The Hamiltonicity and Hamiltonian Connectivity of Some Shaped Supergrid Graphs*

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Abstract—A Hamiltonian path (cycle) of a graph is a simple path (cycle) in which each vertex of the graph is visited exactly once. The Hamiltonian path (cycle) problem is to determine whether or not a graph contains a Hamiltonian path (cycle). A graph is called Hamiltonian if it contains a Hamiltonian cycle, and it is said to be Hamiltonian connected if there exists a Hamiltonian path between any two distinct vertices. Supergrid graphs were first introduced by us and include grid graphs and triangular grid graphs as their subgraphs. These problems on supergrid graphs can be applied to compute the stitching traces of computerized sewing machines. In the past, we have proved the Hamiltonian path (cycle) problem on supergrid graphs to be NP-complete. Recently, we showed that rectangular supergrid graphs are Hamiltonian except one trivial forbidden condition. In this paper, we will verify the Hamiltonicity and Hamiltonian connectivity of some shaped supergrid graphs, including triangular, parallelogram, and trapezoid. The results can be used to solve the Hamiltonian problems on some special classes of supergrid graphs in the future.

Index Terms—Hamiltonicity, Hamiltonian connectivity, supergrid graphs, triangular supergrid graphs, parallelogram supergrid graphs, trapezoid supergrid graphs, computer sewing machines.

I. INTRODUCTION

Hamiltonian path of a graph is a simple path in which each vertex of the graph appears exactly once. A Hamiltonian cycle in a graph is a simple cycle with the same property. The Hamiltonian path (resp., cycle) problem involves deciding whether or not a graph contains a Hamiltonian path (resp., cycle). A graph is called to be Hamiltonian if it contains a Hamiltonian cycle. A graph $G$ is said to be Hamiltonian connected if for each pair of distinct vertices $u$ and $v$ of $G$, there exists a Hamiltonian path between $u$ and $v$ in $G$. If $(u, v)$ is an edge of a Hamiltonian connected graph, then a Hamiltonian cycle containing $(u, v)$ does exist. Thus, a Hamiltonian connected graph contains many Hamiltonian cycles, and, hence, the sufficient conditions of Hamiltonian connectivity are stronger than those of Hamiltonicity. It is well known that the Hamiltonian path and cycle problems are NP-complete for general graphs [10], [24]. The same holds true for bipartite graphs [25], split graphs [11], circle graphs [8], undirected path graphs [2], grid graphs [23], triangular grid graphs [12], and supergrid graphs [15]. In the literature, there are many studies for the Hamiltonian connectivity of interconnection networks. Li et al. [26] proved the Hamiltonian connectivity of the recursive dual-net. The hypercomplete network [6] and the arrangement graph [29] were known to be Hamiltonian connected. The popular hypercubes are Hamiltonian but are not Hamiltonian connected. However, many variants of hypercubes, including augmented hypercubes [14], generalized base-b hypercube [20], twisted cubes [22], crossed cubes [21], Möbius cubes [7], and enhanced hypercubes [28], have been shown to be Hamiltonian connected. For more related works and applications, we refer readers to [1], [4], [5], [9], [13], [17], [18], [27], [30], [31], [32], [33], [34].

The two-dimensional integer grid $G^\infty$ is an infinite graph whose vertex set consists of all points of the Euclidean plane with integer coordinates and in which two vertices are adjacent if the (Euclidean) distance between them is equal to 1. The two-dimensional triangular grid $T^\infty$ is an infinite graph obtained from $G^\infty$ by adding all edges on the lines traced from up-left to down-right. A grid graph is a finite, vertex-induced subgraph of $G^\infty$. For a node $v$ in the plane with integer coordinates, let $v_x$ and $v_y$ be the $x$ and $y$ coordinates of node $v$, respectively, denoted by $v = (v_x, v_y)$. If $v$ is a vertex in a grid graph, then its possible neighbor vertices include $(v_x, v_y + 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, and $(v_x, v_y - 1)$. For example, Fig. I(a) shows a grid graph. A triangular grid graph is a finite, vertex-induced subgraph of $T^\infty$. If $v$ is a vertex in a triangular grid graph, then its possible neighbor vertices include $(v_x, v_y + 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y - 1)$, $(v_x - 1, v_y - 1)$, and $(v_x + 1, v_y - 1)$. For instance, Fig. I(b) depicts a triangular grid graph. Thus, triangular grid graphs contain grid graphs as subgraphs. Note that triangular grid graphs defined above are isomorphic to the original triangular grid graphs studied in the literature [12] but these graphs are different when considered as geometric graphs. By the same construction of triangular grid graphs from grid graphs, we have proposed a new class of graphs, namely supergrid graphs, in [15]. The two-dimensional supergrid $S^\infty$ is an infinite graph obtained from $T^\infty$ by adding all edges on the lines traced from up-right to down-left. A supergrid graph is a finite, vertex-induced subgraph of $S^\infty$. The possible adjacent vertices of a vertex $v = (v_x, v_y)$ in a supergrid graph include $(v_x, v_y + 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y - 1)$, $(v_x - 1, v_y - 1)$, $(v_x + 1, v_y - 1)$, and $(v_x - 1, v_y + 1)$. Then, supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. For example, Fig. I(c) shows a supergrid graph. Notice that grid and triangular grid graphs are not subclasses of supergrid graphs, and the converse is also true: these classes of graphs have common elements (points) but in general they are distinct since the edge sets of these graphs are different. Obviously, all grid graphs are

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bipartite [23] but triangular grid graphs and supergrid graphs are not bipartite.

The Hamiltonian problems on supergrid graphs can be applied to control the stitching trace of a computerized sewing machine as stated in [15]. We also proved that the Hamiltonian cycle and path problems are NP-complete for supergrid graphs [15]. Thus, an important line of investigation is to discover the complexities of the Hamiltonian related problems when the input is restricted to be in special subclasses of supergrid graphs. In [17], we showed that the Hamiltonian cycle problem for linear-convex supergrid graphs is linear solvable. Recently, we proved that rectangular supergrid graphs are always Hamiltonian connected except one trivial forbidden condition [18]. In this paper, we will show that some shaped supergrid graphs, including triangular, parallelogram, and trapezoid, are always Hamiltonian and Hamiltonian connected except few trivial forbidden conditions. The results can be applied to the Hamiltonian problems on some special subclasses of supergrid graphs, such as solid and alphabet supergrid graphs.

The rest of the paper is organized as follows. Section II gives some notations and background results. In Section III, we propose constructive proofs to show that triangular and parallelogram supergrid graphs are Hamiltonian and Hamiltonian connected except two or three trivial conditions. Section IV verifies the Hamiltonicity and Hamiltonian connectivity of trapezoid supergrid graphs by using the Hamiltonicity and Hamiltonian connectivity of rectangular, triangular, and parallelogram supergrid graphs. Finally, we make some concluding remarks in Section V.

II. NOTATIONS AND BACKGROUND RESULTS

In this section, we will introduce some notations. Some observations and previously established results for the Hamiltonian problems on rectangular supergrid graphs are also presented. For graph-theoretic terminology not defined in this paper, the reader is referred to [3].

A. Notations

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $S$ be a subset of vertices in $G$, and let $u$ and $v$ be two distinct vertices in $G$. We write $G[S]$ for the subgraph of $G$ induced by $S$, $G−S$ for the subgraph $G[V−S]$, i.e., the subgraph induced by $V−S$. In general, we write $G−v$ instead of $G−\{v\}$. If $(u, v)$ is an edge in $G$, we say that $v$ is adjacent to $u$. A neighbor of $v$ in $G$ is any vertex that is adjacent to $v$. We use $N_G(v)$ to denote the set of neighbors of $v$ in $G$. The subscript $G$ of $N_G(v)$ can be removed from the notation if it has no ambiguity. The degree of vertex $v$, denoted by $deg(v)$, is the number of vertices adjacent to vertex $v$. The notation $u \sim v$ (resp., $u \not\sim v$) means that vertices $u$ and $v$ are adjacent (resp., non-adjacent). Two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are said to be incident if $u_1 \sim v_1$ and $u_2 \sim v_2$ denote this by $e_1 \sim e_2$. A path $P$ of length $|P|$ in $G$, denoted by $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{|P|−1} \rightarrow v_{|P|}$, is a sequence $(v_1, v_2, \ldots, v_{|P|−1}, v_{|P|})$ of vertices such that $(v_i, v_{i+1}) \in E$ for $1 \leq i < |P|$. The first and last vertices visited by $P$ are denoted by $start(P)$ and $end(P)$, respectively. We will use $v_i \in P$ to denote “$P$ visits vertex $v_i$” and use $(v_i, v_{i+1}) \in P$ to denote “$P$ visits edge $(v_i, v_{i+1})$”. A path from $v_1$ to $v_k$ is denoted by $(v_1, v_k)$-path. In addition, we use $P$ to refer to the set of vertices visited by path $P$ if it is understood without ambiguity. On the other hand, a path is called the reversed path, denoted by rev($P$), of path $P$ if it visits the vertices of $P$ from end($P$) to start($P$) in proper sequence; that is, the reversed path rev($P$) of path $P = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{|P|−1} \rightarrow v_{|P|}$ is $v_{|P|−1} \rightarrow v_{|P|−2} \rightarrow \cdots \rightarrow v_2 \rightarrow v_1$. A path $P$ is a cycle if $|V(P)| \geq 3$ and end($P$) $\sim$ start($P$). Two paths (or cycles) $P_1$ and $P_2$ of graph $G$ are called vertex-disjoint if $V(P_1) \cap V(P_2) = \emptyset$. Two vertex-disjoint paths $P_1$ and $P_2$ can be concatenated into a path, denoted by $P_1 \rightarrow P_2$, if end($P_1$) $\sim$ start($P_2$).

Let $S^{\infty}$ be the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if the difference of their $x$ or $y$ coordinates is not larger than 1. A supergrid graph is a finite, vertex-induced subgraph of $S^{\infty}$. For a vertex $v$ in a supergrid graph, let $v_x$ and $v_y$ be respectively $x$ and $y$ coordinates of $v$. We color vertex $v$ to be white if $v_x + v_y \equiv 0$ (mod 2); otherwise, $v$ is colored to be black. Then there are eight possible neighbors of vertex $v$ including four white vertices and four black vertices. Obviously, all grid graphs are bipartite [23] but supergrid graphs are not bipartite. The edge $(u, v)$ in $S^{\infty}$ is said to be horizontal (resp., vertical) if $u_y = v_y$ and $u_x \neq v_x$ (resp., $u_x = v_x$ and $u_y \neq v_y$), and is called skewed if it is neither a horizontal nor a vertical edge. In the figures, we assume that $(1, 1)$ is coordinates of the up-left vertex, i.e., the leftmost vertex of the first row, in a supergrid graph.

Rectangular supergrid graphs first appeared in [15], in which the Hamiltonian cycle problem was solved. Let $R(m, n)$ be the supergrid graph with vertex set $V(R(m, n)) = \{v = (v_x, v_y) \mid 1 \leq v_x \leq m$ and $1 \leq v_y \leq n\}$. That is, $R(m, n)$ contains $m$ columns and $n$ rows of vertices in $S^{\infty}$. A rectangular supergrid graph is a supergrid graph which is isomorphic to $R(m, n)$. Then $m$ and $n$, the dimensions, specify a rectangular supergrid graph up to isomorphism. The size of $R(m, n)$ is defined to be $mn$, and $R(m, n)$ is called $n$-rectangle. Let $v = (v_x, v_y)$ be a vertex in $R(m, n)$. The vertex $v$ is called the up-left (resp., up-right, down-left, down-right) corner of $R(m, n)$ if for any vertex $w = (w_x, w_y) \in R(m, n)$, $w_x \geq v_x$ and $w_y \geq v_y$ (resp., $w_x \leq v_x$ and $w_y \geq v_y$, $w_x \geq v_x$ and $w_y \leq v_y$, $w_x \leq v_x$ and $w_y \leq v_y$). There are four boundaries (borders) in a rectangular supergrid graph $R(m, n)$ with $m, n \geq 2$. The edge in the boundary of $R(m, n)$ is called boundary edge. For example, Fig. 2(a) shows a rectangular supergrid graph $R(10, 10)$ which is called 10-rectangle and contains $4 \times 9 = 36$ boundary edges. Fig. 2(a) also indicates the types of corners.

The triangular supergrid graphs are subgraphs of rectangular supergrid graphs and are defined as follows.

Definition 1. Let $\ell$ be a diagonal line of $R(n, n)$ with $n \geq 2$ from the up-left corner to the down-right corner. Let $\Delta(n, n)$ be the supergrid graph obtained from $R(n, n)$ by removing all vertices under $\ell$. A triangular supergrid graph is a supergrid graph which is isomorphic to $\Delta(n, n)$.

For example, Fig. 2(b) shows a triangular supergrid graph $\Delta(10, 10)$. Each triangular supergrid graph contains three boundaries, namely horizontal, vertical, and skewed, and
Definition 2. Let $P(m,n)$ be the supergrid graph with $m \geq n$ and vertex set $V(P(m,n)) = \{v = (v_x, v_y) \mid 1 \leq v_y \leq n \text{ and } v_y \leq v_x \leq v_y + m - 1\}$ or $\{v = (v_x, v_y) \mid 1 \leq v_y \leq n \text{ and } -v_y + 2 \leq v_x \leq m - (v_y - 1)\}$. A parallelogram supergrid graph is a supergrid graph which is isomorphic to $P(m,n)$.

In the above definition, there are two types of parallelogram supergrid graphs. We can see that they are isomorphic although they are different when considered as geometric graphs. In this paper, we can only consider the parallelogram supergrid graph $P(m,n)$ with $V(P(m,n)) = \{v = (v_x, v_y) \mid 1 \leq v_y \leq n \text{ and } v_y \leq v_x \leq v_y + m - 1\}$. Each parallelogram supergrid graph contains four boundaries, two horizontal boundaries and two skewed boundaries, and these boundaries form a parallelogram, as depicted in Fig. 2(c). The size of $P(m,n)$ is defined to be $mn$, and $P(m,n)$ is called an $n$-parallelogram. The vertex $w$ of $P(m,n)$ is called a parallel corner if $\deg(w) = 2$. We can see from definition that a parallelogram supergrid graph contains two parallel corners and it can be decomposed into two disjoint triangular supergrid subgraphs. For instance, Fig. 2(c) depicts a parallelogram supergrid graph $P(5,4)$ which can be partitioned into two triangular supergrid graphs $\Delta(4,4)$.

Next, we introduce trapezoid supergrid graphs. Let $R(m,n)$ be a rectangular supergrid graph with $m \geq n \geq 2$. A trapezoid supergrid graph $T_1(m,n)$ or $T_2(m,n)$ is obtained from $R(m,n)$ by removing one or two triangular supergrid graphs $\Delta(n-1,n-1)$. The trapezoid supergrid graphs $T_1(m,n)$ and $T_2(m,n)$ are defined as follows.

Definition 3. Let $R(m,n)$ be a rectangular supergrid graph with $m \geq n \geq 2$. A trapezoid supergrid graph $T_1(m,n)$ or $T_2(m,n)$ is obtained from $R(m,n)$ by removing one or two triangular supergrid graphs $\Delta(n-1,n-1)$ from the corner of $R(m,n)$. A trapezoid supergrid graph $T_2(m,n)$ is constructed from $R(m,n)$ with $m \geq 2n$ by removing two triangular supergrid graphs $\Delta(n-1,n-1)$ from the up-left and up-right corners of $R(m,n)$. Fig. 2(d) shows these two types of trapezoid graphs.

In a trapezoid supergrid graph, a vertex $v$ is called a trapezoid corner if $\deg(v) = 2$. We can see that $T_1(m,n)$ contains a trapezoid corner, $T_2(m,n)$ contains two trapezoid corners, $T_1(m,n)$ contains two horizontal, one vertical and one skewed boundaries, and $T_2(m,n)$ contains two horizontal and two skewed boundaries. By definition, each boundary of $T_1(m,n)$ and $T_2(m,n)$ contains at least two vertices. On the other hand, $T_1(m,n)$ and $T_2(m,n)$ are called $nT_1$-trapezoid and $nT_2$-trapezoid, respectively. For instance, Fig. 2(d) depicts $T_1(6,4)$ and $T_2(9,4)$.

Let $G$ be a rectangular, triangular, parallelogram, or trapezoid supergrid graph. A path on one boundary of $G$ is called flat if its vertices are in the boundary and it contains all boundary edges in the boundary. For example, the solid arrow lines in the down boundary of Fig. 2(a) indicate a flat path of $R(10,10)$.

In proving our results, we need to partition a shaped supergrid graph into two disjoint parts. The decomposition
is defined as follows.

**Definition 4.** Let \( S(m, n) \) be a triangular, parallelogram, or trapezoid supergrid graph. A cut operation on \( S(m, n) \) is a line partition through a set \( Z \) of edges so that the removal of \( Z \) from \( S(m, n) \) results in two disjoint supergrid subgraphs \( S_1 \) and \( S_2 \). A cut is called *vertical* (resp., *horizontal*) if it is a vertical (resp., horizontal) line to separate \( S(m, n) \) into \( S_1 \) and \( S_2 \) such that \( S_1 \) is to the left (resp., upper) of \( S_2 \), i.e., \( Z \) is a set of horizontal (resp., vertical) edges.

For instance, the bold dashed line in Fig. 2(c) depicts a vertical cut on \( P(5, 4) \) to partition it into two disjoint triangular supergrid subgraphs \( \Delta(4, 4) \).

In proving our result, we will construct a canonical Hamiltonian cycle and a canonical Hamiltonian path of a triangular, parallelogram, or trapezoid supergrid graph defined as follows.

**Definition 5.** Let \( S(m, n) \) be a triangular, parallelogram, or trapezoid supergrid graph with \( \kappa \) boundaries, and let \( s \) and \( t \) be its two distinct vertices. A Hamiltonian cycle of \( S(m, n) \) is called *canonical* if it contains \( \kappa - 1 \) flat paths on \( \kappa - 1 \) boundaries, and it contains at least one boundary edge in the other boundary. A Hamiltonian \((s, t)\)-path of \( S(m, n) \) is called *canonical* if it contains at least one boundary edge of each boundary in \( S(m, n) \).

B. Background results

In [15], we have showed that rectangular supergrid graphs always contain Hamiltonian cycles except 1-rectangles. Let \( R(m, n) \) be a rectangular supergrid graph with \( m \geq n \), \( C \) be a cycle of \( R(m, n) \), and let \( H \) be a boundary of \( R(m, n) \), where \( H \) is a subgraph of \( R(m, n) \). The restriction of \( C \) to \( H \) is denoted by \( C|H \). If \( |C|H = 1 \), i.e. \( C|H \) is a boundary path on \( H \), then \( C|H \) is called flat face on \( H \). If \( |C|H > 1 \) and \( C|H \) contains at least one boundary edge of \( H \), then \( C|H \) is called concave face on \( H \). A Hamiltonian cycle of \( R(m, 3) \) is called canonical if it contains three flat faces on two shorter boundaries and one longer boundary, and it contains one concave face on the other boundary, where the shorter boundary consists of three vertices. And, a Hamiltonian cycle of \( R(m, n) \) with \( n = 2 \) or \( n \geq 4 \) is said to be canonical if it contains three flat faces on three boundaries, and it contains one concave face on the other boundary. The following lemma states the result in [15] concerning the Hamiltonicity of rectangular supergrid graphs.

**Lemma 1.** (See [15].) Let \( R(m, n) \) be a rectangular supergrid graph with \( m \geq n \geq 2 \). Then, the following statements hold true:
1. if \( n = 3 \), then \( R(m, 3) \) contains a canonical Hamiltonian cycle;
2. if \( n = 2 \) or \( n \geq 4 \), then \( R(m, n) \) contains four distinct canonical Hamiltonian cycles with concave faces being on different boundaries.

Let \((G, s, t)\) denote the supergrid graph \( G \) with two given distinct vertices \( s \) and \( t \). Without loss of generality, we will assume that \( s_x \leq t_x \), i.e., \( s \) is to the left of \( t \), in the rest of the paper. The notation \( L(G, s, t) \) indicates the length of longest path between \( s \) and \( t \) in \( G \), where the length of a path is defined to be the number of vertices visited by the path. We denote a Hamiltonian path between \( s \) and \( t \) in \( G \) by \( HP(G, s, t) \). We say that \( HP(G, s, t) \) exists if there exists a Hamiltonian \((s, t)\)-path of \( G \). By the definition, \( L(G, s, t) = |V(G)| \) if \( HP(G, s, t) \) does exist. The Hamiltonian cycle of \( R(m, n) \) is called canonical if it satisfies Lemma 1. From Lemma 1, we know that \( HP(R(m, n), s, t) \) does exist when \( m, n \geq 2 \) and \( s, t \) is an edge in the canonical Hamiltonian cycle of \( R(m, n) \). In [18], we have proved that \( HP(R(m, n), s, t) \) always exists for \( m, n \geq 3 \). For \((R(m, n), s, t)\) with \( m \geq n \geq 3 \), a Hamiltonian \((s, t)\)-path of \( R(m, n) \) is called canonical if it contains at least one boundary edge of each side (boundary) in \( R(m, n) \). The following lemma is to show the Hamiltonian connectivity of rectangular supergrid graphs.

**Lemma 2.** (See [18].) For \((R(m, n), s, t)\) with \( m \geq n \geq 3 \), \( R(m, n) \) contains a canonical Hamiltonian \((s, t)\)-path, and, hence, \( HP(R(m, n), s, t) \) does exist.

For the 1-rectangle, \( HP(R(1, 1), s, t) \) does not exist if \( s \) or \( t \) is not a corner. On the other hand, \( HP(R(2, 1), s, t) \) does not exist if \((s, t)\) is a vertical and nonboundary edge of \( R(2, 2) \). For \( n = 1 \) or \( 2 \), \( HP(R(m, n), s, t) \) does exist except the above one trivial forbidden condition [18].

We next give some observations on the relations among cycle, path, and vertex. These propositions are presented in [18] and will be used in proving our results. Let \( C_1 \) and \( C_2 \) be two vertex-disjoint cycles of a graph \( G \). If there exist two edges \( e_1 \in C_1 \) and \( e_2 \in C_2 \) such that \( e_1 \approx e_2 \), then \( C_1 \) and \( C_2 \) can be combined into a cycle of \( G \). Then the following proposition holds true.

**Proposition 3.** (See [18].) Let \( C_1 \) and \( C_2 \) be two vertex-disjoint cycles of a graph \( G \). If there exist two edges \( e_1 \in C_1 \) and \( e_2 \in C_2 \) such that \( e_1 \approx e_2 \), then \( C_1 \) and \( C_2 \) can be combined into a cycle of \( G \). (see Fig. 3(a))

Let \( C_1 \) be a cycle and let \( P_1 \) be a path in a graph \( G \) such that \( V(C_1) \cap V(P_1) = \emptyset \). If there exist two edges \( e_1 \in C_1 \) and \( e_2 \in P_1 \) such that \( e_1 \approx e_2 \), then \( C_1 \) and \( P_1 \) can be combined into a path \( P \) of \( G \) with \( start(P) = start(P_1) \) and \( end(P) = end(P_1) \). Fig. 3(b) depicts such a construction, and, hence, the following proposition holds true.

**Proposition 4.** (See [18].) Let \( C_1 \) and \( P_1 \) be a cycle and a path, respectively, of a graph \( G \) such that \( V(C_1) \cap V(P_1) = \emptyset \). If there exist two edges \( e_1 \in C_1 \) and \( e_2 \in P_1 \) such that \( e_1 \approx e_2 \), then \( C_1 \) and \( P_1 \) can be combined into a path of \( G \). (see Fig. 3(b))

The above observation can be extended to a vertex \( x \), where \( P_1 = x \), as shown in Fig. 3(c), and we then have the following proposition.

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**Fig. 3.** A schematic diagram for (a) Proposition 3, (b) Proposition 4, and (c) Proposition 5, where bold dashed lines indicate the cycles (paths) and \( \otimes \) represents the destruction of an edge while constructing a cycle or path.
Proposition 5. (See [17]) Let $C_1$ be a cycle (path) of a graph $G$ and let $x$ be a vertex in $G - V(C_1)$. If there exists an edge $(u_1,v_1)$ in $C_1$ such that $u_1 \sim x$ and $v_1 \sim x$, then $C_1$ and $x$ can be combined into a cycle (path) of $G$. (see Fig. 3(c))

III. THE HAMILTONICY AND HAMILTONIAN CONNECTIVITY OF TRIANGULAR AND PARALLELOGRAM SUPERGRID GRAPHS

A. The Hamiltonicity and Hamiltonian connectivity of triangular supergrid graphs

In this subsection, we will verify the Hamiltonicity and Hamiltonian connectivity (except two trivial conditions) of triangular supergrid graphs. For a triangular supergrid graph, we will construct a canonical Hamiltonian cycle and a canonical Hamiltonian path. Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geq 2$, and let $s,t \in \Delta(n,n)$. Recall that a Hamiltonian cycle of $\Delta(n,n)$ is called canonical if it contains two flat faces on vertical and horizontal boundaries, and it contains at least one boundary edge in skewed boundary. A Hamiltonian $(s,t)$-path of $\Delta(n,n)$ is called canonical if it contains at least one boundary edge in each boundary. The following lemma proves the Hamiltonicity of triangular supergrid graphs.

Lemma 6. Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geq 2$. Then, $\Delta(n,n)$ contains a canonical Hamiltonian cycle.

Proof: We prove this lemma by induction on $n$. Initially, let $n = 2$ or 3. By inspection, $\Delta(2,2)$ and $\Delta(3,3)$ contain Hamiltonian cycles which contain all boundary edges of each boundary. Thus, the lemma holds true for $n = 2$ and 3. Assume that the lemma holds true when $n = k \geq 3$. Then, $\Delta(k-1,k-1)$ and $\Delta(k,k)$ contain canonical Hamiltonian cycles. Now, assume $n = k + 1$. We first make a vertical cut on $\Delta(k+1,k+1)$ to obtain two disjoint subgraphs $\Delta(k-1,k-1)$ and $T'$, where $T'$ is a 2-rectangle attached by a 2-triangle, and the vertical boundary of $\Delta(k-1,k-1)$ is faced to one boundary of $T'$. By induction hypothesis, $\Delta(k-1,k-1)$ contains a canonical Hamiltonian cycle $HC_{k-1}$, which contains two flat faces of vertical and horizontal boundaries. By visiting all boundary edges of $T'$, we can construct a Hamiltonian cycle $HC'$ of $T'$. Then, there exist two edges $e_1 \in HC_{k-1}$ and $e_2 \in HC'$ such that $e_1$ is a vertical boundary edge and contains the nontriangular corner of $\Delta(k+1,k-1)$, $e_2$ is a vertical boundary edge of $T'$, and $e_1 \approx e_2$. By Proposition 3, $HC_{k-1}$ and $HC'$ can be combined into a Hamiltonian cycle $HC$ of $\Delta(k+1,k+1)$ such that $HC$ contains all boundary edges of vertical and horizontal boundaries, and it contains at least one boundary edge of skewed boundary. Thus, $\Delta(k+1,k+1)$ contains a canonical Hamiltonian cycle and the lemma holds true when $n = k + 1$. By induction, $\Delta(n,n)$ contains a canonical Hamiltonian cycle for $n \geq 2$.

Next, we will study the Hamiltonian connectivity of triangular supergrid graphs. We first observe two conditions for $HP(\Delta(n,n),s,t)$ does not exist. These two forbidden conditions are described as follows:

(F1) $\Delta(n,n)$ is a 3-triangle, and $(s,t)$ is a nonboundary edge of $\Delta(n,n)$ (see Fig. 4(a)).

(F2) $\Delta(n,n)$ satisfies $n \geq 3$, and $(s,t)$ is an edge of $\Delta(n,n)$ such that $s$ and $t$ are adjacent to a triangular corner $w$ of $\Delta(n,n)$ (see Fig. 4(b)).

The conditions of (F1) and (F2) are called forbidden for $HP(\Delta(n,n),s,t)$. Note that $|V(\Delta(n,n))| = \frac{n(n+1)}{2}$. The following lemma computes the longest $(s,t)$-path with length $L(\Delta(n,n), s, t)$ when $(\Delta(n,n), s, t)$ satisfies condition (F1) or (F2).

Lemma 7. Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geq 3$, and let $s$ and $t$ be two distinct vertices of $\Delta(n,n)$. If $(\Delta(n,n), s, t)$ satisfies condition (F1) or (F2), then $L(\Delta(n,n), s, t) = \frac{n(n+1)}{2} - 1$.

Proof: By inspection, the lemma holds true when $n = 3$. In the following, assume that $n \geq 4$. Then, $(\Delta(n,n), s, t)$ satisfies condition (F2), and, hence, $(s,t)$ is an edge of $\Delta(n,n)$ such that $s$ and $t$ are adjacent to a triangular corner $w$ of $\Delta(n,n)$. By Lemma 6, $\Delta(n,n)$ contains a canonical Hamiltonian cycle $HC$. Since $\deg(w) = 2$, edges $(s,w)$ and $(w,t)$ are in $HC$. By removing $w$ from $HC$, we obtain a $(s,t)$-path $P$ with length $\frac{n(n+1)}{2} - 1$. Clearly, $HP(\Delta(n,n), s, t)$ does not exist, and, hence, the length of any $(s,t)$-path is less than $\frac{n(n+1)}{2}$. Thus, $P$ is the longest $(s,t)$-path. In addition, $P$ contains all boundary edges (except $(s,w)$ or $(w,t)$) of vertical and horizontal boundaries, and it contains at least one boundary edge of skewed boundary in $\Delta(n,n)$. Then, $L(\Delta(n,n), s, t) = |V(\Delta(n,n))| - 1 = \frac{n(n+1)}{2} - 1$, and, hence, the lemma holds true.

We have computed the longest $(s,t)$-path of $\Delta(n,n)$ when $(\Delta(n,n), s, t)$ satisfies forbidden condition (F1) or (F2). When $(\Delta(n,n), s, t)$ does not satisfy conditions (F1) and (F2), we will construct a canonical Hamiltonian $(s,t)$-path of $\Delta(n,n)$ as follows.

Lemma 8. Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geq 3$, and let $s$ and $t$ be two distinct vertices of $\Delta(n,n)$. If $(\Delta(n,n), s, t)$ does not satisfy conditions (F1) and (F2), then $\Delta(n,n)$ contains a canonical Hamiltonian $(s,t)$-path, and, hence, $HP(\Delta(n,n), s, t)$ does exist.

Proof: We will prove this lemma by induction on $n$. Initially, let $n = 3$ or 4. By inspecting every case, we can verify the lemma when $n = 3$ and 4. Fig. 5(a) and Fig. 5(b) depict the possible constructed canonical Hamiltonian

Fig. 4. Triangular supergrid graph in which there exists no Hamiltonian $(s,t)$-path for (a) condition (F1), and (b) condition (F2), where dotted lines indicate the forbidden edges $(s,t)$. (F1) $\Delta(n,n)$ is a 3-triangle, and $(s,t)$ is a nonboundary edge of $\Delta(n,n)$ (see Fig. 4(a)).

(F2) $\Delta(n,n)$ satisfies $n \geq 3$, and $(s,t)$ is an edge of $\Delta(n,n)$ such that $s$ and $t$ are adjacent to a triangular corner $w$ of $\Delta(n,n)$ (see Fig. 4(b)).

The conditions of (F1) and (F2) are called forbidden for $HP(\Delta(n,n),s,t)$. Note that $|V(\Delta(n,n))| = \frac{n(n+1)}{2}$. The following lemma computes the longest $(s,t)$-path with length $L(\Delta(n,n), s, t)$ when $(\Delta(n,n), s, t)$ satisfies condition (F1) or (F2).

Lemma 7. Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geq 3$, and let $s$ and $t$ be two distinct vertices of $\Delta(n,n)$. If $(\Delta(n,n), s, t)$ satisfies condition (F1) or (F2), then $L(\Delta(n,n), s, t) = \frac{n(n+1)}{2} - 1$.

Proof: By inspection, the lemma holds true when $n = 3$. In the following, assume that $n \geq 4$. Then, $(\Delta(n,n), s, t)$ satisfies condition (F2), and, hence, $(s,t)$ is an edge of $\Delta(n,n)$ such that $s$ and $t$ are adjacent to a triangular corner $w$ of $\Delta(n,n)$. By Lemma 6, $\Delta(n,n)$ contains a canonical Hamiltonian cycle $HC$. Since $\deg(w) = 2$, edges $(s,w)$ and $(w,t)$ are in $HC$. By removing $w$ from $HC$, we obtain a $(s,t)$-path $P$ with length $\frac{n(n+1)}{2} - 1$. Clearly, $HP(\Delta(n,n), s, t)$ does not exist, and, hence, the length of any $(s,t)$-path is less than $\frac{n(n+1)}{2}$. Thus, $P$ is the longest $(s,t)$-path. In addition, $P$ contains all boundary edges (except $(s,w)$ or $(w,t)$) of vertical and horizontal boundaries, and it contains at least one boundary edge of skewed boundary in $\Delta(n,n)$. Then, $L(\Delta(n,n), s, t) = |V(\Delta(n,n))| - 1 = \frac{n(n+1)}{2} - 1$, and, hence, the lemma holds true.

We have computed the longest $(s,t)$-path of $\Delta(n,n)$ when $(\Delta(n,n), s, t)$ satisfies forbidden condition (F1) or (F2). When $(\Delta(n,n), s, t)$ does not satisfy conditions (F1) and (F2), we will construct a canonical Hamiltonian $(s,t)$-path of $\Delta(n,n)$ as follows.

Lemma 8. Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geq 3$, and let $s$ and $t$ be two distinct vertices of $\Delta(n,n)$. If $(\Delta(n,n), s, t)$ does not satisfy conditions (F1) and (F2), then $\Delta(n,n)$ contains a canonical Hamiltonian $(s,t)$-path, and, hence, $HP(\Delta(n,n), s, t)$ does exist.

Proof: We will prove this lemma by induction on $n$. Initially, let $n = 3$ or 4. By inspecting every case, we can verify the lemma when $n = 3$ and 4. Fig. 5(a) and Fig. 5(b) depict the possible constructed canonical Hamiltonian
(s, t)-path of $\Delta(3, 3)$ and $\Delta(4, 4)$, respectively.

Now, assume that the lemma holds true when $n = k \geq 4$.

Then, there exists a canonical Hamiltonian $(s^*, t^*)$-path $P^*$ of $\Delta(k-1, k-1)$ if $(\Delta(k-1, k-1), s^*, t^*)$ does not satisfy conditions (F1) and (F2). Consider that $n = k + 1$. Let $w$ and $w'$ be two triangular corners of $\Delta(k+1, k+1)$ such that $w'$ is in vertical boundary. Let $s$ and $t$ be two distinct vertices of $\Delta(k+1, k+1)$ such that $(\Delta(k+1, k+1), s, t)$ does not satisfy forbidden condition (F2). We then make a vertical cut on $\Delta(k+1, k+1)$ to partition it into two disjoint subgraphs $(\Delta(k+1, k+1), s, t)$ such that there is a vertical boundary edge $e$ in $P$. Since there exists a boundary edge $(u, v)$ in $HC'$ such that $u \sim w^*$ and $v \sim w^*$, by Proposition 5, $HC'$ and $w^*$ can be merged into a canonical Hamiltonian cycle $HC^*$ of $T' \cup \{w^*\}$. Then, there exists an edge $e^*$ in $HC^*$ such that $e^* \approx e$. By Proposition 4, $P$ and $HC^*$ can be combined into a canonical Hamiltonian $(s, t)$-path of $\Delta(k+1, k+1)$.

Case 1.2: $(\Delta(k-1, k-1), s, t)$ does not satisfy conditions (F1) and (F2). By induction hypothesis, there exists a canonical Hamiltonian $(s, t)$-path $P$ of $\Delta(k-1, k-1)$. Then, there exist two edges $e \in P$ and $e' \in HC'$ such that $e \approx e'$. By Proposition 4, $P$ and $HC'$ can be combined into a canonical Hamiltonian $(s, t)$-path of $\Delta(k+1, k+1)$.

Case 2: $s \in \Delta(k-1, k-1)$ and $t \in T'$. Let $p$ be a vertex in $\Delta(k-1, k-1)$ such that $p = w^*$ if $s \neq w^*$, and $(p, w^*)$ is a vertical boundary edge of $\Delta(k-1, k-1)$ otherwise. Then, $(\Delta(k-1, k-1), s, p)$ does not satisfy conditions (F1) and (F2). Let $q$ be a vertex in $T'$ such that $q \sim p$ and $(q, t)$ is not a horizontal edge of $T'$. Since $T'$ is a 2-rectangle attached by a triangle, such a vertex $q$ can be easily found. Then, $HP(T', q, t)$ exist (see [18]). Let $P'$ be the canonical Hamiltonian $(q, t)$-path of $T'$. By induction hypothesis, there exists a canonical Hamiltonian $(s, p)$-path $P^*$ of $\Delta(k-1, k-1)$. Then, $P^* \approx P'$ forms a canonical Hamiltonian $(s, t)$-path of $\Delta(k+1, k+1)$.

Case 3: $s, t \in T'$. Let $T' = R' \cup \{w^*\}$, where $R' = R(2, k)$. Then, $R'$ is a 2-rectangle. Since $(\Delta(k+1, k+1), s, t)$ does not satisfy condition (F2), $s \sim w^*$ or $t \sim w^*$. Suppose that $s \sim t$ or $(s, t)$ is not a horizontal and nonboundary edge in $R'$. In [18], there exists a canonical Hamiltonian $(s, t)$-path $P^*$ of $R'$. By Proposition 5, $w^*$ can be merged into $P^*$ to form a canonical Hamiltonian $(s, t)$-path $P^*$ of $T'$. By Lemma 6, $(\Delta(k-1, k-1), s, t)$ contains a canonical Hamiltonian cycle $HC$. Then, there exist two edges $e \in HC$ and $e^* \in P^*$ such that $e \approx e^*$. By Proposition 4, $P^*$ and $HC'$ can be combined into a canonical Hamiltonian $(s, t)$-path of $\Delta(k+1, k+1)$. On the other hand, suppose that $(s, t)$ is a horizontal and nonboundary edge in $R'$. Without loss of generality, assume that $s_x < t_x$. Let $p_1, p_2 \in R'$ and $q_1, q_2 \in \Delta(k-1, k-1)$ such that $(s, p_1)$ and $(s, p_2)$ are two vertical and boundary edges in $R'$, $p_1$ is to the upper of $s$, $p_1 \sim q_1$, and $p_2 \sim q_2$. Then, we can easily construct two disjoint $(s, p_1)$-path $P_1$ and $(p_2, t)$-path $P_2$ of $T'$ so that $P_1 \cup P_2$ visits all vertices of $T'$ and contains at least one boundary edge in each boundary.
of $T'$. We can see that $(\Delta(k - 1, k - 1), q_1, q_2)$ does not satisfy conditions (F1) and (F2). By induction hypothesis, there exists a canonical Hamiltonian $(q_1, q_2)$-path $P^*$ of $\Delta(k - 1, k - 1)$. Then, $P_1 \Rightarrow P^* \Rightarrow P_2$ forms a canonical Hamiltonian $(s, t)$-path of $\Delta(k + 1, k + 1)$.

It immediately follows from the above cases that $\Delta(k + 1, k + 1)$ contains a canonical Hamiltonian $(s, t)$-path. Thus, the lemma holds true when $n = k + 1$. By induction, $\Delta(n, n)$ contains a canonical Hamiltonian $(s, t)$-path for $n \geq 3$, and hence $HP(\Delta(n, n), s, t)$ does exist. This completes the proof of the lemma.

B. The Hamiltonicity and Hamiltonian connectivity of parallelogram supergrid graphs

In this subsection, we will prove the Hamiltonicity and Hamiltonian connectivity of parallelogram supergrid graphs. In a parallelogram supergrid graph $P(m, n)$, we only consider that $V(P(m, n)) = \{v = (v_x, v_y) \mid 1 \leq v_x \leq m \text{ and } v_y \leq v_x + m - 1\}$. The other type of parallelograms can be verified by the same arguments. Note that there are two horizontal and two skewed boundaries in $P(m, n)$. We first provide a constructive proof to show that any parallelogram supergrid graph $P(m, n)$ with $m \geq n \geq 2$ contains a Hamiltonian cycle. We then prove that $P(m, n)$ always contains a canonical $(s, t)$-path except three trivial conditions. The following lemma first appears in [15] and shows the Hamiltonicity of parallelogram supergrid graphs.

**Lemma 9.** Let $P(m, n)$ be a parallelogram supergrid graph with $m \geq n \geq 2$. Then, $P(m, n)$ contains a canonical Hamiltonian cycle.

**Proof:** By inspection, the lemma can be easily verified when $3 \geq m$. For example, Fig. 6(a) shows a canonical Hamiltonian cycle of $P(3, 3)$. In the following, assume that $m \geq n \geq 4$. Note that $P(m, n)$ consists of $m$ columns and $n$ rows of vertices. Let $a_{ij}$ be the vertex located at $i$-th row and $j$-th column of $P(m, n)$, where $n \geq i \geq 1$ and $m \geq j \geq 1$. Consider the following two cases:

**Case 1:** $n$ is even. Let $P_1 = a_{11} \Rightarrow a_{12} \Rightarrow \cdots \Rightarrow a_{1(m-1)} \Rightarrow a_{1m}$, $P_2 = a_{21} \Rightarrow a_{32} \Rightarrow \cdots \Rightarrow a_{(m-1)} \Rightarrow a_{mn}$ for $n \geq i \geq 1$ and let $P_{n+1} = a_{21} \Rightarrow a_{32} \Rightarrow \cdots \Rightarrow a_{(n-1)} \Rightarrow a_{n1}$. Let $HC = P_1 \Rightarrow rev(P_2) \Rightarrow rev(P_3) \Rightarrow \cdots \Rightarrow P_{n-1} \Rightarrow rev(P_n) \Rightarrow rev(P_{n+1})$, where $j$ is an odd and $n-1 \geq j \geq 1$. Then, $HC$ is a canonical Hamiltonian cycle of $P(m, n)$. For instance, Fig. 6(b) depicts a canonical Hamiltonian cycle of $P(5, 4)$.

**Case 2:** $n$ is odd. In this case, $n \geq 5$. Let $P_1 = a_{11} \Rightarrow a_{12} \Rightarrow \cdots \Rightarrow a_{1(m-1)} \Rightarrow a_{1m}$, $P_2 = a_{21} \Rightarrow a_{23} \Rightarrow \cdots \Rightarrow a_{i(m-1)} \Rightarrow a_{im}$ for $n - 3 \geq i \geq 2$,
(F5) $P(m, n)$ satisfies $m \geq n \geq 2$, and $(s, t)$ is an edge of $P(m, n)$ such that $s \sim w$ and $t \sim w$ for any parallel corner $w$ of $P(m, n)$, where $s \neq w, t \neq w$, and $\deg(w) = 2$ (see Fig. 7(c)).

When $(P(m, n), s, t)$ satisfies condition (F5), we can compute the longest $(s, t)$-path by removing the vertex $w$ from the canonical Hamiltonian cycle of $P(m, n)$ constructed in Lemma 9. Thus, we have the following lemma.

**Lemma 11.** Let $(P(m, n))$ be a parallelogram supergrid graph with $m \geq n \geq 2$, and let $s$ and $t$ be its two distinct vertices. If $(P(m, n), s, t)$ satisfies condition (F5), then $L(P(m, n), s, t) = mn - 1$, and the longest $(s, t)$-path contains at least one boundary edge of each boundary in $P(m, n)$ when $n \geq 3$.

In the following, we consider that $(P(m, n), s, t)$ does not satisfy conditions (F3)–(F5). Then, we will construct a canonical Hamiltonian $(s, t)$-path of $P(m, n)$. We first consider 3-parallelogram $(P(m, 3))$ as follows.

**Lemma 12.** Let $P(m, n)$ be a 3-parallelogram with $n = 3$ and $m \geq 3$, and let $s$ and $t$ be two distinct vertices of $P(m, n)$ with $s_2 \leq t_2$. If $(P(m, n), s, t)$ does not satisfy condition (F5), then $(P(m, n))$ contains a canonical Hamiltonian $(s, t)$-path, and, hence, $HP(P(m, 3), s, t)$ does exist.

Proof: Let $w$ and $w'$ be two parallel corners of $(P(m, 3))$. Since $(P(m, 3), s, t)$ does not satisfy condition (F5), we get that $s \sim w$ or $t \sim w$ for $w = w'$. Consider the following cases:

**Case 1:** $m = n = 3$. We first make a vertical cut on $P(3, 3)$ to obtain two disjoint triangular supergrid subgraphs $\Delta_1 = \Delta(3, 3)$ and $\Delta_2 = \Delta(2, 2)$, as depicted in Fig. 8(a).

Without loss of generality, assume that $w \in \Delta_1$ and $w' \in \Delta_2$. Let $w_1$ and $w_2$ be respectively parallel corners of $\Delta_1$ and $\Delta_2$ different from $w$ and $w'$. There are three subcases:

**Case 1.1:** $s, t \in \Delta_1$. By visiting all boundary edges of $\Delta_2$, we obtain a Hamiltonian cycle $H_{22}$ of $\Delta_2$. Suppose that $(\Delta_1, s, t)$ does not satisfy condition (F1). By Lemma 8, $\Delta_1$ contains a canonical Hamiltonian $(s, t)$-path $P_1$ (see Fig. 5(a)). Then, there exist two edges $e_1 \in P_1$ and $e_2 \in H_{22}$ such that $e_1 \approx e_2$. By Proposition 4, $P_1$ and $H_{22}$ can be combined into a canonical Hamiltonian $(s, t)$-path of $(P(3, 3))$.

On the other hand, suppose that $(\Delta_1, s, t)$ satisfies condition (F1). Then, $(s, t)$ is a nonboundary edge of $\Delta_1$, and $s \sim w$ or $t \sim w$ (see Fig. 4(a)). By inspecting every case, we can construct a Hamiltonian $(s, t)$-path $P^*_1$ of $\Delta_1 - w$ such that it contains a vertical boundary edge $e_1$ of $\Delta_1$. Let $w^*$ be the vertex of $\Delta_1 - \{w, w_2\}$. Then, $H_{22}$ contains vertical boundary edge $(w^*, w_2)$ of $\Delta_2$ such that $w_1 \sim w^*$ and $w_1 \sim w_2$. By Proposition 5, $w_1$ can be merged into $H_{22}$ to form a Hamiltonian cycle $H_{22}$ of $\Delta_2 \cup \{w_1\}$. Then, there exists an edge $e_2 \in H_{22}$ such that $e_1 \approx e_2$. By Proposition 4, $P_1$ and $H_{22}$ can be combined into a canonical Hamiltonian $(s, t)$-path of $(P(3, 3))$. Fig. 8(a) depicts such a constructed Hamiltonian $(s, t)$-path.

**Case 1.2:** $s, t \in \Delta_2$. By Lemma 6, $\Delta_1$ contains a canonical Hamiltonian cycle $H_{C1}$. Since $(P(3, 3), s, t)$ does not satisfy condition (F5), $s \sim w$ or $t \sim w$. Thus, $w' = s$ or $t$. Since $s_2 \leq t_2$, $w^* = t$. Then, $\Delta_2$ contains a Hamiltonian $(s, t)$-path $P_2$ such that it contains the vertical boundary edge $e_2$ of $\Delta_2$. Thus, there exist two edges $e_1 \in H_{C1}$ and $e_2 \in P_2$ with $e_1 \approx e_2$. By Proposition 4, we can combine $P_2$ and $H_{C1}$ into a canonical Hamiltonian $(s, t)$-path of $(P(3, 3))$.

**Case 2:** $m = n + 1 = 4$. In this case, we first make a vertical cut on $P(4, 3)$ to get two disjoint triangular supergrid subgraphs $\Delta_1 = \Delta(3, 3)$ and $\Delta_2 = \Delta(3, 3)$, as depicted in Fig. 8(b). Then, there are the following two subcases:

**Case 2.1:** $s, t \in \Delta_1$ or $\Delta_2$. By symmetry, we can only consider that $s, t \in \Delta_1$. By similar arguments in proving Case 1.1, a canonical Hamiltonian $(s, t)$-path of $(P(4, 3))$ can be constructed.

**Case 2.2:** $s \in \Delta_1$ and $t \in \Delta_2$. Let $p \in \Delta_1$ and $q \in \Delta_2$ such that $p \neq s, q \neq t, (\Delta_1, s, p)$ and $(\Delta_2, q, t)$ do not satisfy condition (F1), and $p \sim q$. Consider that $p$ and $q$ do exist. By Lemma 8, $\Delta_1$ and $\Delta_2$ contain canonical Hamiltonian $(s, p)$-path $P_1$ and $(q, t)$-path $P_2$, respectively. Then, $P_1 \Rightarrow P_2$ forms a canonical Hamiltonian $(s, t)$-path of $(P(4, 3))$. On the other hand, consider that $p$ or $q$ does not exist. By inspecting every case for the locations of $s$ and $t$, only one case occurs about that $p$ and $q$ do not exist. The location of $s$ and $t$ is shown in Fig. 8(b). Then, a canonical Hamiltonian $(s, t)$-path of $P(4, 3)$ can be easily constructed, as depicted in Fig. 8(b).

**Case 3:** $m = n + 2 = 5$. We first perform two vertical cuts on $P(5, 3)$ to partition it into three disjoint supergrid subgraphs, $\Delta_1 = \Delta(3, 3)$, $\Delta_2 = \Delta(2, 2)$, and $R = R(2, 3)$, as depicted in Fig. 8(c). Let $w_1$ and $w_2$ be respectively parallel corners of $\Delta_1$ and $\Delta_2$ different from $w$ and $w'$. There are four subcases:

**Case 3.1:** $s, t \in \Delta_1$. By visiting all boundary edges of $R = R(2, 3)$, we get a Hamiltonian cycle $H_{C2}$ of $R$ such that it contains four flat paths of $R$. By visiting all boundary edges of $\Delta_2$, we obtain a canonical Hamiltonian cycle $H_{C2}$ of $\Delta_2$. Then, there exist two edges $e_R \in H_{C2}$ and $e_2 \in H_{C2}$ such that $e_R \approx e_2$. By Proposition 3, $H_{C2}$ and $H_{C2}$ can be combined into a Hamiltonian cycle $H_{C0}$ of $U \cup \Delta_1$ such that $H_{C0}$ contains one flat face of $R$ that is placed to face $\Delta_1$. Suppose that $(\Delta_1, s, t)$ does not satisfy condition (F1). By Lemma 8, $\Delta_1$ contains a canonical Hamiltonian $(s, t)$-path $P_1$. Then, there exist two edges $e_1 \in P_1$ and $e^* \in H_{C0}$ such that $e_1 \approx e^*$. By Proposition 4, $P_1$ and $H_{C0}$ can be combined into a canonical Hamiltonian $(s, t)$-path of $(P(m, n))$. On the other hand, suppose that $(\Delta_1, s, t)$ satisfies condition (F1). Then, $(s, t)$ is a nonboundary edge of $\Delta_1$, and $s \sim w$ or $t \sim w$ (see Fig. 4(a)). By inspecting every case, we can construct a Hamiltonian $(s, t)$-path $P^*_1$ of $\Delta_1 - w_1$ such that it contains a vertical boundary edge $e_1$ of $\Delta_1$. Let $w^*$ be the down-left corner of $R$ and let $(w^*, p)$ be the vertical edge in $R$. Then, $w_1 \sim w^*$ and $w_1 \sim p$. By Proposition 5, $w_1$ can be merged into $H_{C0}$ to form a Hamiltonian cycle $H_{C0}$ of $\Delta_2 \cup U \cup \{w_1\}$. By Lemma 7, $\Delta_1 - (w_1)$ contains a canonical Hamiltonian $(s, t)$-path $P^*_1$. Then, there exist
two edges $e_1^* \in P^*_R$ and $e' \in H C'$ such that $e_1^* \approx e'$. By Proposition 4, $P^*_R$ and $H C'$ can be combined into a canonical Hamiltonian $(s, t)$-path of $P(5, 3)$. The subcase of $s, t \in \Delta_2$ can be proved similarly.

Case 3.2: $s, t \in R$. By Lemma 6, $\Delta_1$ contains a canonical Hamiltonian cycle $H C_1$. By visiting all boundary edges of $\Delta_2$, $\Delta_2$ has a Hamiltonian cycle $H C_2$ which contains all boundary edges. Suppose that $(s, t)$ is a horizontal and nonboundary edge of $R$. Then, $H P(R, s, t)$ does not exist. We then perform a horizontal cut on $R$ to obtain two disjoint subgraphs $R_1$ and $R_2$, as illustrated in Fig. 8(c). By visiting all boundary edges of $R_1$ except $(s, t)$, we obtain a Hamiltonian $(s, t)$-path $P_R$ of $R_1$. For every vertex $v \in R_2$, $v$ is incident to one edge of $H C_1$ or $H C_2$. By Proposition 5, the vertices of $R_2$ can be merged into $H C_1$ or $H C_2$. Let the merged cycles of $R_2$ into $H C_1$ and $H C_2$ be $H C'_1$ and $H C'_2$, respectively. Then, there exist four edges $e_1', e_2' \in H C'_1$, $e_2'^* \in H C'_2$, and $e_1^*, e_2^* \in P_R$ such that $e_1' \approx e_1^*$ and $e_2' \approx e_2^*$. By Proposition 4, $P_R$, $H C'_1$, and $H C'_2$ can be combined into a canonical Hamiltonian $(s, t)$-path of $P(5, 3)$. For example, Fig. 8(c) shows a such canonical Hamiltonian $(s, t)$-path of $P(5, 3)$. On the other hand, suppose that $s \sim t$ or $(s, t)$ is a horizontal and nonboundary edge of $R$. Then, $R$ contains a canonical Hamiltonian $(s, t)$-path $P_R$ constructed in [18]. Thus, there exist four edges $e_1 \in H C_1$, $e_2 \in H C_2$, and $e_1^*, e_2^* \in P_R$ such that $e_1 \approx e_1^*$ and $e_2 \approx e_2^*$. By Proposition 4, $P_R$, $H C_1$, and $H C_2$ can be combined into a canonical Hamiltonian $(s, t)$-path of $P(5, 3)$.

Case 3.3: $s$ and $t$ are in the different partitioned subgraphs. Note that $s_2 \leq t_2$. We have the following subcases:

Case 3.3.1: $(s \in \Delta_1$ and $t \in R)$ or $(s \in R$ and $t \in \Delta_2)$. Consider that $s \in \Delta_1$ and $t \in R$. Let $p$ be a vertex in $\Delta_1$ such that $p = w_1$ if $s \neq w_1$, and $(p, w_1)$ is a vertical boundary edge of $\Delta_1$ otherwise. Let $q \in R$ such that $q \neq t$, $q \sim p$, and $(q, t)$ is not a horizontal nonboundary edge of $R$. Then, $(\Delta_1, s, p)$ does not satisfy condition (F1), and $R$ contains a canonical Hamiltonian $(q, t)$-path $P_R$ constructed in [18]. By visiting all boundary edges of $\Delta_2$, we get a Hamiltonian cycle $H C_2$ of $\Delta_2$ which contains all boundary edges. Then, there exist two edges $e_R \in P_R$ and $e_2 \in H C_2$ such that $e_R \approx e_2$. By Proposition 4, $P_R$ and $H C_2$ can be combined into a Hamiltonian $(q, t)$-path $P_R \cup \Delta_2$. By Lemma 8, $\Delta_1$ contains a canonical Hamiltonian $(s, p)$-path $P_1$. Then, $P_1 \Rightarrow P_R$ forms a canonical Hamiltonian $(s, t)$-path of $P(5, 3)$. The subcase of $s \in R$ and $t \in \Delta_2$ can be verified by the same arguments.

Case 3.3.2: $s \in \Delta_1$ and $t \in \Delta_2$. Let $p$ be a vertex in $\Delta_1$ such that $p = w_1$ if $s \neq w_1$, and $(p, w_1)$ is a vertical boundary edge of $\Delta_1$ otherwise. Let $q \in R$ such that $q = w_1$ if $t \neq w_2$, and $(q, w_2)$ is a vertical boundary edge of $\Delta_2$ otherwise. Let $r_1, r_2 \in R$ such that $r_1 \sim p$, $r_2 \sim q$, and $(r_1, r_2)$ is a horizontal nonboundary edge of $R$. By inspecting any case, $p, q$, and $r_1, r_2$ do exist. Then, $(\Delta_1, s, p)$ does not satisfy condition (F1), $H P(\Delta_2, q, t)$ does exist, and $H P(R, r_1, r_2)$ does exist. By Lemma 8, $\Delta_1$ contains a canonical Hamiltonian $(s, p)$-path $P_1$. Let $P_R$ be the canonical Hamiltonian $(r_1, r_2)$-path of $R$ constructed in [18], and let $P_2$ be the Hamiltonian $(q, t)$-path of $\Delta_2$. Then, $P_1 \Rightarrow P_R \Rightarrow P_2$ forms a canonical Hamiltonian $(s, t)$-path of $P(5, 3)$.

Case 4: $m \geq n + 3 = 6$. We first make two vertical cuts on $P(m, 3)$ to partition it into three disjoint supergrid subgraphs, $\Delta_1 = \Delta(3, 3)$, $R = R(m - 3, 3)$, and $\Delta_2 = \Delta(3, 3)$, as depicted in Fig. 8(d). By Lemma 2, $R$ is Hamiltonian connected. Then, a canonical Hamiltonian $(s, t)$-path of $P(m, 3)$ can be constructed by similar arguments in proving Case 3. For instance, Fig. 8(d) depicts a canonical Hamiltonian $(s, t)$-path of $P(m, 3)$ when $s \in \Delta_1$ and $t \in \Delta_2$.

We have considered any case to construct a canonical Hamiltonian $(s, t)$-path of $P(m, 3)$ for $m \geq 3$. This completes the proof of the lemma.

By similar arguments in proving Lemma 12, we can prove the Hamiltonian connectivity of parallelogram supergrid graph $P(m, n)$ with $m \geq n \geq 4$ as follows.

Lemma 13. Let $P(m, n)$ be a parallelogram supergrid graph with $m \geq n \geq 4$, and let $s$ and $t$ be two distinct vertices of $P(m, n)$ with $s_2 \leq t_2$. If $(P(m, n), s, t)$ does not satisfy condition (F5), then $P(m, n)$ contains a canonical Hamiltonian $(s, t)$-path, and, hence, $H P(P(m, n), s, t)$ does exist.

Proof: We will prove this lemma by constructing a canonical Hamiltonian $(s, t)$-path of $P(m, n)$. Let $w$ and $w'$ be the two parallel corners of $P(m, n)$. Since $(P(m, n), s, t)$ does not satisfy condition (F5), $s \sim w$ or $t \sim w$ for $w = w'$ or $w'$. Since $n \geq 4$, $n - 1 \geq 3$. The considered cases are the same as Lemma 12 and are discussed as follows:

Case 1: $m = n$. We first make a vertical cut on $P(m, n)$ to get two disjoint triangular supergrid subgraphs $\Delta_1 = \Delta(n, n)$ and $\Delta_2 = \Delta(n - 1, n - 1)$, as depicted in Fig. 9(a). Without loss of generality, assume that $w \in \Delta_1$ and $w' \in \Delta_2$. Let $w_1$ and $w_2$ be respectively corners of $\Delta_1$ and $\Delta_2$ different from $w$ and $w'$. There are three subcases:

Case 1.1: $s, t \in \Delta_1$ or $\Delta_2$. By Lemma 6, $\Delta_1$ and $\Delta_2$ contain canonical Hamiltonian cycles. Then, a canonical Hamiltonian $(s, t)$-path of $P(m, n)$ can be constructed by similar arguments in proving Case 1.1 of Lemma 12.

Case 1.2: $s \in \Delta_1$ and $t \in \Delta_2$. Let $q$ be a vertex in $\Delta_2$ such that $q = w_2$ if $t \neq w_2$, and $(q, w_2)$ is a vertical boundary edge of $\Delta_2$ otherwise. Then, $(\Delta_2, q, t)$ does not
Fig. 9. The constructed canonical Hamiltonian $(s, t)$-path of $P(m, n)$ with $m \geq n \geq 4$ for (a) $m = n$, and $s \in \Delta_1$, $t \in \Delta_2$, (b) $m = n + 1$, and $s, t \in \Delta_1$, (c) $m = n + 2$, $s, t \in R(2, m)$, and $H(P(R, s, t))$ does not exist, and (d) $m \geq n + 3$, and $s \in \Delta_1$, $t \in \Delta_2$, where bold dashed lines represent the cut operations on $P(m, n)$, solid lines indicate the constructed Hamiltonian $(s, t)$-path, and $\Box$ represents the destruction of an edge while constructing a Hamiltonian $(s, t)$-path.

satisfy conditions (F1) and (F2). Let $p \in \Delta_1$ such that $p \neq s$, $p \sim q$, and $(\Delta_1, s, p)$ does not satisfy condition (F2). Since $n \geq 4$, $p$ and $q$ do exist. By Lemma 8, $\Delta_1$ and $\Delta_2$ contain canonical Hamiltonian $(s, p)$-path $P_1$ and $(q, t)$-path $P_2$, respectively. Then, $P_1 \Rightarrow P_2$ forms a canonical Hamiltonian $(s, t)$-path of $P(m, n)$. Fig. 9(a) depicts a such canonical Hamiltonian $(s, t)$-path.

Case 2: $m = n + 1$. In this case, we first perform a vertical cut on $P(m, n)$ to partition it into two disjoint triangular supergrid subgraphs $\Delta_1 = \Delta(n, n)$ and $\Delta_2 = \Delta(n, n)$, as depicted in Fig. 9(b). By similar arguments in proving Case 1, we can construct a canonical Hamiltonian $(s, t)$-path of $P(m, n)$. For instance, Fig. 9(b) shows a constructed canonical Hamiltonian $(s, t)$-path when $s, t \in \Delta_1$ and $(\Delta_1, s, t)$ does not satisfy condition (F2).

Case 3: $m = n + 2$. We first make two vertical cuts on $P(m, n)$ to partition it into three disjoint supergrid subgraphs, $\Delta_1 = \Delta(n, n)$, $R = R(2, m)$, and $\Delta_2 = \Delta(n - 1, n - 1)$, as depicted in Fig. 9(c). There are the following three subcases:

Case 3.1: $s, t \in \Delta_1$ or $\Delta_2$. By Lemma 6, $\Delta_1$ and $\Delta_2$ contain canonical Hamiltonian cycles. By similar arguments in proving Case 3.1 of Lemma 12, a canonical Hamiltonian $(s, t)$-path of $P(m, n)$ can be constructed.

Case 3.2: $s, t \in R$. By similar arguments in proving Case 3.2 of Lemma 12, we can construct a canonical Hamiltonian $(s, t)$-path of $P(m, n)$. For instance, Fig. 9(c) depicts a constructed canonical Hamiltonian $(s, t)$-path of $P(m, n)$ when $H(P(R, s, t))$ does not exist.

Case 3.3: $s$ and $t$ are not in the same partitioned subgraph. This subcase can be verified by similar arguments in proving Case 3.3 of Lemma 12.

Case 4: $m \geq n + 3$. We first perform two vertical cuts on $P(m, n)$ to partition it into two disjoint supergrid subgraphs, $\Delta_1 = \Delta(n, n)$, $R = R(m - n, n)$, and $\Delta_2 = \Delta(n, n)$, as depicted in Fig. 9(d). Then, a canonical Hamiltonian $(s, t)$-path of $P(m, n)$ can be constructed by similar arguments in proving Case 3.3 of Lemma 12. For instance, Fig. 9(d) shows a canonical Hamiltonian $(s, t)$-path of $P(m, n)$ when $s \in \Delta_1$ and $t \in \Delta_2$.

It follows from the above cases that a canonical Hamiltonian $(s, t)$-path of $P(m, n)$ with $m \geq n \geq 4$ can be constructed, and, hence, $H(P(m, n), s, t)$ does exist.

It immediately follows from Lemmas 12 and 13 that we can verify the Hamiltonicity of trapezoid supergrid graphs.

Theorem 14. Let $P(m, n)$ be a parallelogram supergrid graph with $m \geq n \geq 1$, and let $s$ and $t$ be two distinct vertices of $P(m, n)$. If $P(m, n), s, t$ does not satisfy conditions (F3)–(F5), then $P(m, n)$ contains a canonical Hamiltonian $(s, t)$-path, and, hence, $H(P(m, n), s, t)$ does exist.

IV. THE HAMILTONICITY AND HAMILTONIAN CONNECTIVITY OF TRAPEZOID SUPERGRID GRAPHS

In this section, we will prove the Hamiltonicity and Hamiltonian connectivity (except two trivial conditions) of trapezoid supergrid graphs. There are two types of trapezoid supergrid graphs $T_1(m, n)$ and $T_2(m, n)$. By similar arguments in proving Lemma 9, we can verify the Hamiltonicity of trapezoid supergrid graphs as follows.

Lemma 15. Let $T(m, n)$ be a trapezoid supergrid graph. Then, $T(m, n)$ contains a canonical Hamiltonian cycle.

Proof: Consider $T(m, n) = T_1(m, n)$, i.e., $T(m, n)$ is a $n \times r$, trapezoid. By inspection, the lemma can be easily verified when $3 \geq n$. In the following, assume that $n \geq 4$. By definition of $T_1(m, n)$, $m \geq n + 1 \geq 5$. Note that $T_1(m, n)$ consists of $m$ columns and $n$ rows of vertices. Let $a_{ij}$ be the vertex located at $i$-th row and $j$-th column of $T_1(m, n)$, where $1 \leq i \leq n$ and $1 \leq j \leq m - i + 1$. Consider the following two cases:

Case 1: $n$ is even. Let $P_1 = a_{11} \rightarrow a_{12} \rightarrow \cdots \rightarrow a_{1(m-n+1)}$ and $P_2 = a_{12} \rightarrow a_{13} \rightarrow \cdots \rightarrow a_{1(m-n+1)} \rightarrow a_{1(n+1)}$ such that $n \geq 2$, and let $P_{n+1} = a_{21} \rightarrow a_{31} \rightarrow \cdots \rightarrow a_{n-1,1} \rightarrow a_{n,1}$. Let $HC = P_1 \Rightarrow rev(P_2) \Rightarrow P_3 \Rightarrow rev(P_4) \Rightarrow \cdots \Rightarrow P_j \Rightarrow rev(P_{j+1}) \Rightarrow \cdots \Rightarrow P_{n-1} \Rightarrow rev(P_n) \Rightarrow rev(P_{n+1})$, where $j$ is an odd and $n - 1 \geq j \geq 1$. Then, $HC$ is a canonical Hamiltonian cycle of $T_1(m, n)$. For example, Fig. 10(a) depicts a canonical Hamiltonian cycle of $T_1(8, 4)$.

Case 2: $n$ is odd. In this case, $n \geq 5$. Let $P_1 = a_{11} \rightarrow a_{12} \rightarrow \cdots \rightarrow a_{1(m-n)} \rightarrow a_{1(n+1)}$ and $P_2 = a_{2} \rightarrow a_{3} \rightarrow \cdots \rightarrow a_{(m-n+1)} \rightarrow a_{(n+1)}$ for $n - 3 \geq \tau \geq 2$.

Fig. 10. The canonical Hamiltonian cycle of (a) $T_1(8, 4)$, (b) $T_1(9, 5)$, (c) $T_2(11, 4)$, and $T_2(13, 5)$, where arrow lines indicate the edges in the Hamiltonian cycle.
prove $T_1(m, n)$ to be Hamiltonian connected as follows.

**Lemma 17.** Let $T_1(m, n)$ be a trapezoid supergrid graph with $m - 1 \geq n \geq 2$, and let $s$ and $t$ be two distinct vertices of $T_1(m, n)$. If $(T_1(m, n), s, t)$ does not satisfy conditions (F6)–(F7), then $T_1(m, n)$ contains a canonical Hamiltonian $(s, t)$-path, and, hence, $HP(T_1(m, n), s, t)$ does exist.

**Proof:** When $n = 2$, i.e., $T_1(m, n)$ is a 2-trapezoid, $HP(R_1, s, t)$ does exist [18] and hence $HP(T_1(m, n), s, t)$ can be easily constructed. In the following, assume that $n \geq 3$. By definition of $T_1(m, n)$, $m \geq n + 1 \geq 4$. We first make a vertical cut on $T_1(m, n)$ to obtain two disjoint subgraphs $R_1 = R(m - n + 1, n)$ and $\Delta_1 = \Delta(n - 1, n - 1)$, as shown in Fig. 12(a). Depending on the locations of $s$ and $t$, we consider the following three cases:

**Case 1:** $s, t \in R_1$. In this case, we consider whether $R_1$ is a 2-rectangle as follows:

**Case 1.1:** $m - n + 1 = 2$. In this subcase, $R_1$ is a 2-rectangle. Suppose that $(s, t)$ is not a horizontal and nonboundary edge of $R_1$. In [18], $R_1$ contains a canonical Hamiltonian $(s, t)$-path $P_1$. By Lemma 6, $\Delta_1$ contains a canonical Hamiltonian cycle $C_1$. Then, there exist two edges $e_1 \in P_1$ and $e_2 \in C_1$ such that $e_1 \approx e_2$. By Proposition 4, $P_1$ and $C_1$ can be merged into a Hamiltonian $(s, t)$-path of $T_1(m, n)$. On the other hand, suppose that $(s, t)$ is a horizontal and nonboundary edge of $R_1$. Then, $R_1$ contains no Hamiltonian $(s, t)$-path. We next preform two horizontal cuts on $R_1$ to get three disjoint rectangular supergrid subgraphs $R_{11}, R_{12}$ and $R_{13}$ so that $R_{12}$ contains only $s$ and $t$, as depicted in Fig. 12(b). Let $p_1, q_1 \in R_{11}, p_2, q_2 \in R_{13}, r_1, r_2 \in \Delta_1$ such that $(s, p_1)$ and $(s, p_2)$ are vertical edges, $(t, q_1)$ and $(t, q_2)$ are vertical edges, and $(q_1, r_1)$ and $(q_2, r_2)$ are horizontal edges in $T_1(m, n)$, as shown in Fig. 12(b). We can easily construct a Hamiltonian $(p_1, q_1)$-path $P_1$ of $R_{11}$ and a Hamiltonian $(q_2, p_2)$-path $P_2$ of $R_{13}$ such that $P_1$ (resp., $P_2$) visits all boundary edges of $R_{11}$ (resp., $R_{13}$) except $(p_1, q_1)$ (resp., $(q_2, p_2)$) if $|V(R_{11})| > 2$ (resp., $|V(R_{13})| > 2$). We can see that $(\Delta_1, r_1, r_2)$ does not satisfy conditions (F1) and (F2). By Lemma 8, $\Delta_1$ contains a canonical Hamiltonian $(r_1, r_2)$-path $P_3$. Then, $s = p_1 \Rightarrow p_3 \Rightarrow p_2 \Rightarrow t$ forms a canonical Hamiltonian $(s, t)$-path of $T_1(m, n)$.

**Case 1.2:** $m - n + 1 > 2$. By Lemma 2, $R_1$ contains a canonical Hamiltonian $(s, t)$-path $P_1$. By Lemma 6, $\Delta_1$ contains a canonical Hamiltonian cycle $C_1$. Then, there exist two edges $e_1 \in P_1$ and $e_2 \in C_1$ such that $e_1 \approx e_2$. By Proposition 4, $P_1$ and $C_1$ can be combined into a canonical Hamiltonian $(s, t)$-path of $T_1(m, n)$.

**Case 2:** $s, t \in \Delta_1$. Let $w$ be the trapezoid corner of $T_1(m, n)$, and let $w'$ be a trapezoid corner of $\Delta_1$ different from $w$. Since $(T_1(m, n), s, t)$ does not satisfy conditions (F6)–(F7), we get that $s \sim w$ or $t \sim w$. Suppose that $(\Delta_1, s, t)$ satisfies condition (F1) or (F2). Let $\Delta_1' = \Delta_1 - \{w\}$ and let $R_1' = R_1 \cup \{w\}$. By Lemma 7, $\Delta_1'$ contains a canonical Hamiltonian $(s, t)$-path $P_1'$. By Lemma 1, $R_1'$ contains a canonical Hamiltonian cycle $C_1$. Then, there exists an edge $(u, v)$ in $C_1$ such that $u \sim w'$ and $v \sim w'$. By Proposition 5, $C_1$ and $w'$ can be merged into a Hamiltonian cycle $C_1'$. We can easily find two edges $e_1 \in C_1'$ and $e_2 \in P_1'$ such that $e_1 \approx e_2$. By Proposition 4, $P_1'$ and

![Fig. 11. Trapezoid supergrid graph in which there exists no Hamiltonian (s, t)-path for (a) condition (F6), and (b) condition (F7), where the solid lines indicate the longest (and let $P_n = a_{2n} \rightarrow a_{3n} \rightarrow \cdots \rightarrow a_{mn}$, and let $P_n = a_{2n} \rightarrow a_{3n} \rightarrow \cdots \rightarrow a_{mn}$. Thus, the lemma holds true.

We have proved the lemma holds true for $T(m, n) = T_1(m, n)$. For the type of trapezoid supergrid graphs $T_2(m, n)$, we can verify their Hamiltonicity by the same construction. For instance, Fig. 10(c) and Fig. 10(d) depict the canonical Hamiltonian cycles of $T_2(11, 4)$ and $T_2(13, 5)$, respectively. Thus, the lemma holds true.

Next, we will study the Hamiltonian connectivity of trapezoid supergrid graphs. Let $T(m, n)$ be a trapezoid supergrid graph, where $T(m, n) = T_1(m, n)$ or $T(m, n) = T_2(m, n)$. We first observe the conditions so that $HP(T(m, n), s, t)$ does not exist. For a 2$T_1$-trapezoid or 2$T_2$-trapezoid, the following condition implies that $HP(T(m, 2), s, t)$ does not exist.

(F6) $T(m, n)$ is a 2$T_1$-trapezoid or 2$T_2$-trapezoid, and $(s, t)$ is a vertical and nonboundary edge of $T(m, n)$ (see Fig. 11(a)).

For a trapezoid corner $w$ of $T(m, n)$, we can easily see that $HP(T(m, n), s, t)$ does not exist when $s, t \neq w, s \sim w$, and $t \sim w$.

(F7) $T(m, n)$ is a trapezoid supergrid graph for $n \geq 2$, $w$ is a trapezoid corner of $T(m, n)$, $s, t \neq w, s \sim w$, and $t \sim w$ (see Fig. 11(b)).

By similar arguments in proving Lemma 7, the following lemma can be verified.

**Lemma 16.** Let $T(m, n)$ be a trapezoid supergrid graph with $n \geq 2$, and let $s$ and $t$ be two distinct vertices of $T(m, n)$. Then, the following statements hold true:

1. If $(T(m, n), s, t)$ satisfies condition (F6), then $L(T_1(m, n), s, t) = \max\{2(m - s_2 + 1) - 1, 2s_2\}$ and $L(T_2(m, n), s, t) = \max\{2(m - s_2 + 1) - 1, 2s_2\} + 1$.

2. If $(T(m, n), s, t)$ satisfies condition (F7), then $L(T(m, n), s, t) = |V(T(m, n))| - 1$.

In the following, we will assume that $(T(m, n), s, t)$ does not satisfy conditions (F6) and (F7). Then, we will construct a canonical Hamiltonian $(s, t)$-path of $T(m, n)$. We first
does not satisfy conditions (F1) and (F2). By Lemma 8, \( T = T(s, t) \) is a Hamiltonian cycle for \( s, t \in T(m-n+1, n) \).

When \( n = 1 \), it is easy to construct a canonical Hamiltonian \( (s, t) \)-path of \( T(m, n) \). Thus, \( \Delta_1 \) contains a canonical Hamiltonian \( (s, t) \)-path \( P_1 \). By Lemma 1, \( R_1 \) contains a canonical Hamiltonian cycle \( C_1 \). Then, there exist two edges \( e_1 \in P_1 \) and \( e_2 \in C_1 \) such that \( e_1 \approx e_2 \). By Proposition 4, \( P_1 \) and \( C_1 \) can be combined into a canonical Hamiltonian \( (s, t) \)-path of \( T(m, n) \).

Case 3: \( s \in R_1 \) and \( t \in \Delta_1 \). In this case, we first find two vertices \( p \in R_1 \) and \( q \in \Delta_1 \) to satisfy that \( HP(R_1, s, p) \) and \( HP(\Delta_1, q, t) \) do exist, and \( p \approx q \). The vertices \( p \) and \( q \) can be easily computed. Let \( P_1 = HP(R_1, s, p) \) and \( Q_1 = HP(\Delta_1, q, t) \) be canonical Hamiltonian \( (s, p) \)-path and \( (q, t) \)-path of \( R_1 \) and \( \Delta_1 \), respectively. Then, \( P_1 \Rightarrow Q_1 \) forms a canonical Hamiltonian \( (s, t) \)-path of \( T(m, n) \).

We have considered any case to construct a canonical Hamiltonian \( (s, t) \)-path of \( T(m, n) \). This completes the proof of the lemma.

Next, we consider the other type of trapezoid supergrid graph \( T_2(m, n) \) as follows.}

**Lemma 18.** Let \( T_2(m, n) \) be a trapezoid supergrid graph with \( \frac{m}{n} \geq \frac{2}{n} \geq 2 \), and let \( s \) and \( t \) be two distinct vertices of \( T_2(m, n) \). If \( T_2(m, n) \) does not satisfy conditions (F6)–(F7), then \( T_2(m, n) \) contains a canonical Hamiltonian \( (s, t) \)-path, and, hence, \( HP(T_2(m, n), s, t) \) does exist.

**Proof:** By inspection, \( HP(T_2(m, 2), s, t) \) does exist when \( T_2(m, n) \) does not satisfy condition (F6). In the following, assume that \( n \geq 3 \). We first perform a vertical cut on \( T_2(m, n) \) to partition it into two disjoint subgraphs \( \Delta_2 = \Delta(n-1, n-1) \) and \( T_1(m-n+1, n) \), as depicted in Fig. 12(c). Let \( w \) and \( w' \) be the trapezoid corners of \( T_2(m, n) \) such that \( w \in \Delta_2 \) and \( w' \in T_1(m-n+1, n) \). Depending on the locations of \( s \) and \( t \), there are three cases:

**Case 1:** \( s, t \in \Delta_2 \). Let \( w_1 \) be a triangular corner of \( \Delta_2 \) different from \( w \). Suppose that \( (\Delta_2, s, t) \) satisfies condition (F1) or (F2). Since \( T_2(m, n) \) does not satisfy condition (F7), we get that \( s \sim w_1 \) and \( t \sim w_1 \). Let \( \Delta' = \Delta_2 - \{ w_1 \} \), and let \( T_1' = T_1(m-n+1, n) \cup \{ w_1 \} \). By Lemma 7, \( \Delta_2 \) contains a canonical Hamiltonian \( (s, t) \)-path \( P' \). By Lemma 15, \( T_1(m-n+1, n) \) contains a canonical Hamiltonian cycle \( C_1 \). Then, there exists an edge \( (u, v) \in C_1 \) such that \( u \sim w_1 \) and \( v \sim w_1 \). By Proposition 5, \( w_1 \) can be combined into \( C_1 \) to form a canonical Hamiltonian cycle \( C_1' \) of \( T_1' \).

Then, there exist two edges \( e' \in P' \) and \( e_1 \in C_1 \) such that \( e' \approx e_1 \). By Proposition 4, \( P' \) and \( C_1 \) can be combined into a canonical Hamiltonian \( (s, t) \)-path of \( T_2(m, n) \). On the other hand, suppose that \( (\Delta_2, s, t) \) does not satisfy conditions (F1) and (F2).

**Case 2:** \( s \in \Delta_2 \) and \( t \in T_1(m-n+1, n) \). Let \( p \in \Delta_2 \) and \( q \in T_1(m-n+1, n) \) such that \( HP(\Delta_2, s, p) \) and \( HP(T_1(m-n+1, n), q, t) \) do exist, and \( p \approx q \). The vertices \( p \) and \( q \) can be easily computed. Let \( P \) and \( Q \) be the constructed canonical Hamiltonian \( (s, p) \)-path and Hamiltonian \( (q, t) \)-path of \( \Delta_2 \) and \( T_1(m-n+1, n) \), respectively. Then, \( P \Rightarrow Q \) forms a canonical Hamiltonian \( (s, t) \)-path of \( T_2(m, n) \).

**Case 3:** \( s, t \in T_1(m-n+1, n) \). By Lemma 17, \( T_1(m-n+1, n) \) contains a canonical Hamiltonian \( (s, t) \)-path \( P \). By Lemma 6, \( \Delta_2 \) contains a canonical Hamiltonian cycle \( C \). Then, there exist two edges \( e \in P \) and \( e_1 \in C \) such that \( e \approx e_1 \). By Proposition 4, \( P \) and \( C \) can be combined into a canonical Hamiltonian \( (s, t) \)-path of \( T_2(m, n) \). For instance, Fig. 12(d) depicts the construction of such a canonical Hamiltonian \( (s, t) \)-path.

It follows from the above cases that a canonical Hamiltonian \( (s, t) \)-path of \( T_2(m, n) \) is constructed. Thus, the lemma holds true.

It immediately follows from Lemmas 17–18 that the following theorem holds true.

**Theorem 19.** Let \( T(m, n) \) be a trapezoid supergrid graph with \( n \geq 2 \), and let \( s \) and \( t \) be two distinct vertices of \( T(m, n) \), where \( T(m, n) = T_1(m, n) \) or \( T_2(m, n) \). If \( T(m, n), s, t \) does not satisfy conditions (F6)–(F7), then \( T(m, n) \) contains a canonical Hamiltonian \( (s, t) \)-path, and, hence, \( HP(T(m, n), s, t) \) does exist.

V. CONCLUDING REMARKS

In this paper, we provide constructive proofs to show that some shaped supergrid graphs, including triangular, parallelogram, and trapezoid, are Hamiltonian and Hamiltonian connected except few trivial conditions. These constructive proofs give linear time algorithms to construct the longest paths or Hamiltonian paths between any two distinct vertices of shaped supergrid graphs. A supergrid graph is called alphabet if its boundaries form an alphabet. There are 26 types of alphabet supergrid graphs. We can see from the structures of alphabet supergrid graphs that they can be
decomposed into triangular, parallelogram, and trapezoid supergrid subgraphs. In the future, we would like to apply our results to study the Hamiltonian connectivity of alphabet supergrid graphs.

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