Embedding Fault-Free Cycles of Various Lengths in k-ary n-cubes with Faulty Edges

Ying Zhou, Yang Liu, and Guodong Zhao

***Abstract*—Some parallel application such as image or signal processing is originally designated on cycle architecture owing to the simple structure and low degree. Thus it is important to have fault tolerant cyclic embedding in a host network. In this paper, we investigate the faulty embedding of circles onto a k-ary n-cube, denoted as Q_n^k with odd $k \geq 3$ and $n \geq 3$ which is not bipartite. The faulty k-ary n-cube is considered that each vertex is incident with at least two healthy edges. We prove that there exist fault free cycles of every length varying from k to k^n in Q_n^k even if Q_n^k contains up to $4n-5$ faulty edges.**

***Index Terms*—Interconnection network, conditional edge fault, k-ary n-cubes, cyclic embedding.**

I. Introduction

IN recent decades, Very Large Scale Integration (VLSI) systems which have brought the parallel and distributed systems of thousands of processors to reality, have become widely used in data centers. There are quite a few interconnection networks proposed to serve as the underlying topologies of large scale multiprocessor systems [1], [2]. The topology is one of the crucial factors for an interconnection network because it determines the performance of the network or the distributed systems. So such a network usually has a regular degree. For example, every node is incident with the same number of links.

While numerous topologies have been proposed over the years, almost many networks have actually been constructed using topologies derived from a main family which is named torus or k-ary n-cubes [3], [4]. Networks such as torus or mesh, and k-ary n-cubes (see Figure 1), pack $N = k^n$ nodes in a regular n-dimensional grid with k nodes in each dimension and edges between nearest neighbors. They span a range of networks from rings ($n = 1$) to binary n-cubes ($k = 2$), which is also known as hyper cubes. A network [5], [6], [7], [8] based on k-ary n-cubes is such that each node is incident with $2n$ edges, and consequently k can be increased, in order to incorporate more processors, while keeping n constant.

Another important advantage of increased distributed systems is a network's ability to handle faults, such as failed vertices or edges. In the interconnection network, fault edges are inevitable. One measure of a network's ability to handle faults is the number of edge disjoint or node disjoint paths allowed by the routing function between each source destination pair or among cycles. Studying faulty k-ary n-cubes has a rich history that spans many decades. Ashir and Stewart [3] studied the problem of embedding cycles in healthy k-ary n-cubes. Stewart and Xiang [9] showed that healthy k-ary n-cubes are edge-bipancyclic for arbitrary $k \geq 3$ and $n \geq 2$. They also showed that the healthy k-ary n-cubes with odd $k \geq 3$ contains a cycle of every possible length between $k-1$ and k^n. In [4] Ashir and Stewart studied the problem of *Hamiltonian cycle embedding* in a k-ary n-cubes with a possibility of edge failures. Yang et al. [10] studied the problem of *Hamiltonian path and linear array embedding* in faulty k-ary n-cubes with odd $k \geq 3$. They proved that for two arbitrary distinct healthy vertices of a faulty k-ary n-cubes, there exist a fault free Hamiltonian path connecting these two vertices if the number of faulty vertices or edges is at most $2n-3$. For even $k \geq 4$, Stewart and Xiang [11] considered the problem of *embedding long paths* in the k-ary n-cubes with faulty vertices and edges.

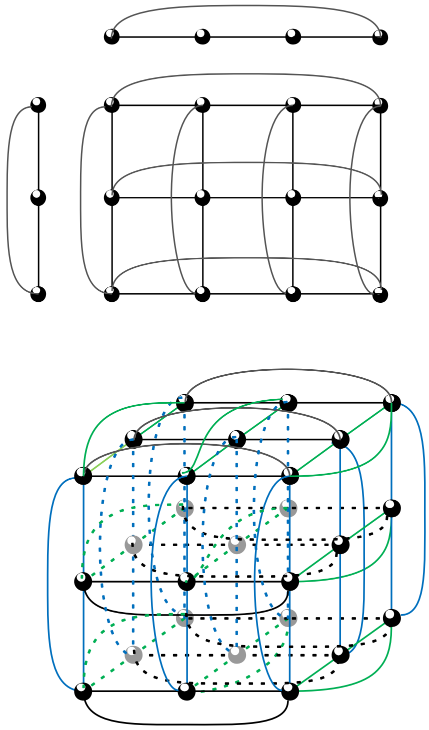

Fig. 1. The torus and k-ary n-cubes

Cheng and Hao [12] considered an n-dimensional hypercube denoted by Q_n with faulty edges $f_e \leq 3n-8$ and $n \geq 5$. The hypercube is under the condition that each vertex is incident to at least two fault free edges, and every 4-cycle does not have any pair of non-adjacent vertices whose degrees are both two after removing the faulty edges. They proved that Q_n has a fault free cycle of every even length from 4 to 2^n. In [13] Dong et al. consider the problem

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of embedding cycles and paths into faulty 3-ary n-cubes. They show that when the faulty vertices and edges satisfy $f_v + f_e \leq 2n-2$, there exists a cycle of any length from 3 to $|V(Q_n^3 - f_v - f_e)|$. Yang et al. [14] investigated the problem of embedding cycles of various lengths passing through prescribed paths in the k-ary n-cubes. They proved every path with length h $(1 \leq h \leq 2n-1)$ in the k-ary n-cube lying on cycles of every length from $h+(k-1)(n-1)/2+k$ to k^n inclusive for $n \geq 2$ and $k \geq 5$ with k odd. In another work [15], Zhang et al. considered the problem of a fault-free hamiltonian cycle passing through prescribed edges in a k-ary n-cube Q_n^k with some faulty edges. For any $n \geq 2$ and $k \geq 3$, let $F \subset E(Q_n^k)$ and $P \subset E(Q_n^k) \setminus F$ with $|P| \leq 2n-2$ and $|F| \leq 2n - (|P|+2)$. Then there exists a hamiltonian cycle passing through all edges of P in $Q_n^k \setminus F$ if and only if the subgraph induced by P consists of pairwise vertex-disjoint paths.

In recent years, the Hamiltonian cycle, path embedding or extra connectivity of the k-ary n-cubes have been researched in many literatures (see, for example, [16], [17], [18]). Under similar conditions, let Q_n^k be a non-bipartite k-ary n-cubes for $k \geq 3$ and $n \geq 3$ with k odd, in which each vertex is incident with at least two healthy edges. In this paper we will prove that Q_n^k with at most $4n-5$ faulty edges has fault free cycles of every length between k and k^n so that we call F as a conditional faulty edges set.

Ashir and Stewart [4] showed that, with only edge faults and under the condition that every node is incident with at least two fault-free edges, a wounded k-ary n-cubes still has a Hamiltonian circuit, provided that there are no more than $4n-5$ faulty edges.

Theorem 1.1: (see [4]) Let $k \geq 4$ and $n \geq 2$, or let $k = 3$ and $n \geq 3$. If Q_n^k has at most $4n-5$ faulty edges, and every vertex is incident with at least two healthy edges, then Q_n^k has a Hamiltonian circuit.

II. Definition and terminology

Generally, an interconnection network is represented by an *undirected simple* graph G. Given a graph G, we denote it as $G = (V, E)$ where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set respectively. We say that a graph is regular if the degree of every vertex $v \in V(G)$ is equal which can be expressed as $d_G(v) = k$. A graph G is *bipartite* if $V(G)$ can be divided into two partite sets such that every edge has two end vertices indifferent partite sets.

A *path* denoted by $< v_1, v_2, \cdots, v_k >$ is a sequence of adjacent vertices where all the vertices are distinct but with a possibility of $v_1 = v_k$. We say that a path is a *Hamiltonian path* if it traverses all the vertices of G exactly once. A *cycle* is a path that begins and ends with the same vertex. A *Hamiltonian cycle* is a cycle which includes all the vertices of G. A graph G is *Hamiltonian connected* if, for any two arbitrary vertices u and v in G, there is a Hamiltonian path connecting u and v. A graph G is *pan connected* if, for any two arbitrary vertices x and y in G, there is a path of length from $d_G(x, y)$ to $|V(G)| - 1$ connecting x and y.

A graph G is *pan cyclic* if it contains cycles of every length from the shortest cycle length of G as $g(G)$ to $|V(G)|$ and *edge pan cyclic* if every edge lies on a cycle of every length from $g(G)$ to $|V(G)|$. A bipartite graph G is *bipancyclic* if it contains cycles of every even length from $g(G)$ to $|V(G)|$ and *edge bipancyclic* if every edge lies on cycle of every even length from $g(G)$ to $|V(G)|$.

The k-ary n-cube, denoted by Q_n^k ($k \geq 2$ and $n \geq 2$), is a graph consisting of k^n vertices, each of which has the form $u = u_{n-1}u_{n-2}\cdots u_0$, where $u_i \in \{0, 1, \cdots, k-1\}$ for $i \in \{0, 1, \cdots, n-1\}$. Two vertices $u = u_{n-1}u_{n-2}\cdots u_0$ and $v = v_{n-1}v_{n-2}\cdots v_0$ are adjacent if and only if there exists an integer $j \in \{0, 1, \cdots, n-1\}$ such that $u_j = v_j \pm 1 (modulo\, k)$ and for every $i \in \{0, 1, \cdots, j-1, j+1, \cdots, n-1\}$ there exists $u_i = v_i$. Such an edge (u, v) is called a j-dimensional edge.

We can partition Q_n^k along the dimension j, by deleting all the j-dimensional edges, into k disjoint sub cubes as $Q_n^k[0]$, $Q_n^k[1], \cdots$, $Q_n^k[k-1]$, (for ease of notation, abbreviated as $Q[0]$, $Q[1], \cdots$, $Q[k-1]$, if there is no ambiguity). If $Q[i]$, for every $i \in \{0, 1, \cdots, k-1\}$, is a sub graph of Q_n^k induced by the vertices labeled by $u_{n-1}\cdots u_{j+1}iu_{j-1}\cdots u_0$ (see Figure 2), then it is clear that each $Q[i]$ is isomorphic to Q_{n-1}^k for $0 \leq i \leq k-1$. Note that Q_n^k can be divided into k disjoint copies of Q_n^k along n different dimensions. And vice versa we can combine k k-ary $(n-1)$-cubes in order to construct a k-ary n-cubes.

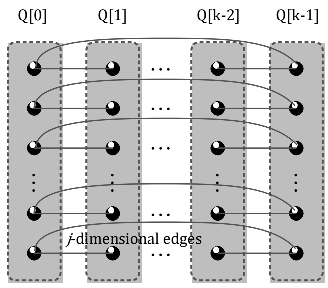

Fig. 2. Q_n^k is divided into $Q[0]$, $Q[1]$, ..., $Q[k-2]$, $Q[k-1]$.

We consider the fault-tolerance of a graph G. The following definitions are cited from the reference [19]. Let F be a set of faulty edges of Q_n^k . We call F a conditional faulty edge set of Q_n^k if every vertex in $Q_n^k - F$ is incident with at least two healthy edges. F^j indicates the set of faulty j-dimensional edges for $j \in \{0, 1, \cdots, n-1\}$. Then we refer to F as $\bigcup_{j=0}^{n-1} F^j$. For $p, r \in \{0, 1, \cdots, k-2, k-1\}$ and $p = r \pm 1 (modulo\ k)$, we use $\mathcal{F}^{s,r} \in F^j$ to denote the set of faulty j-dimensional edges between p and its neighbor r sub cubes. On the other hand, for each $i \in \{0, 1, \cdots, k-1\}$, we refer to F_i as $F \cap E(Q[i])$.

III. Pan cyclic embedding in the conditional faulty k-ary n-cubes

Let us now proceed to the proof of our main theorem. We begin by proving the inductive step, and then we return to the base cases of the induction.

A. Preliminaries

Theorem 3.1: (see [20]) Let k be an odd integer with $k \geq 3$, and let $n \geq 3$ be an integer. Let Q_n^k be a k-ary n-cube with faulty vertices f_v and faulty edges f_e where $0 \leq f_v + f_e \leq 2n-3$. We call F as a faulty vertex and edge set of Q_n^k if every vertex in $Q_n^k - F$ is incident with at least two healthy edges. For two arbitrary healthy vertices, there exists a path whose length is from $(n(k-1)-1)$ to $|V(Q_n^k - F) - 1|$ connecting these two vertices in the faulty Q_n^k. The k-ary n-cubes is also referred as $2n-3$ faults $(n(k-1)-1)$ pan connected.

Yang et al. investigate the fault-tolerant capabilities of the k-ary n-cubes for odd integer k with respect to the *Hamiltonian* and *Hamiltonian connected* properties. By a simple mathematical induction of Theorem 8 in [10], we have the following theorem.

Theorem 3.2: (see [10]) Let F be a faulty set with vertices and edges, and let $k \geq 3$ be an odd integer. When $|F| \leq 2n-2$, it is showed that there exists a Hamiltonian cycle in a wounded k-ary n-cube. In addition, when $|F| \leq 2n-3$, it is proved that, for two arbitrary nodes, there exists a Hamiltonian path connecting these two nodes in the wounded k-ary n-cubes.

In the following lemmas, namely *Lemmas 3.1-3.6*, which are useful for the proof of the main theorem, we shall construct cycles of various lengths in conditional faulty k-ary $(n-1)$-cube. Before going any further, we will consider an arrangement of the $4n-5$ edge faults in the k-ary n-cubes.

Lemma 3.1: Let Q_n^k be a k-ary n-cubes with $n \geq 3$ and $4n-5$ edge faults, then there is an m-dimension where exist the most faulty edges in Q_n^k and the number of the most faulty edges is no less than three denoted by $|\mathcal{F}^m| \geq 3$.

Proof: Suppose the most number of faulty edges which is denoted by $|\mathcal{F}|$ is in the m-dimension. To consider the limit of $|\mathcal{F}^m| \geq \lceil \frac{4n-5}{n} \rceil = \lceil 4 - \frac{5}{n} \rceil$, and when it satisfies $n \geq 3$ hence we compute $|\mathcal{F}^m| \geq 3$. It means that the number of the most faulty edges in the m-dimension is at least three. ■

We say that Q_n^k is partitioned along the dimension m for some $m \in \{0, 1, \cdots, n-1\}$ by deleting all the m-dimension edges into k disjoint sub cubes $Q[0], Q[1], \cdots, Q[k-1]$. Let u be a vertex in $Q[i]$, we denote it as u_i. We also use u_j which is adjacent to u_i to stand for the vertex belongs to $Q[j]$. Furthermore, if (u_i, v_i) is an edge of $Q[i]$, then (u_j, v_j) is the edge which belongs to $Q[j]$.

Let F be a set of faulty edges of Q_n^k. Assume that $F_i = F \cap E(Q[i])$ where $i \in \{0, 1, \cdots, k-2, k-1\}$. Obviously after deleting all the m-dimension edges, we can estimate the number of fault edges $|F_0 \cup F_1 \cup \cdots \cup F_{k-2} \cup F_{k-1}| = 4n-5-|\mathcal{F}^m| \leq 4n-8$ in the k disjoint sub cubes.

We may suppose $Q[0]$ is the sub cube with the most faulty edges while suppose $Q[s]$ and $Q[t]$ are the second and the third most faulty edges sub cubes where $s, t \in \{1, 2, \cdots, k-2, k-1\}$. Taking a step back, without loss of generality we assume that $|F_0| \geq |F_s| \geq |F_t|$ as illustrated in Figure 3.

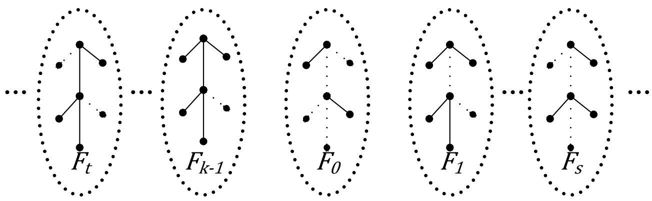

Fig. 3. Arrangement of $4n-5$ edge faults in the k-ary n-cubes

Lemma 3.2: Let F be a set of faulty edges of Q_n^k. Assume that $F_i = F \cap E(Q[i])$ where $i \in \{0, 1, \cdots, k-2, k-1\}$. If $Q[0]$ is the sub cube with the most faulty edges, then there exist $|F_s| \leq 2n-4$ and $|F_t| \leq 2n-5$ where $s, t \in \{1, 2, \cdots, k-2, k-1\}$.

Proof: We will give a proof by contradiction. Suppose that $|F_s| \geq 2n-3$ then we have $|F_0| \geq |F_s| \geq 2n-3$. Clearly, there exists $|F_0 \cup F_s| \geq 4n-6$. Note that, Lemma 3.1 implies that $|F_0 \cup F_1 \cup \cdots \cup F_{k-2} \cup F_{k-1}| \leq 4n-8$, so we obtain a contradiction. Hence, we have the assertion of $|F_s| \leq 2n-4$ where $s \in \{1, 2, \cdots, k-2, k-1\}$.

We may use a similar construction in the proof of $|F_{k-1}| \leq 2n-5$. By assuming $|F_t| \geq 2n-4$ we have $|F_s| \geq |F_t| \geq 2n-4$. Similarly, there exists $|F_s \cup F_t| \geq 4n-8$ which contradicts that $|F_0 \cup F_1 \cup \ldots \cup F_{k-2} \cup F_{k-1}| \leq 4n-8$ since we also have the assumption of $|F_0| \geq |F_s| \geq 2n-3$. This concludes the proof of this lemma. ■

B. Cycles of length from k to $2 \times k^{n-1}$ embedding in conditional fault k-ary n-cubes

In this section, we will prove the main assertion.

Lemma 3.3: Given an odd integer $k \geq 3$, let F be a set of faulty edges of Q_n^k. If each vertex of the k-ary n-cubes is incident with at least two healthy edges, then there exist a cycle of length from k to $2 \times k^{n-1}$ in the fault $Q_n^k - F$ if it has at most $4n-5$ edge faults.

Proof:

From Lemma 3.2 there exists the implication of $|F_s| \leq 2n-4 = 2(n-1)-2$. As $Q[s]$ is isomorphic to Q_{n-1}^k where $n-1 \geq 2$, thus by Theorem 3.2, it is showed that there exists cycles in the faulty $Q[s] - F_s$ whose length is from k to k^{n-1}. The k-ary $(n-1)$-cube is also referred as *pan cyclic*.

Consequently, we will prove the following assertion that there exist a cycle of length from $k^{n-1}+1$ to $2 \times k^{n-1}$ in the faulty $Q_n^k - F$.

Case 1: $|F_s| = 2n-4$

In this case, $|F_0| \geq |F_s| = 2n-4$ and $|F_0 \cup F_1 \cup \ldots \cup F_{k-2} \cup F_{k-1}| = 4n-5-|\mathcal{F}^m| \leq 4n-8$, suffice it to say that there exists $|F_0| = |F_s| = 2n-4$ and $\mathcal{F}^m = 3$. By the induction hypothesis, the other $|F_i| = 0$ can be constructed provided that $i \in \{1, 2, \cdots, k-2, k-1\}$ and $i \neq s$

From Theorem 3.2, there exist a cycle C_s of length from k to k^{n-1} in the faulty $Q[s] - F_s$. On the other hand, we suppose that its neighbor is r sub cubes where $r = s \pm 1(\textit{modulo } k)$. We select two adjacent vertices such as u_s, v_s lying on C_s which satisfy $(u_s, u_r), (v_s, v_r) \notin \mathcal{F}^{s,r}$.

According to Theorem 3.2, with the aid of path $P[u_s, v_s] = C_s - (u_s, v_s)$, the length l_s of $P[u_s, v_s]$ is from $k-1$ to $k^{n-1}-1$. For an illustrative example of $|F_r| = 0$, by Theorem 3.1 there exists a path $P[v_r, u_r]$ in $Q[r]$ whose length is l_r holding for $l_r \in \{(k-1)(n-1)-1, \ldots, k^{n-1}-1\}$. Connect l_s, l_r as illustrated in Figure 4, so that we get a cycle whose length is $l = l_s + l_r + 2 \in \{k + (k-1)(n-1), \ldots, 2 \times k^{n-1}\}$.

When $n \geq 3$ and an odd integer $k \geq 3$, it will be seen that $k + (k-1)(n-1) \leq k^{n-1}+1$. As a result, we can get a cycles whose length is from $k^{n-1}+1$ to $2 \times k^{n-1}$ in the faulty $Q_n^k - F$.

Case 2: $|F_s| \leq 2n-5$

If $n \geq 3$ and $|V(Q[s])| - |\mathcal{F}^{s,r}| \geq k^{n-1} - (4n-5) \geq 2$, without loss of generality, we select two vertices such as u_s,

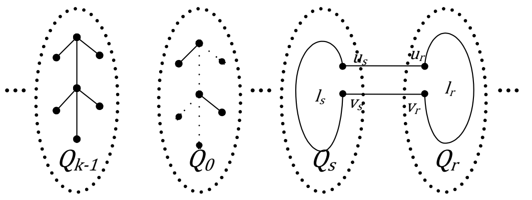

Fig. 4. The cycle whose length from $k + (k-1)(n-1)$ to $2 \times k^{n-1}$

v_s where it satisfies the requirement of $(u_s, u_r), (v_s, v_r) \notin \mathcal{F}^{s,r}$.

Since $|F_s|, |F_r| \leq 2n-5 = 2(n-1)-3$, by Theorem 3.1, we note l_s in $Q[s]-F_s$ and l_r in $Q[r]-F_r$ as $((n-1)(k-1)-1)$ fault *pan connected* whose length is $(k-1)(n-1)-1 \leq l_s, l_r \leq k^{n-1}-1$. Connecting l_s, l_r then we get that $l = l_s + l_r + 2 \in \{2(k-1)(n-1), \ldots, 2 \times k^{n-1}\}$.

If there exist $n \geq 3$ and $2(k-1)(n-1) \leq k^{n-1}+1$, then the Lemma is as required. This completes the proof. ■

C. Cycles of length from $2 \times k^{n-1}+1$ to k^n embedding in conditional fault k-ary n-cubes

Lemma 3.4: Assume $|F_s| \leq 2n-5$ for $s \in \{1, 2, \cdots, k-2, k-1\}$. If there is a cycle C_0 whose length is $k^{n-1}-1$ or k^{n-1} in $Q[0]$, then there exists a cycle of length from $2 \times k^{n-1}+1$ to $k^n - 1$ in the conditional fault $Q_n^k - F$.

Proof:

We say that the length of C_0 is l_0. Then there is $l_0 = k^{n-1}-1$ or $l_0 = k^{n-1}$.

Suppose first that $l_0 = k^{n-1}-1$ and it is enough to consider that the length of C_0 is more than the number of the largest faulty edges in the m-dimension which can be denoted by $l_0 = k^{n-1}-1 \geq 4n-5 \geq |\mathcal{F}^m|$. Hence, there would exist some edge named (u_0, v_0) where (u_0, u_1), $(v_0, v_1) \notin \mathcal{F}^{0,1}$ or (u_0, u_{k-1}), $(v_0, v_{k-1}) \notin \mathcal{F}^{0,k-1}$.

Without loss of generality, assume (u_0, u_1), $(v_0, v_1) \notin \mathcal{F}^{0,1}$. Using $|F_1| \leq 2n-5 = 2(n-1)-3$ in $Q[1]$ and $k^{n-1}-(2n-3) \geq (k-1)(n-1)-1$, by Theorem 3.1,we connect these two vertices in the faulty $Q[1]$ and consequently get a path $P[u_1, v_1]$ whose length is from $((n-1)(k-1)-1)$ to $|V(Q[1])-1|$. We use l_1 to stand for the length of path $P[u_1, v_1]$ where $(k-1)(n-1)-1 \leq l_1 \leq k^{n-1}-1$.

Suppose next $P[u_0, v_0] = C_0 - (u_0, v_0)$. If connect the paths of $P[v_1, u_1]$ and $P[u_0, v_0]$ then we can obtain a cycle C_1 whose length is $L_1 = (l_0 - 1) + l_1 + 2 = l_0 + l_1 + 1 \in \{(k-1)(n-1) + k^{n-1} - 1, \ldots, 2 \times k^{n-1} - 1\}$, and it is shown in Figure 5.

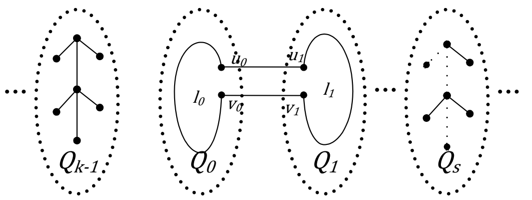

Fig. 5. The cycle C_1 whose length is L_1

Suppose now that an edge (x_0, y_0) lies on $P[u_0, v_0]$ where it satisfies the requirements of $(x_0, x_{k-1}), (y_0, y_{k-1}) \notin \mathcal{F}^{0,k-1}$. However, from Lemma 3.2 there exist $|F_{k-1}| \leq 2n-5 = 2(n-1)-3$ and $k^{n-1}-(2n-3) \geq (k-1)(n-1)-1$. By Theorem 3.1 connecting these two vertices in the faulty $Q[k-1]$ we consequently get a path $P[x_{k-1}, y_{k-1}]$ whose length is from $((n-1)(k-1)-1)$ to $|V(Q[k-1])-1|$. We use l_{k-1} to stand for the length of path $P[x_{k-1}, y_{k-1}]$ where $(k-1)(n-1)-1 \leq l_{k-1} \leq k^{n-1}-1$.

Suppose eventually that $P[x_0, y_0] = C_0 - (x_0, y_0)$ whose length is $l_0 - 1 = k^{n-1} - 2$. Connect $P[x_{k-1}, y_{k-1}]$ and $P[x_0, y_0]$ then we can obtain a cycle C_{k-1} whose length is as follows: For a given L_{k-1}, there exists $L_{k-1} = (L_1 - 1) + l_{k-1} + 2 = l_0 + l_1 + l_{k-1} + 2 \in 2(k-1)(n-1) + k^{n-1} - 1, \ldots, 3 \times k^{n-1} - 1$. Figure 6 illustrates such an example.

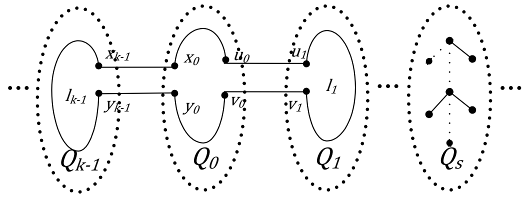

Fig. 6. The cycle C_{k-1} whose length is L_{k-1}

Finally, the following cases can be done as this: connect $P[x_{k-2}, y_{k-2}]$ and $P[u_2, v_2]$ and so on which is shown in Figure, we can obtain a whole cycle C whose length is $L = l_0 + l_1 + \ldots + l_{k-1} + k - 1 \in \{(k-1)(k-1)(n-1) + k^{n-1} - 1, \ldots, k \times k^{n-1} - 1\} = \{k-1)^2(n-1) + k^{n-1} - 1, \ldots, k^n - 1\}$.

Because of $(k-1)^2(n-1) + k^{n-1} - 1 \leq 2 \times k^{n-1}$, it implies that $L \in \{2 \times k^{n-1}, 2 \times k^{n-1}+1, \ldots, k^n - 1\}$. Then the conclusion is as required. ■

From Lemma 3.4, the following corollary is immediate.

Corollary 3.1: Let $|F_s| \leq 2n-5$ and a path $P[u_0, v_0]$ of length $k^{n-1}-2$ or k^{n-1} in $Q[0]$ where (u_0, u_1), $(v_0, v_1) \notin \mathcal{F}^{0,1}$ or $(u_0, u_{k-1}), (v_0, v_{k-1}) \notin \mathcal{F}^{0,k-1}$, then there exist a cycle of length from $2 \times k^{n-1}$ to $k^n - 1$ in the conditional fault $Q_n^k - F$.

Lemma 3.5: Let $|F_0| = 4n-8$ then there exist an edge $(u_0, v_0) \in F_0$ where $(u_0, u_1), (v_0, v_1) \notin \mathcal{F}^{0,1}$ or $(u_0, u_{k-1}), (v_0, v_{k-1}) \notin \mathcal{F}^{0,k-1}$.

Proof: As $|\mathcal{F}^m| \geq 3$ from Lemma 3.1, clearly, we only need to consider the case for $|\mathcal{F}^m| \leq 4n-5-|F_0| = 3$ and so $|\mathcal{F}^m| = 3$.

First, assume that there exist two distinct and non-adjacent fault edges in $Q[0]$, the conclusion is true apparently. Suppose next that any two fault edges are adjacent, then they are incident with the same vertex. Assume this vertex is u_0, the degree of u_0 is $2n-2$ in $Q[0]$ after deleting all the m-dimension edges, this also mean that u_0 is incident with $2n-2$ both healthy and faulty edges. By taking into consideration of $4n-8 \leq 2n-2$ and $n \geq 3$, hence, there is $n = 3$. Since $|\mathcal{F}^m| = 3$, there exist at least one healthy edge of (u_0, u_1) . We now consider the conditional faulty k-ary n-cubes that each vertex is incident with at least two healthy edges. So if u_0 is incident with at most four faulty edges, and then there existing a faulty edge of (u_0, v_0) and an edge of (v_0, v_1) which is an inevitable healthy edge as shown in Figure 7.

■

Lemma 3.6: There exist a cycle of length from $2 \times k^{n-1}+1$ to k^n in the conditional fault $Q_n^k - F$.

Proof: According to Theorem1.1 there exist a cycle of length k^n in the faulty $Q_n^k - F$. As a result, we need find

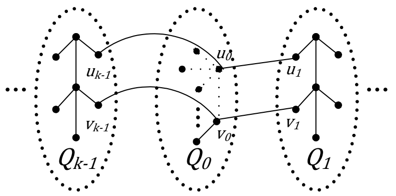

Fig. 7. The degree of u_0 is at least two

the cycle of length from $2 \times k^{n-1} + 1$ to $k^n - 1$ in the faulty $Q_n^k - F$.

Case 1: By assumption, there exists a vertex w_0 that is incident with at most one healthy edge in $Q[0]$.

Assume F_0^w is a set of faulty edges which are incident with w_0 in the faulty $Q[0]$. Therefore, there are $|F_0^w| \geq 2n-3$ and $|F_s| \leq 4n - 8 - (2n-3) = 2n-5$ where $s \in \{1, 2, \cdots, k-2, k-1\}$. Suppose w_0 as a temporary faulty vertex. If $|(F_0 - F_0^w) \cup \{w_0\}| \leq 4n - 8 - (2n-3) + 1 = 2(n-1) - 2$, from Theorem 3.2, then there exists a cycle C_0 of length $k^{n-1} - 1$ in the faulty $Q[0] - ((F_0 - F_0^w) \cup \{w_0\})$.

Since $|F_s| \leq 2n - 5$, from Lemma 3.4, we can find a cycle C_0 whose length is $k^{n-1} - 1$ in $Q[0]$, then there exist a cycle of length from $2 \times k^{n-1}$ to $k^n - 1$ in the conditional fault $Q_n^k - F$. Figure 8 shows an example of this case.

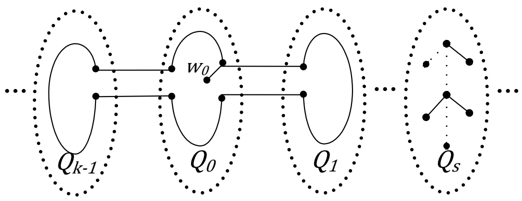

Fig. 8. The cycle C_0 whose length is $k^{n-1} - 1$ in $Q[0]$

Case 2: Suppose that each vertex of $Q[0]$ is incident with at least two healthy edges and $|F_0| = 4n - 8$.

Since $|F_0| = 4n - 8$, by Lemma 3.5, we can get the results of $|\mathcal{F}^m| = 3$ and $|F_1| = |F_2| = \ldots = |F_{k-1}| = 0$. Suppose next that there is a Hamiltonian cycle in $Q[0] - F_0$, from Lemma 3.4, the conclusion is proved. Consequently, assume that there is not any Hamiltonian cycle in $Q[0] - F_0$. From Lemma 3.5, then there exist a faulty edge $(u_0, v_0) \in F_0$ for (u_0, u_1), $(v_0, v_1) \notin \mathcal{F}^{0,1}$ or $(u_0, u_{k-1}), (v_0, v_{k-1}) \notin \mathcal{F}^{0,k-1}$.

Suppose (u_0, v_0) is a pseudo-healthy edge, by Theorem 1.1, given an integer of $k \geq 3$, the conditional faulty Q_{n-1}^k with at most $4(n-1) - 5$ faulty edges is Hamiltonian. Then there is a Hamiltonian cycle in the modified $Q[0] - F_0$ which has $4n-9$ faulty edges containing pseudo-healthy edge of (u_0, v_0). So the $Q[0] - F_0$ has a Hamiltonian path from u_0 to v_0 whose length is $k^{n-1} - 1$. According to corollary 3.1, we present the conclusion.

Case 3: Let each vertex of $Q[0]$ be incident with at least two healthy edges and $|F_0| \leq 4n - 9$.

Case 3.1: $|F_s| \leq 2n - 5$

Given $|F_0| \leq 4n - 9$ and $n \geq 3$, from Theorem 1.1, there is a cycle of length of k^{n-1} in $Q[0]$. By Lemma 3.4, if $|F_s| \leq 2n - 5$, then there exist a cycle of length from $2 \times k^{n-1}$ to $k^n - 1$ in the conditional fault $Q_n^k - F$.

Case 3.2: $|F_s| = 2n - 4$

If $|F_0| \geq |F_s|$ and $|F_0 \cup F_1 \cup \ldots \cup F_{k-2} \cup F_{k-1}| \leq 4n-5-|\mathcal{F}^m| \leq 4n-8$, then there exists $|F_0| = |F_s| = 2n-4$ and $|\mathcal{F}^m| = 3$. From Theorem3.2, given an odd integer of $k \geq 3$, where $|F_0| = |F_s| = 2n-4$, there exists a cycle C_0 of length of k^{n-1} in the fault $Q[0] - F_0$ and a cycle C_s of length of k^{n-1} in the faulty $Q[s] - F_s$ for $s \in \{1, 2, \cdots, k-2, k-1\}$.

In contrast, without loss of generality consider the interconnections between $Q[0]$ and $Q[1]$ shown in Figure 9. If there are two healthy edges of (u_0, v_0) and (v_0, w_0) on the cycle C_0, then it is satisfied that (u_0, u_1), (v_0, v_1) and (w_0, w_1) are three healthy edges. Suppose that (u_1, v_1) and (v_1, w_1) are healthy edges in $Q[1]$, with the aid of marking some edges which are incident with v_1 as temporary faulty ones except (u_1, v_1) and (v_1, w_1), so as to have v_1 be incident with at most three healthy edges. Given $|F_1| \leq 2n - 4$, therefore, there is $|F_1| \leq (2n-4) + (2n-5) = 4n - 9$ after joining the new temporary fault edges. According to Theorem1.1, then there exists a Hamiltonian cycle where (u_1, v_1) or (v_1, w_1) lie on in $Q[1]$.

On the other hand, we provide an assumption that either (u_1, v_1) or (v_1, w_1) is a faulty edge and without loss of generality we may assume (u_1, v_1) is the faulty one.

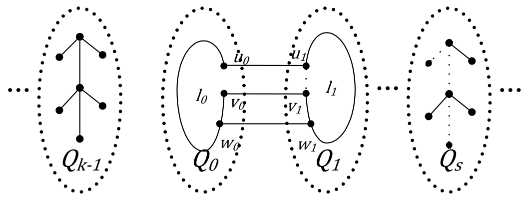

Fig. 9. Interconnections between $Q[0]$ and $Q[1]$

Next, we suppose that (u_1, v_1) is a pseudo-healthy edge, so it is easy to see that $|F_1 - \{u_1, v_1\}| \leq 2n-5$. By Theorem 3.1 there exists a path $P[u_1, v_1]$ in $Q[1]$ whose length is l_1 holding for $l_1 \in \{(k-1)(n-1) - 1, \ldots, k^{n-1} - 1\}$. Connecting C_0 and $P[u_1, v_1]$ with (u_0, u_1) and (v_0, v_1), then we get that $L_1 = l_0 + l_1 + 1 \in \{k^{n-1} + (k-1)(n-1), \ldots, 2 \times k^{n-1}\}$.

Finally, the following cases can be done as this: connect $P[u_{k-1}, v_{k-1}]$ and $P[u_2, v_2]$ and so on we can obtain a whole cycle C whose length is $L \in \{(k-1)(k-1)(n-1) + k^{n-1}, \ldots, k \times k^{n-1}\} = \{k-1)^2(n-1) + k^{n-1}, \ldots, k^n\}$.

Because of $(k-1)^2(n-1) + k^{n-1} \leq 2 \times k^{n-1}$, it implies that $L \in \{2 \times k^{n-1}, 2 \times k^{n-1} + 1, \ldots, k^n - 1\}$.

By the above cases, we complete the proof. ■

IV. Conclusion

In conclusion, by Lemma 3.3 and Lemma 3.6, the fault-tolerant *pan cyclicity* of Q_n^k is given in the following theorem.

Theorem 4.1: Let Q_n^k be a k-ary n-cube with odd $k \geq 3$ and $n \geq 3$ which is not bipartite. We consider the faulty k-ary n-cubes that each vertex is incident with at least two healthy edges. We prove that there exist a fault-free cycle of every length from k to k^n in Q_n^k even if it has up to $4n - 5$ edge faults. It also means that the Q_n^k is conditional $(4n-5)$ edge fault *pan cyclic*.

In this paper, we investigate the k-ary n-cubes for an odd $k \geq 3$ with some faulty edges such that each vertex is incident with at least two healthy edges. We proved that such a k-ary n-cube with at most $4n-5$ faulty edges contains a cycle whose length varies in different k to k^n. We show that the conditional fault-tolerant capability of the k-ary n-cubes is excellent in terms of pan cyclic embedding which can be used to develop corresponding applications on the distributed-memory parallel system in the environment of k-ary n-cubes. Our further work is to consider whether the above result is optimal in conditional faulty k-ary n-cubes with odd $k \geq 3$.

Interconnection networks are often composed of hundreds (or thousands) of components, just like routers, channels, and connectors. They collectively have failure rates higher than is acceptable for the application. Thus, these networks must employ error control to continue operation without interruption. The k-ary n-cubes are particularly easy to map to physical space with uniformly short edges. The simplest case is when the network is a cycle with the same number of dimensions as the physical dimensions of the packaging technology. A k-ary n-cube can be transformed into an express cube network augmented with a number of long or express cycles. By routing packets that must traverse a long distance in a dimension over the express channels. The header latency can be reduced to nearly the channel latency limit. Because the number of express channels can be controlled to match the bisection width of the network, this reduction in header latency can be achieved without increasing serialization latency. So these are also used to provide fault tolerance.

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