Embedding Fault-Free Cycles of Various Lengths in $k$-ary $n$-cubes with Faulty Edges

Ying Zhou, Yang Liu, and Guodong Zhao

Abstract—Some parallel application such as image or signal processing is originally designated on cycle architecture owing to the simple structure and low degree. Thus it is important to have fault tolerant cyclic embedding in a host network. In this paper, we investigate the faulty embedding of circles onto a $k$-ary $n$-cube, denoted as $Q^n_k$, with odd $k \geq 3$ and $n \geq 3$ which is not bipartite. The faulty $k$-ary $n$-cube is considered that each vertex is incident with at least two healthy edges. We prove that there exist fault free cycles of every length varying from $k$ to $k^n$ in $Q^n_k$ even if $Q^n_k$ contains up to $4n - 5$ faulty edges.

Index Terms—Interconnection network, conditional edge fault, $k$-ary $n$-cubes, cyclic embedding.

I. INTRODUCTION

In recent decades, Very Large Scale Integration (VLSI) systems which have brought the parallel and distributed systems of thousands of processors to reality, have become widely used in data centers. There are quite a few interconnection networks proposed to serve as the underlying topologies of large scale multiprocessor systems [1], [2]. The topology is one of the crucial factors for an interconnection network because it determines the performance of the network or the distributed systems. So such a network usually has a regular degree. For example, every node is incident with the same number of links.

While numerous topologies have been proposed over the years, almost many networks have actually been constructed using topologies derived from a main family which is named torus or $k$-ary $n$-cubes [3], [4]. Networks such as torus or mesh, and $k$-ary $n$-cubes (see Figure 1), pack $N = k^n$ nodes in a regular $n$-dimensional grid with $k$ nodes in each dimension and edges between nearest neighbors. They span a range of networks from rings ($n = 1$) to binary $n$-cubes ($k = 2$), which is also known as hyper cubes. A network [5], [6], [7], [8] based on $k$-ary $n$-cubes is such that each node is incident with $2n$ edges, and consequently $k$ can be increased, in order to incorporate more processors, while keeping $n$ constant.

Another important advantage of increased distributed systems is a network’s ability to handle faults, such as failed vertices or edges. In the interconnection network, fault edges are inevitable. One measure of a network’s ability to handle faults is the number of edge disjoint or node disjoint paths allowed by the routing function between each source destination pair or among cycles. Studying faulty $k$-ary $n$-cubes has a rich history that spans many decades. Ashir and Stewart [3] studied the problem of embedding cycles in healthy $k$-ary $n$-cubes. Stewart and Xiang [9] showed that healthy $k$-ary $n$-cubes are edge-bipancyclic for arbitrary $k \geq 3$ and $n \geq 2$. They also showed that the healthy $k$-ary $n$-cubes with odd $k \geq 3$ contains a cycle of every possible length between $k - 1$ and $k^n$. In [4] Ashir and Stewart studied the problem of Hamiltonian cycle embedding in a $k$-ary $n$-cubes with a possibility of edge failures. Yang et al. [10] studied the problem of Hamiltonian path and linear array embedding in faulty $k$-ary $n$-cubes with odd $k \geq 3$. They proved that for two arbitrary distinct healthy vertices of a faulty $k$-ary $n$-cubes, there exist a fault free Hamiltonian path connecting these two vertices if the number of faulty vertices or edges is at most $2n - 3$. For even $k \geq 4$, Stewart and Xiang [11] considered the problem of embedding long paths in the $k$-ary $n$-cubes with faulty vertices and edges.

Cheng and Hao [12] considered an $n$-dimensional hypercube denoted by $Q_n$, with faulty edges $f_e \leq 3n - 8$ and $n \geq 5$. The hypercube is under the condition that each vertex is incident to at least two fault free edges, and every 4-cycle does not have any pair of non-adjacent vertices whose degrees are both two after removing the faulty edges. They proved that $Q_n$ has a fault free cycle of every even length from 4 to $2^n$. In [13] Dong et al. consider the problem...
of embedding cycles and paths into faulty 3-ary n-cubes. They show that when the faulty vertices and edges satisfy \( f_v + f_e \leq 2n - 2 \), there exists a cycle of any length from 3 to \( |V(Q_n^k) - f_v - f_e| \). Yang et al. [14] investigated the problem of embedding cycles of various lengths passing through prescribed paths in the k-ary n-cubes. They proved every path with length \( h (1 \leq h \leq 2n-1) \) in the k-ary n-cube lying on cycles of every length from \( h + (k-1)(n-1)/2 + k \) to \( k^n \) inclusive for \( n \geq 2 \) and \( k \geq 2 \) with \( k \) odd. In another work [15], Zhang et al. considered the problem of a fault-free hamiltonian cycle passing through prescribed edges in a k-ary n-cube \( Q_n^k \) with some faulty edges. For any \( n \geq 2 \) and \( k \geq 3 \), let \( F \subseteq E(Q_n^k) \) and \( P \subseteq E(Q_n^k) \setminus F \) with \( |F| \leq 2n-2 \) and \( |F| \leq 2n - \left( |P| + 2 \right) \). Then there exists a hamiltonian cycle passing through all edges of \( P \) in \( Q_n^k \setminus F \) if and only if the subgraph induced by \( P \) consists of pairwise vertex-disjoint paths.

In recent years, the Hamiltonian cycle, path embedding or extra connectivity of the k-ary n-cubes have been researched in many literatures (see, for example, [16], [17], [18]). Under similar conditions, let \( Q_n^k \) be a non-bipartite k-ary n-cubes for \( k \geq 3 \) and \( n \geq 3 \) with \( k \) odd, in which each vertex is incident with at least two healthy edges. In this paper we will prove that \( Q_n^k \) with at most \( 4n-5 \) faulty edges has fault free cycles of every length between \( k \) and \( k^n \) so that we call \( F \) as a conditional faulty edges set.

Ashir and Stewart [4] showed that, with only edge faults and under the condition that every node is incident with at least two fault-free edges, a wounded k-ary n-cubes still has a Hamiltonian circuit, provided that there are no more than \( 4n-5 \) faulty edges.

Theorem 1.1: (see [4]) Let \( k \geq 4 \) and \( n \geq 2 \), or let \( k = 3 \) and \( n \geq 3 \). If \( Q_n^k \) has at most \( 4n-5 \) faulty edges, and every vertex is incident with at least two healthy edges, then \( Q_n^k \) has a Hamiltonian circuit.

II. DEFINITION AND TERMINOLOGY

Generally, an interconnection network is represented by an undirected simple graph \( G \). Given a graph \( G \), we denote it as \( G = (V, E) \) where \( V = V(G) \) is the vertex set and \( E = E(G) \) is the edge set respectively. We say that a graph is regular if the degree of every vertex \( v \in V(G) \) is equal which can be expressed as \( d_G(v) = k \). A graph \( G \) is bipartite if \( V(G) \) can be divided into two partite sets such that every edge has two end vertices indifferent partite sets.

A path denoted by \( < v_1, v_2, \ldots, v_k > \) is a sequence of adjacent vertices where all the vertices are distinct but with a possibility of \( v_1 = v_k \). We say that a path is a Hamiltonian path if it traverses all the vertices of \( G \) exactly once. A cycle is a path that begins and ends with the same vertex. A Hamiltonian cycle is a cycle which includes all the vertices of \( G \). A graph \( G \) is Hamiltonian connected if, for any two arbitrary vertices \( u \) and \( v \) in \( G \), there is a Hamiltonian path connecting \( u \) and \( v \). A graph \( G \) is pan connected if, for any two arbitrary vertices \( x \) and \( y \) in \( G \), there is a path of length from \( d_G(x,y) \) to \( |V(G)| - 1 \) connecting \( x \) and \( y \).

A graph \( G \) is pan cyclic if it contains cycles of every length from the shortest cycle length of \( G \) as \( g(G) \) to \( |V(G)| \) and edge pan cyclic if every edge lies on a cycle of every length from \( g(G) \) to \( |V(G)| \). A bipartite graph \( G \) is bipan cyclic if it contains cycles of every even length from \( g(G) \) to \( |V(G)| \) and edge bipan cyclic if every edge lies on cycle of every even length from \( g(G) \) to \( |V(G)| \).

The k-ary n-cube, denoted by \( Q_n^k \) (\( k \geq 2 \) and \( n \geq 2 \)), is a graph consisting of \( k^n \) vertices, each of which has the form \( u = u_{n-1}u_{n-2} \cdots u_0 \), where \( u_i \in \{ 0, 1, \ldots, k-1 \} \) for \( i \in \{ 0, 1, \ldots, n-1 \} \). Two vertices \( u = u_{n-1}u_{n-2} \cdots u_0 \) and \( v = v_{n-1}v_{n-2} \cdots v_0 \) are adjacent if and only if there exists an integer \( j \in \{ 0, 1, \ldots, n-1 \} \) such that \( u_j = v_j \pm 1 \) (modulo \( k \)) and for every \( i \in \{ 0, 1, \ldots, j-1, j+1, \ldots, n-1 \} \) there exists \( u_i = v_i \). Such an edge \( (u, v) \) is called a \( j \)-dimensional edge.

We can partition \( Q_n^k \) along the dimension \( j \), by deleting all the \( j \)-dimensional edges, into \( k \) disjoint sub cubes as \( Q_n^k[0], Q_n^k[1], \ldots, Q_n^k[k-1] \), (for ease of notation, abbreviated as \( Q[0], Q[1], \ldots, Q[k-1] \), if there is no ambiguity). If \( Q[x] \), for every \( i \in \{ 0, 1, \ldots, k-1 \} \), is a sub graph of \( Q_n^k \) induced by the vertices labeled by \( u_{n-1}u_{n-2} \cdots u_ju_{j+1} \cdots u_0 \) (see Figure 2), then it is clear that each \( Q[x] \) is isomorphic to \( Q_{n-1}^k \) for \( 0 \leq i \leq k-1 \). Note that \( Q_n^k \) can be divided into \( k \) disjoint copies of \( Q_n^k \) along \( n \) different dimensions. And vice versa we can combine \( k \) k-ary \((n-1)\)-cubes in order to construct a k-ary n-cubes.

Fig. 2. \( Q_n^k \) is divided into \( Q[0], Q[1], \ldots, Q[k-2], Q[k-1] \).

We consider the fault-tolerance of a graph \( G \). The following definitions are cited from the reference [19]. Let \( F \) be a set of faulty edges of \( Q_n^k \). We call \( F \) a conditional faulty edge set of \( Q_n^k \) if every vertex in \( Q_n^k - F \) is incident with at least two healthy edges. \( F^j \) indicates the set of faulty \( j \)-dimensional edges for \( j \in \{ 0, 1, \ldots, n-1 \} \). Then we refer to \( F \) as \( \bigcup_{j=0}^{n-1} F^j \). For \( p, r \in \{ 0, 1, \ldots, k-2, k-1 \} \) and \( p = r \pm 1 \) (modulo \( k \)), we use \( F^{p+r} \in F^j \) to denote the set of faulty \( j \)-dimensional edges between \( p \) and its neighbor \( r \) sub cubes. On the other hand, for each \( i \in \{ 0, 1, \ldots, k-1 \} \), we refer to \( F_i \) as \( F \cap E(Q[i]) \).

III. PAN CYCLIC EMBEDDING IN THE CONDITIONAL FAULTY K-ARY N-CUBES

Let us now proceed to the proof of our main theorem. We begin by proving the inductive step, and then we return to the base cases of the induction.

A. Preliminaries

Theorem 3.1: (see [20]) Let \( k \) be an odd integer with \( k \geq 3 \), and let \( n \geq 3 \) be an integer. Let \( Q_n^k \) be a k-ary n-cube with faulty vertices \( f_v \) and faulty edges \( f_e \) where \( 0 \leq f_v + f_e \leq 2n - 3 \). We call \( F \) as a faulty vertex and edge set of \( Q_n^k \) if every vertex in \( Q_n^k - F \) is incident with at least two healthy edges.
edges. For two arbitrary healthy vertices, there exists a path whose length is from \((n(k - 1) - 1)\) to \([V(Q^k_n - F)]\) - 1 connecting these two vertices in the faulty \(Q^k_n\). The k-ary n-cubes is also referred to as 2n - 3 faults \((n(k - 1) - 1)\) pan connected.

Yang et al. investigate the fault-tolerant capabilities of the k-ary n-cubes for odd integer k with respect to the Hamiltonian and Hamiltonian connected properties. By a simple mathematical induction of Theorem 8 in [10], we have the following theorem.

**Theorem 3.2:** (see [10]) Let \(F\) be a faulty set with vertices and edges, and let \(k \geq 3\) be an odd integer. When \(|F| \leq 2n - 2\), it is shown that there exists a Hamiltonian cycle in a wounded k-ary n-cube. In addition, when \(|F| \leq 2n - 3\), it is proved that, for two arbitrary nodes, there exists a Hamiltonian path connecting these two nodes in the wounded k-ary n-cubes.

In the following lemmas, namely Lemmas 3.1-3.6, which are useful for the proof of the main theorem, we shall construct cycles of various lengths in conditional faulty k-ary (n - 1)-cube. Before going any further, we will consider an arrangement of the \(4n - 5\) edge faults in the k-ary n-cubes.

**Lemma 3.1:** Let \(Q^k_n\) be a k-ary n-cubes with \(n \geq 3\) and \(4n - 5\) edge faults, then there is an m-dimension where exist the most faulty edges in \(Q^k_n\) and the number of the most faulty edges is no less than three denoted by \(|F^m| \geq 3\).

**Proof:** Suppose the most number of faulty edges which is denoted by \(|F|\) is in the m-dimension. To consider the limit of \(|F^m| \geq \left\lceil \frac{4n - 5}{n} \right\rceil = \left\lceil \frac{4 - \frac{5}{n}}{1} \right\rceil\) and when it satisfies \(n \geq 3\) hence we compute \(|F^m| \geq 3\). It means that the number of the most faulty edges in the m-dimension is at least three.

We say that \(Q^k_n\) is partitioned along the dimension m for some \(m \in \{0, 1, \ldots, n - 1\}\) by deleting all the m-dimension edges into k disjoint sub cubes \(Q[0], Q[1], \ldots, Q[k - 1]\). Let \(v\) be a vertex in \(Q[i]\), we denote it as \(u_i\). We also use \(u\) which is adjacent to \(u_j\) to stand for the vertex belongs to \(Q[j]\). Furthermore, if \((u_i, v)\) is an edge of \(Q[i]\), then \((u_j, v)\) is the edge which belongs to \(Q[j]\).

Let \(F\) be a set of faulty edges of \(Q^k_n\). Assume that \(F_1 = F \cap E(Q[i])\) where \(i \in \{0, 1, \ldots, k - 2, k - 1\}\). Obviously after deleting all the m-dimension edges, we can estimate the number of fault edges \(|F_0 \cup F_1 \cup \cdots \cup F_{k - 2} \cup F_{k - 1}| = 4n - 5 - |F^m| \leq 4n - 8\) in the k disjoint sub cubes.

We may suppose \(Q[0]\) is the sub cube with the most faulty edges while suppose \(Q[s]\) and \(Q[t]\) are the second and the third most faulty edges sub cubes where \(s, t \in \{1, 2, \ldots, k - 2, k - 1\}\). Taking a step back, without loss of generality we assume that \(|F_0| \geq |F_1| \geq |F_t|\) as illustrated in Figure 3.

![Fig. 3. Arrangement of 4n - 5 edge faults in the k-ary n-cubes](https://example.com/figure3.png)

**Lemma 3.2:** Let \(F\) be a set of faulty edges of \(Q^k_n\). Assume that \(F_i = F \cap E(Q[i])\) where \(i \in \{0, 1, \ldots, k - 2, k - 1\}\). If \(Q[0]\) is the sub cube with the most faulty edges, then there exist \(|F_s| \leq 2n - 4\) and \(|F_t| \leq 2n - 5\) where \(s, t \in \{1, 2, \ldots, k - 2, k - 1\}\).

**Proof:** We will give a proof by contradiction. Suppose that \(|F_s| \geq 2n - 3\) then we have \(|F_t| \geq 2n - 3\). Clearly, there exists \(|F_0 \cup F_s| \geq 4n - 6\). Note that, Lemma 3.1 implies that \(|F_0 \cup F_1 \cup \ldots \cup F_{k - 2} \cup F_{k - 1}| \leq 4n - 8\), so we obtain a contradiction. Hence, we have the assertion of \(|F_s| \leq 2n - 4\) where \(s \in \{1, 2, \ldots, k - 2, k - 1\}\).

We may use a similar construction in the proof of \(|F_{k - 1}| \leq 2n - 5\). By assuming \(|F_t| \geq 2n - 4\) we have \(|F_s| \geq |F_t| \geq 2n - 4\). Similarly, there exists \(|F_0 \cup F_t| \geq 4n - 8\) which contradicts that \(|F_0 \cup F_1 \cup \ldots \cup F_{k - 2} \cup F_{k - 1}| \leq 4n - 8\) since we also have the assumption of \(|F_0| \geq |F_s| \geq 2n - 3\). This concludes the proof of this lemma.

**B. Cycles of length from k to \(2 \times k^{n - 1}\) embedding in conditional faulty k-ary n-cubes**

In this section, we will prove the main assertion.

**Lemma 3.3:** Given an odd integer \(k \geq 3\), let \(F\) be a set of faulty edges of \(Q^k_n\). If each vertex of the k-ary n-cubes is incident with at least two healthy edges, then there exist a cycle of length from \(k\) to \(2 \times k^{n - 1}\) in the fault \(Q^k_n - F\) if it has at most \(4n - 5\) edge faults.

**Proof:**

From Lemma 3.2 there exists the implication of \(|F_s| \leq 2n - 4 = 2(n - 1) - 2\). As \(Q[s]\) is isomorphic to \(Q_{n - 1}^k\) where \(n - 1 \geq 2\), thus by Theorem 3.2, it is shown that there exists cycles in the faulty \(Q[s] - F_s\) whose length is from \(k\) to \(k^{n - 1}\). The k-ary (n - 1)-cube is also referred as pan cyclic.

Consequently, we will prove the following assertion that there exist a cycle of length from \(k^{n - 1} + 1\) to \(2 \times k^{n - 1}\) in the faulty \(Q^k_n - F\).

**Case 1:** \(|F_s| = 2n - 4\)

In this case, \(|F_0| \geq |F_s| = 2n - 4\) and \(|F_0 \cup F_1 \cup \ldots \cup F_{k - 2} \cup F_{k - 1}| = 4n - 5 - |F^m| \leq 4n - 8\), suffice it to say that there exists \(|F_0| = |F_1| = 2n - 4\) and \(F^m = 3\). By the induction hypothesis, the other \(|F| = 0\) can be constructed provided that \(i \in \{1, 2, \ldots, k - 2, k - 1\}\) and \(i \neq s\). From Theorem 3.2, there exist a cycle \(C_c\) of length from \(k\) to \(k^{n - 1}\) in the faulty \(Q[s] - F_s\). On the other hand, we suppose that its neighbor is \(r\) sub cubes where \(r = s \pm 1\) (modulo k). We select two adjacent vertices such as \(u_s, v_s\) lying on \(C_c\) which satisfy \((u_s, u_r), (v_s, v_r) \notin F^{m,r}\).

According to Theorem 3.2, with the aid of path \(P[u_s, v_s] = C_s - (u_s, v_s)\), the length \(l_s\) of \(P[u_s, v_s]\) is from \(k - 1\) to \(k^{n - 1} - 1\). For an illustrative example of \(F_0 = 0\), by Theorem 3.1 there exists a path \(P[u_s, v_s]\) in \(Q[r]\) whose length is \(l_r\) holding for \(l_r \in \{(k - 1)(n - 1) - 1, \ldots, k^{n - 1} - 1\}\). Connect \(l_s, l_r\) as illustrated in Figure 4, so that we get a cycle whose length is \(l = l_s + l_r + 2 \in \{(k + (k - 1)(n - 1) - 1), \ldots, 2 \times k^{n - 1}\}\).

When \(n \geq 3\) and an odd integer \(k \geq 3\), it will be seen that \(k + (k - 1)(n - 1) \leq k^{n - 1} + 1\). As a result, we can get a cycles whose length is from \(k^{n - 1} + 1\) to \(2 \times k^{n - 1}\) in the faulty \(Q^k_n - F\).

**Case 2:** \(|F_s| \leq 2n - 5\)

If \(n \geq 3\) and \(|V(Q[s])| - |F^{m,r}| \geq k^{n - 1} - (4n - 5) \geq 2\), without loss of generality, we select two vertices such as \(u_s, v_s\),
Since $|F_s|, |F_r| \leq 2n - 5 = 2(n - 1) - 3$, by Theorem 3.1, we note $l_s$ in $Q[s] - F_s$ and $l_r$ in $Q[r] - F_r$ as $((n - 1)(k - 1) - 1)$ fault path connected whose length is $(k - 1)(n - 1) - 1 \leq l_s, l_r \leq k^{n-1} - 1$. Connecting $l_s, l_r$ then we get that $l = l_s + l_r + 2 \in \{2(k - 1)(n - 1), \ldots, 2 \times k^{n-1} + 1\}$.

If there exist $n \geq 3$ and $2(k - 1)(n - 1) \leq k^{n-1} + 1$, then the Lemma is as required. This completes the proof. $

\textbf{C. Cycles of length from } 2 \times k^{n-1} + 1 \text{ to } k^n \text{ embedding in conditional fault } k\text{-ary } n\text{-cubes}

\textbf{Lemma 3.4:} Assume $|F_s| \leq 2n - 5$ for $s \in \{1, 2, \ldots, k - 2, k - 1\}$. If there is a cycle $C_0$ whose length is $k^{n-1} - 1$ or $k^{n-1}$ in $Q[0]$, then there exists a cycle of length from $2 \times k^{n-1} + 1$ to $k^n - 1$ in the conditional fault $Q[k] - F$.

\textbf{Proof:}

We say that the length of $C_0$ is $l_0$. Then there is $l_0 = k^{n-1} - 1$ or $l_0 = k^{n-1}$.

Suppose first that $l_0 = k^{n-1} - 1$ and it is enough to consider that the length of $C_0$ is more than the number of the largest faulty edges in the $m$-dimension which can be denoted by $l_0 = k^{n-1} - 1 \geq 4n - 5 \geq |F[0]|$. Hence, there would exist some edge name $(u_0, v_0)$ where $(u_0, u_1), (v_0, v_1) \not\in F[0]$ or $(u_0, u_{k-1}), (v_0, v_{k-1}) \not\in F[k-1]$.

Without loss of generality, assume $(u_0, u_1), (v_0, v_1) \not\in F[0], F[1]$. Using $|F[1]| \leq 2n - 5 = 2(n - 1) - 3$ in $Q[1]$ and $k^{n-1} - (2n - 5) \geq (k - 1)(n - 1) - 1$, by Theorem 3.1 we connect these two vertices in the faulty $Q[1]$ and consequently get a path $P[u_1, v_1]$ whose length is from $(n - 1)(k - 1) - 1$ to $|V[Q[1]]| - 1$. We use $l_1$ to stand for the length of path $P[u_1, v_1]$ where $(k - 1)(n - 1) - 1 \leq l_1 \leq k^{n-1} - 1$.

Suppose next $P[u_0, v_0] = C_0 - (u_0, v_0)$. If connect the paths of $P[u_1, v_1]$ and $P[u_0, v_0]$ then we can obtain a cycle $C_1$ whose length is $l_1 = (l_0 - 1) + l_1 + 2 = l_0 + l_1 + 1 \in \{(k - 1)(n - 1) + k^{n-1} - 1, \ldots, 2 \times k^{n-1} + 1\}$, and it is shown in Figure 5.

Finally, the following cases can be done as this: connect $P[x_{k-2}, y_{k-2}]$ and $P[u_2, v_2]$ and so on which is shown in Figure, we can obtain a whole cycle $C$ whose length is $L = l_0 + l_1 + \ldots + l_{k-1} + k - 1 \in \{(k - 1)(n - 1) + \ldots, k^{n-1} - 1, \ldots, \}$. The conclusion is as required.

From Lemma 3.4, the following corollary is immediate.

\textbf{Corollary 3.1:} Let $|F_s| \leq 2n - 5$ and a path $P[u_0, v_0]$ of length $k^{n-1} - 2$ or $k^{n-1}$ in $Q[0]$ where $(u_0, u_1), (v_0, v_1) \not\in F[0], F[1]$ or $(u_0, u_{k-1}), (v_0, v_{k-1}) \not\in F[k-1]$, then there exist a cycle of length from $2 \times k^{n-1}$ to $k^n - 1$ in the conditional fault $Q[k] - F$.

\textbf{Lemma 3.5:} Let $|F[0]| = 4n - 8$ then there exist an edge $(u_0, v_0)$ in $F[0]$ where $(u_0, u_1), (v_0, v_1) \not\in F[0], F[1]$ or $(u_0, u_{k-1}), (v_0, v_{k-1}) \not\in F[k-1]$.

\textbf{Proof:} As $|F[0]| \geq 3$ from Lemma 3.1, clearly, we only need to consider the case for $|F[0]| = 4n - 8 - |F[0]| = 3$.

First, assume that there exist two distinct and non-adjacent fault edges in $Q[0]$, the conclusion is true apparently. Suppose next that any two fault edges are adjacent, then they are incident with the same vertex. Assume this vertex is $u_0$, the degree of $u_0$ is $2n - 2$ in $Q[0]$ after deleting all the $m$-dimension edges, this also mean that $u_0$ is incident with $2n - 2$ both healthy and faulty edges. By taking into consideration of $4n - 8 \leq 2n - 2$ and $n \geq 3$, hence, there is $n = 3$. Since $|F[0]| = 3$, there exist at least one healthy edge of $(u_0, u_1)$. We now consider the conditional faulty $k$-ary $n$-cubes that each vertex is incident with at least two healthy edges. So if $u_0$ is incident with at most four faulty edges, and then there existing a faulty edge of $(u_0, v_0)$ and an edge of $(u_0, v_1)$ which is an inevitable healthy edge as shown in Figure 7.

\textbf{Lemma 3.6:} There exist a cycle of length from $2 \times k^{n-1} + 1$ to $k^n$ in the conditional fault $Q[k] - F$.

\textbf{Proof:} According to Theorem 1 there exist a cycle of length $k^n$ in the faulty $Q[k] - F$. As a result, we need find
If $|F_0| \geq |F_s|$ and $|F_0 \cup F_1 \cup \ldots \cup F_{k-2} \cup F_{k-1}| \leq 4n-5-|F_m| \leq 4n-8$, then there exists $|F_0| = |F_s| = 2n-4$ and $|F_m| = 3$. From Theorem 3.2, given an odd integer of $k \geq 3$, where $|F_0| = |F_s| = 2n-4$, there exists a cycle $C_0$ of length of $k^{n-1}$ in the fault $Q_0 - F_0$ and a cycle $C_s$ of length of $k^{n-1}$ in the faulty $Q_s - F_s$ for $s \in \{1, 2, \ldots, k-2, k-1\}$.

In contrast, without loss of generality consider the interconnections between $Q[0]$ and $Q[1]$ shown in Figure 9. If there are two healthy edges of $(u_0, v_0)$ and $(v_0, w_0)$ on the cycle $C_0$, then it is satisfied that $(u_0, u_1)$, $(v_0, v_1)$ and $(u_1, w_1)$ are three healthy edges. Suppose that $(u_1, v_1)$ and $(v_1, w_1)$ are healthy edges in $Q[1]$, with the aid of marking some edges which are incident with $v_1$ as temporary faulty ones except $(u_1, v_1)$ and $(v_1, w_1)$, so as to have $v_1$ be incident with at most three healthy edges. Given $|F_1| \leq 2n-4$, therefore, there is $|F_1| \leq (2n-4) + (2n-5) = 4n-9$ after joining the new temporary fault edges. According to Theorem 1.1, then there exists a Hamiltonian cycle where $(u_1, v_1)$ or $(v_1, w_1)$ lies on in $Q[1]$.

On the other hand, we provide an assumption that either $(u_1, v_1)$ or $(v_1, w_1)$ is a faulty edge and without loss of generality we may assume $(u_1, v_1)$ is the faulty one.

Next, we suppose that $(u_1, v_1)$ is a pseudo-healthy edge, so it is easy to see that $|F_1 - \{u_1, v_1\}| \leq 2n-5$. By Theorem 3.1, there exists a path $P[u_1, v_1]$ in $Q[1]$ whose length is $l_1$ holding for $l_1 \in \{(k-1)(n-1) - 1, \ldots, k^{n-1} - 1\}$. Connecting $C_0$ and $P[u_1, v_1]$ with $(u_0, u_1)$ and $(v_0, v_1)$, then we get that $L_1 = l_0 + l_1 + 1 \in \{k^{n-1} + (k-1)(n-1), \ldots, 2k^{n-1}\}$.

Finally, the following cases can be done as this: connect $P[u_0, v_0]$ and $P[v_0, w_0]$ and so on we can obtain a whole cycle $C$ whose length is $L \in \{(k-1)(k-1)(n-1) + k^{n-1}, \ldots, k \times k^{n-1}\} = \{(k-1)^2(n-1) + k^{n-1}, \ldots, k^{n}\}$.

Because of $(k-1)^2(n-1) + k^{n-1} \leq 2k^{n-1}$, it implies that $L \in \{2k^{n-1}, 2k^{n-1} + 1, \ldots, k^{n}\}$.

By the above cases, we complete the proof.

IV. CONCLUSION

In conclusion, by Lemma 3.3 and Lemma 3.6, the fault-tolerant pan cyclicity of $Q_n$ is given in the following theorem.

Theorem 4.1: Let $Q_n^k$ be a $k$-ary $n$-cube with odd $k \geq 3$ and $n \geq 3$ which is not bipartite. We consider the faulty $k$-ary $n$-cubes that each vertex is incident with at least two healthy edges. We prove that there exist a fault-free cycle of every length from $k$ to $k^n$ in $Q_n^k$ even if it has up to $4n-5$ edge faults. It also means that the $Q_n^k$ is conditional $(4n-5)$ edge fault pan cyclic.

In this paper, we investigate the $k$-ary $n$-cubes for an odd $k \geq 3$ with some faulty edges such that each vertex is incident with at least two healthy edges. We proved that such a $k$-ary $n$-cube with at most $4n-5$ faulty edges contains

(Advance online publication: 20 November 2017)
a cycle whose length varies in different $k$ to $k^n$. We show that the conditional fault-tolerant capability of the $k$-ary $n$-cubes is excellent in terms of pan cyclic embedding which can be used to develop corresponding applications on the distributed-memory parallel system in the environment of $k$-ary $n$-cubes. Our further work is to consider whether the above result is optimal in conditional faulty $k$-ary $n$-cubes with odd $k \geq 3$.

Interconnection networks are often composed of hundreds (or thousands) of components, just like routers, channels, and connectors. They collectively have failure rates higher than is acceptable for the application. Thus, these networks must employ error control to continue operation without interruption. The $k$-ary $n$-cubes are particularly easy to map to physical space with uniformly short edges. The simplest case is when the network is a cycle with the same number of dimensions as the physical dimensions of the packaging technology. A $k$-ary $n$-cube can be transformed into an express cube network augmented with a number of long or express cycles. By routing packets that must traverse a long distance in a dimension over the express channels. The header latency can be reduced to nearly the channel latency limit. Because the number of express channels can be controlled to match the bisection width of the network, this reduction in header latency can be achieved without increasing serialization latency. So these are also used to provide fault tolerance.

ACKNOWLEDGMENT

This work was supported in part by the Science and Technology Development Strategy Research Fund of Tianjin under Grant No. 15ZLLZF00070.

REFERENCES


