{i+1,j}h^x_i\right) - \tfrac{3}{2}u^x_{i,j}F_{i+1,j},\right. \\ & \tau^x_{i,j}F_{i+1,j} + 2v^x_{i,j}\left(3F_{i+1,j} - 2D^x_{i+1,j}h^x_i\right) - u^x_{i,j}F_{i+1,j}, \\ & \left.\tau^x_{i,j}F_{i+1,j} + v^x_{i,j}\left(3F_{i+1,j} - 2D^x_{i+1,j}h^x_i\right)\right\} \\ & \geq \tau^x_{i,j}F_{i+1,j} + v^x_{i,j}\left(F_{i+1,j} - 2D^x_{i+1,j}h^x_i\right) - \tfrac{3}{2}u^x_{i,j}F_{i+1,j}. \end{aligned}$$

Therefore, we can see that the following conditions are sufficient to ensure the constrains (14) hold

$$\begin{cases} \tau^x_{i,j}F_{i,j} + u^x_{i,j}\left(F_{i,j} + 2D^x_{i,j}h^x_i\right) - \frac{3}{2}v^x_{i,j}F_{i,j} \geq 0, \\ \tau^x_{i,j}F_{i+1,j} + v^x_{i,j}\left(F_{i+1,j} - 2D^x_{i+1,j}h^x_i\right) - \frac{3}{2}u^x_{i,j}F_{i+1,j} \geq 0. \end{cases} \tag{15}$$

If $D^x_{i,j} \geq 0$ and $D^x_{i+1,j} \leq 0$, then it can be easy to check that the following conditions are sufficient to ensure the constrains (14) hold.

$$\begin{cases} \tau^x_{i,j}F_{i,j} - \frac{3}{2}v^x_{i,j}F_{i,j} \geq 0, \\ \tau^x_{i,j}F_{i+1,j} - \frac{3}{2}u^x_{i,j}F_{i+1,j} \geq 0. \end{cases} \tag{16}$$

From the above analysis, we can obtain the following sufficient conditions to ensure $F(x,y_j) - \frac{1}{2}b_0(t)F_{i,j} - \frac{1}{2}b_1(t)F_{i+1,j} > 0$

$$\begin{cases} u^x_{i,j} > 0, \quad v^x_{i,j} > 0, \\ \tau^x_{i,j} = \max\left\{-\frac{u^x_{i,j}\left(F_{i,j} + 2D^x_{i,j}h^x_i\right)}{F_{i,j}} + \frac{3}{2}v^x_{i,j},\right. \\ \qquad -\frac{v^x_{i,j}\left(F_{i+1,j} - 2D^x_{i+1,j}h^x_i\right)}{F_{i+1,j}} + \frac{3}{2}u^x_{i,j}, \\ \qquad \left.\frac{3}{2}u^x_{i,j}, \frac{3}{2}v^x_{i,j}\right\} + w^x_{i,j}, \end{cases} \tag{17}$$

where $w^x_{i,j} \geq 0$.

In the same fashion, we can also derive similar sufficient conditions for $F(x,y_{j+1}) - \frac{1}{2}b_0(t)F_{i,j+1} - \frac{1}{2}b_1(t)F_{i+1,j+1} > 0$, $F(x_i,y) - \frac{1}{2}b_0(s)F_{i,j} - \frac{1}{2}b_1(s)F_{i,j+1} > 0$ and $F(x_{i+1},y) - \frac{1}{2}b_0(s)F_{i+1,j} - \frac{1}{2}b_1(s)F_{i+1,j+1} > 0$. In conclusion, for a positive data set, we can obtain the following sufficient conditions for $F(x,y) > 0, \forall (x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$

$$\begin{cases} u^x_{i,j} > 0, \quad v^x_{i,j} > 0, \\ u^y_{i,j} > 0, \quad v^y_{i,j} > 0, \\ \tau^x_{i,j} = \max\Big\{ -\frac{u^x_{i,j}\left(F_{i,j}+2D^x_{i,j}h^x_i\right)}{F_{i,j}} + \frac{3}{2}v^x_{i,j}, \\ \quad -\frac{v^x_{i,j}\left(F_{i+1,j}-2D^x_{i+1,j}h^x_i\right)}{F_{i+1,j}} + \frac{3}{2}u^x_{i,j}, \\ \quad \frac{3}{2}u^x_{i,j}, \frac{3}{2}v^x_{i,j}\Big\} + w^x_{i,j}, \\ \tau^y_{i,j} = \max\Big\{ -\frac{u^y_{i,j}\left(F_{i,j}+2D^y_{i,j}h^y_j\right)}{F_{i,j}} + \frac{3}{2}v^y_{i,j}, \\ \quad -\frac{v^y_{i,j}\left(F_{i,j+1}-2D^y_{i,j+1}h^y_j\right)}{F_{i,j+1}} + \frac{3}{2}u^y_{i,j}, \\ \quad \frac{3}{2}v^y_{i,j}, \frac{3}{2}u^y_{i,j}\Big\} + w^y_{i,j}. \end{cases} \tag{18}$$

where $i = 1, 2, \cdots, n-1$, $j = 1, 2, \cdots, m-1$ and $w^x_{i,j}$,$w^y_{i,j}$ are arbitrary nonnegative real numbers and serve as free parameters.

It is worth to mention that in [17], [18], [19], for $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, the positivity of the interpolation surfaces $F(x, y)$ is based on the positivity of the four boundary functions $F(x, y_j), F(x, y_{j+1}), F(x_i, y)$ and $F(x_{i+1}, y)$. However, there lack theory proving that the constrain conditions $F(x, y_j) > 0, F(x, y_{j+1}) > 0, F(x_i, y) > 0$ and $F(x_{i+1}, y) > 0$ are sufficient to ensure $F(x, y) > 0$ for any $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ generally. The new constrain conditions given in (13) are different to those used in [17], [18], [19] and they can ensure the positivity of the resulting interpolation surface everywhere in the domain.

We shall give several numerical examples to show that the conditions developed in Theorem 6.1 of [17] are not sufficient for constructing positivity-preserving interpolation surfaces. As comparisons, we will also show that the new proposed C^1 interpolation surface $F(x, y)$ given in (10) can be used to nicely visualize the shape of 3D positive data on rectangular grids. In the following figures, the given data points have been marked with solid black dots.

Fig. 3 shows the interpolation surfaces generated by using method [17] for the 3D positive data set given in Tab. III. Fig. 3(A) shows the interpolation surface generated by using the sufficient conditions given in Theorem 6.1 of [17] with all the free parameters $a_{ij} = b_{ij} = c_{ij} = l_{ij} = m_{ij} = n_{ij} = o_{ij} = p_{ij} = 0.8$. Fig. 3(D) shows the interpolation surface generated by changing all the free shape parameters from 0.8 to 8. From the results, we can see that both of the two interpolation surfaces fail to preserve the shape of the 3D positive data set given in Tab. III.

As a comparison, for the 3D positive data set given in Tab. III, fig. 4 shows the positivity-preserving interpolation surfaces $F_1(x, y)$ and $F_2(x, y)$ generated by using our method. Fig. 4(A) shows the interpolation surface $F_1(x, y)$ generated by using the sufficient conditions given in (18) with all the free parameters $u^x_{i,j} = v^x_{i,j} = u^x_{i,j} = v^y_{i,j} = 1$ and $w^x_{i,j} = w^y_{i,j} = 0$. Fig. 4(D) shows the interpolation surface $F_2(x, y)$ generated by changing all the free shape parameters $w^x_{i,j}, w^x_{i,j}$ from 0 to 1. From the results, we can see that both of the two visually pleasing interpolation surfaces preserve the shape of the 3D positive data set given in Tab. III genially.

Fig. 5 shows the interpolation surfaces generated by using method [17] for the 3D positive data set given in Tab. IV. Fig. 5(A) shows the interpolation surface generated by using the sufficient conditions given in Theorem 6.1 of [17] with all the free parameters $a_{ij} = b_{ij} = c_{ij} = l_{ij} = m_{ij} = n_{ij} = o_{ij} = p_{ij} = 2$. And fig. 5(D) shows the interpolation surface generated by changing all the free shape parameters from 2 to 4. It can be observed from the fig. 5 that both of the two interpolation surfaces loose the shape of the positive data set given in Tab. IV.

For the same 3D positive data set given in Tab. IV, fig. 6 shows the positivity-preserving interpolation surfaces $F_3(x, y)$ and $F_4(x, y)$ generated by using our method. Fig. 6(A) shows the interpolation surface $F_3(x, y)$ generated by using the sufficient conditions given in (18) with all the free parameters $u^x_{i,j} = v^x_{i,j} = u^x_{i,j} = v^y_{i,j} = 1$ and $w^x_{i,j} = w^y_{i,j} = 0$. Fig. 6(D) shows the interpolation surface $F_3(x, y)$ generated by changing all the free parameters $w^x_{i,j}, w^x_{i,j}$ from 0 to 10. As can be seen from Fig. 6, both of the two interpolation surfaces visualize the shape of the 3D positive data set given in Tab. IV well.

Fig. 5 shows the interpolation surfaces generated by using method [17] for the 3D positive data set given in Tab. V. Fig. 7(A) shows the interpolation surface generated by using the sufficient conditions given in Theorem 6.1 of [17] with all the free parameters $a_{ij} = b_{ij} = c_{ij} = l_{ij} = m_{ij} = n_{ij} = o_{ij} = p_{ij} = 5$. And fig. 7(D) shows the interpolation surface generated by changing all the free shape parameters from 2 to 10. From the fig. 7, it is obvious that both the two interpolation surfaces fail to preserve the shape of the positive data set given in Tab. V.

As a comparison, for the 3D monotonic data set given in Tab. V, fig. 8 shows the positivity-preserving interpolation surfaces $F_5(x, y)$ and $F_6(x, y)$ generated by using our method. Fig. 8(A) shows the interpolation surface $F_5(x, y)$ generated by using the sufficient conditions given in (18) with all the free parameters $u^x_{i,j} = v^x_{i,j} = u^x_{i,j} = v^y_{i,j} = 4$ and $w^x_{i,j} = w^y_{i,j} = 0$. Fig. 6(D) shows the interpolation surface $F_6(x, y)$ generated by changing the free control parameters $w^x_{i,j}$ and $w^x_{i,j}$, $i = 3, 4, j = 3, 4$ from 0 to 5. As can be seen from the Fig. 8, both of the two interpolation surfaces visualize the positive shape of the data given in Tab. V nicely and the shape of the interpolation surfaces can be locally adjusted by using the local free parameters.

TABLE III
THE 3D POSITIVE DATA SET GIVEN IN [17].

y/x	-3	-2	-1	0	1	2	3
-3	1	26	65	82	65	26	1
-2	26	1	10	17	10	1	26
-1	65	10	1	2	1	10	65
0	82	17	2	1	2	17	82
1	65	10	1	2	1	10	65
2	26	1	10	17	10	1	26
3	1	26	65	82	65	26	1

TABLE V
THE 3D POSITIVE DATA SET GIVEN IN [15].

y/x	-3	-2	-1	1	2	3
-3	0.0124	0.0238	0.0404	0.0404	0.0238	0.0124
-2	0.0238	0.0635	0.1667	0.1667	0.0635	0.0238
-1	0.0404	0.1667	1.3333	1.3333	0.1667	0.0404
1	0.0404	0.1667	1.333	1.3333	0.1667	0.0404
2	0.0238	0.0635	0.1667	0.1667	0.0635	0.0238
3	0.0124	0.0238	0.0404	0.0404	0.0238	0.0124

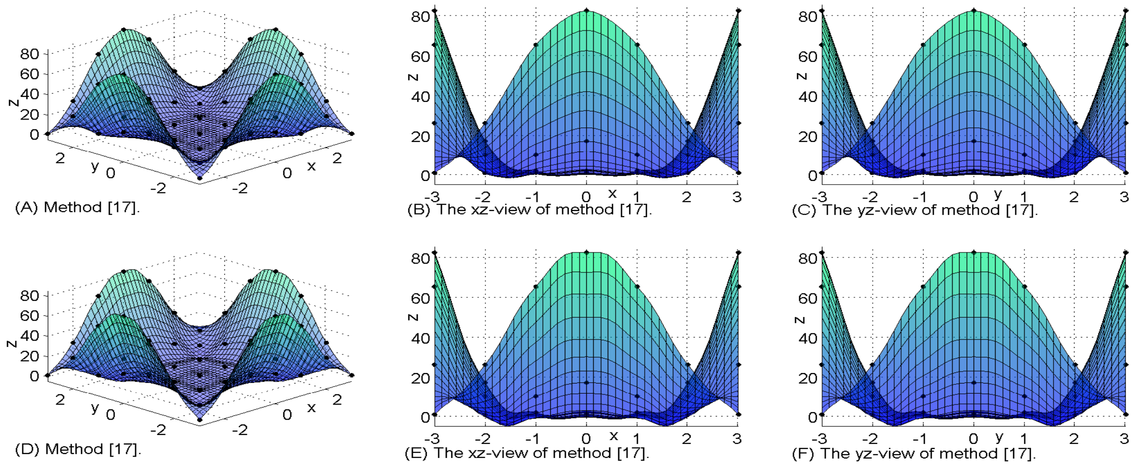

Fig. 3. Interpolation surfaces generated by using the method [17] for the 3D positive data set given in Tab. III.

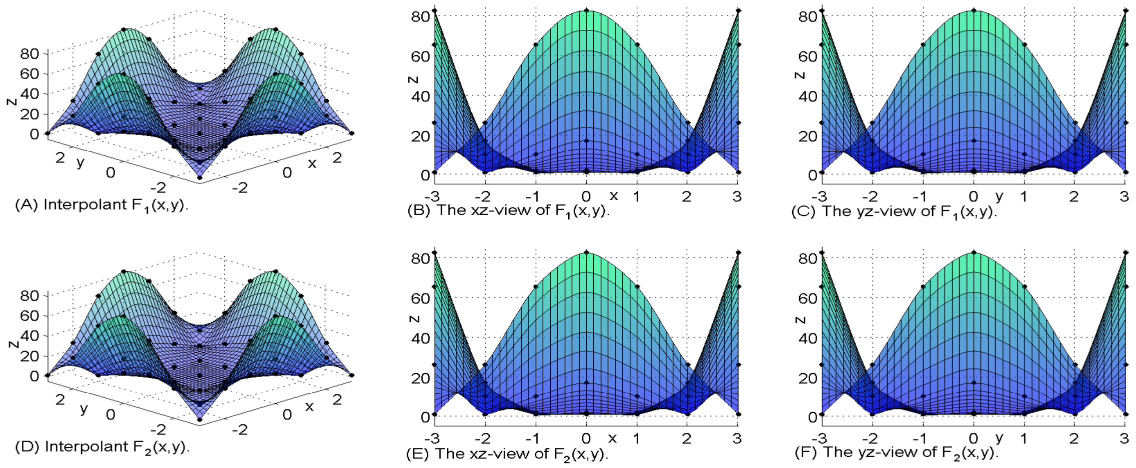

Fig. 4. Positivity-preserving interpolation surfaces for the 3D positive data set given in Tab. III.

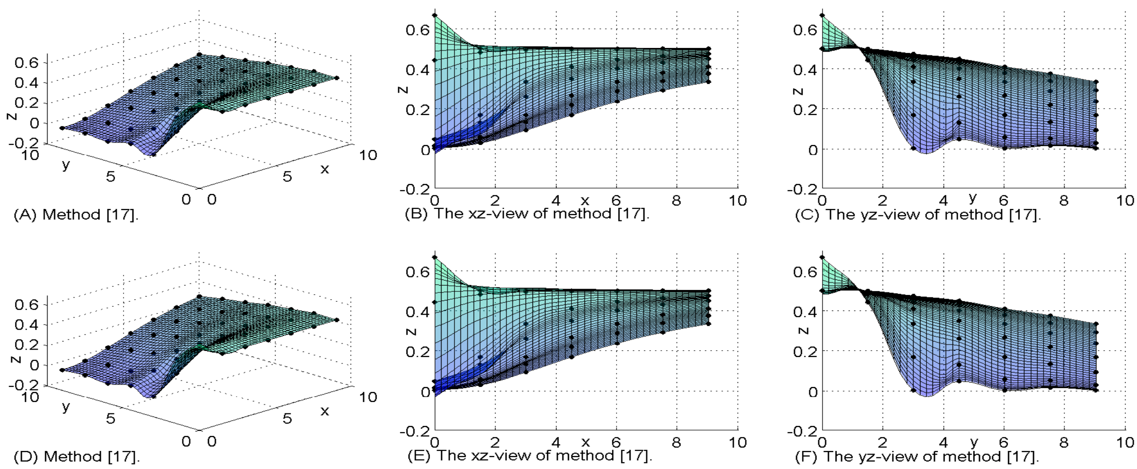

Fig. 5. Interpolation surfaces generated by using the method [17] for the 3D positive data set given in Tab. IV.

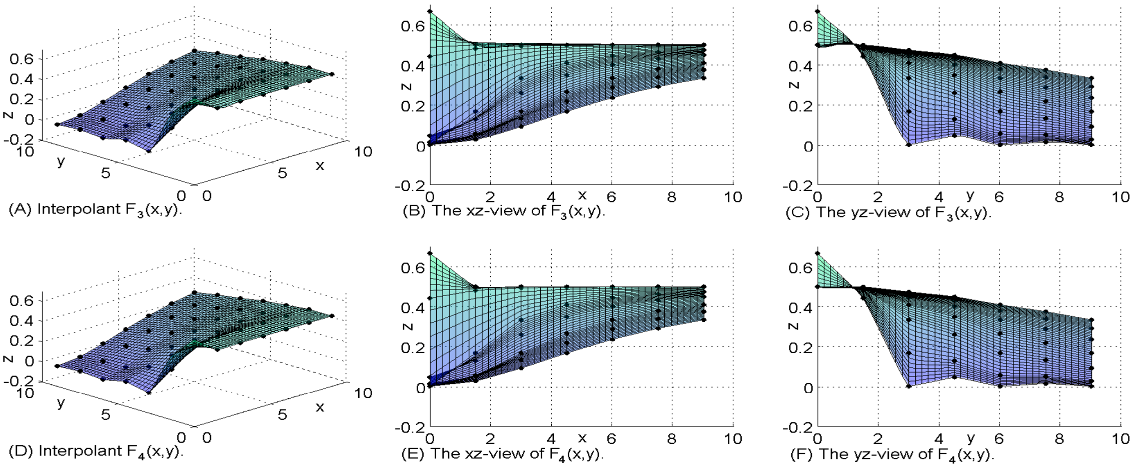

Fig. 6. Positivity-preserving interpolation surfaces for the 3D positive data set given in Tab. IV.

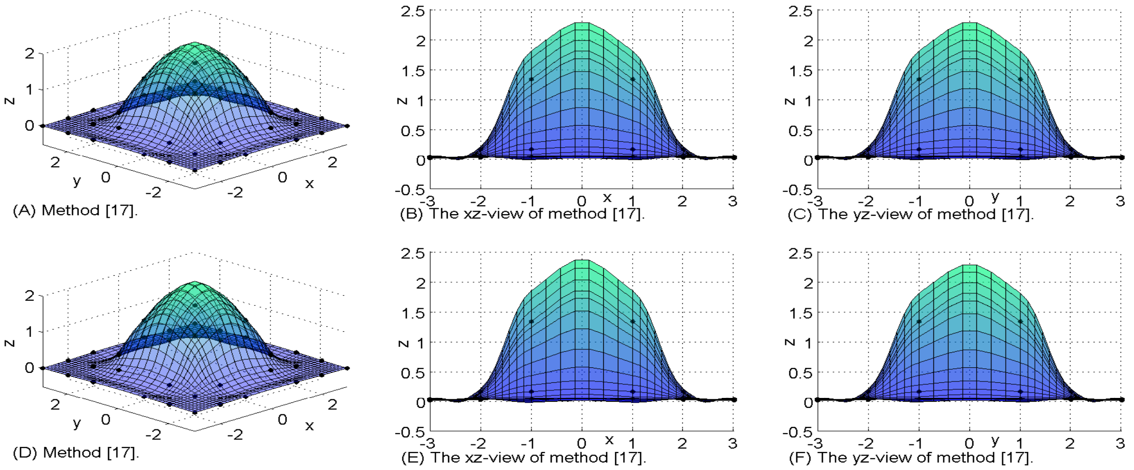

Fig. 7. Interpolation surfaces generated by using the method [17] for the 3D positive data set given in Tab. V.

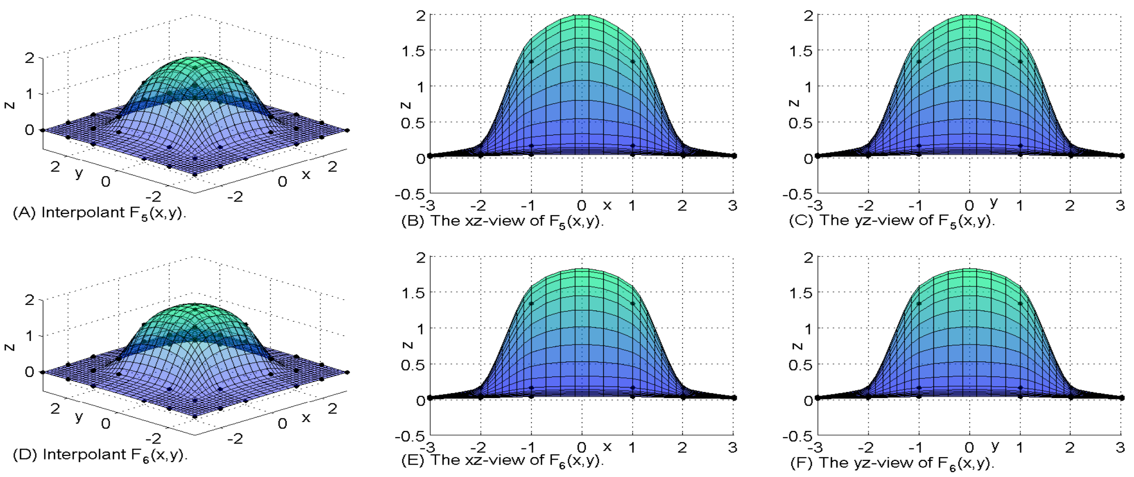

Fig. 8. Positivity-preserving interpolation surfaces for the 3D positive data set given in Tab. V.

TABLE IV
THE 3D POSITIVE DATA SET GIVEN IN [17].

y/x	0.0001	1.5	3	4.5	6	7.5	9
0.0001	0.6667	0.5	0.5	0.5	0.5	0.5	0.5
1.5	0.4422	0.4807	0.4936	0.4970	0.4982	0.4989	0.4992
3	0.0022	0.1681	0.3341	0.4095	0.4447	0.4631	0.4738
4.5	0.0472	0.1295	0.2603	0.3491	0.4006	0.4309	0.4497
6	0.0022	0.0575	0.1681	0.2657	0.3341	0.3793	0.4095
7.5	0.0156	0.0515	0.1331	0.2184	0.2876	0.3385	0.3752
9	0.0021	0.0283	0.0926	0.1681	0.2364	0.2916	0.3340

IV. CONCLUSION

As stated above, the conditions given in Theorems 4.1 and 6.1 of [17] were not sufficient to generate positivity-preserving interpolation curves and surfaces and we have successfully overcome there shortcomings by introducing a new local tension parameter τ_i into the cubic/quadratic interpolation spline given in [17]. The constructed bi-cubic partially blended rational cubic/quadratic interpolation surfaces with six families of local free parameters can be C^1 continuous without making use of the first mixed partial derivatives at the data points. For 3D positive data on rectangular grids, by developing new constrain conditions on the boundary curves of each local interpolation surface patch, differing from those used in [17], [18], [19], we theoretically deduce simple sufficient data dependent conditions on the local free parameters to generate positivity-preserving interpolation surfaces everywhere in the domains. The given method also allows extensions to generate C^1 positivity-preserving interpolation surfaces for 3D non-gridded data. There are still some problems worthy of further study, such as the construction of convexity-preserving interpolation surfaces with local free parameters. These will be our future work.

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