Convergence and Stability of Some Modified Iterative Processes for a Class of Generalized Contractive-like Operators

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***Abstract*—In this paper, we establish that the modified Jungck-Ishikawa iterative scheme converges strongly to the common fixed point of a pair of weakly compatible mappings satisfying generalized contractive-like conditions in Banach spaces. Furthermore, we study the stability of the iterative scheme. Our results extend and generalize some results in the literature.**

***Index Terms*—Banach space, contractive-like operator, modified Jungck-Ishikawa iterative scheme, modified Jungck-Mann iterative scheme, stability**

I. Introduction

It was established in [1] that the modified Jungck-Mann iterative scheme and modified Jungck-Ishikawa iterative scheme converged faster than the existing Jungck-Mann iterative scheme and Jungck-Ishikawa iterative scheme respectively, in any arbitrary Banach space using the generalized Zamfirescu contractive conditions . In the same reference, they proved the strong convergence of the coincidence point of the operators. Motivated by the work of Sadib and Iqbal [1], using the modified Jungck-Ishikawa iterative scheme, we prove the strong convergence of the common fixed points of generalized contractive-like operator in any arbitrary Banach space. We further prove that the iterative scheme is (S, T)-stable.

II. PRELIMINARIES

In 1976, Jungck [2] introduced an iterative scheme which was adopted to approximate the common fixed point of the Jungck contraction map.

Let (X, d) be a metric space and $S,T:Y\to X$ be two mappings such that $T(Y)\subseteq S(Y)$, where Y is an arbitrary set. For any $x_0\in Y$, the jungck iterative scheme is defined as the sequence $\{Sx_n\}_{n=0}^{\infty}$ such that

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$$Sx_{n+1}=Tx_n,\ n\geq 0$$

If S= Identity then we have Picard iterative scheme.
Singh et. al. [3] introduced the Jungck-Mann iterative process and discussed the stability for a pair of contractive maps.

For any $x_0\in Y$, the Jungck-Mann iterative scheme is defined as the sequence $\{Sx_n\}_{n=0}^{\infty}$ such that

$$Sx_{n+1}=(1-\alpha_n)Sx_n+\alpha_n Tx_n$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0, 1) such that $\sum_{n=0}^{\infty}\alpha_n=\infty$.

Olatinwo and Imoru [4] built on the work of Singh et.al.[3 to introduced the Jungck-Ishikawa iterative scheme. For any $x_0\in Y$, the jungck-Ishikawa iterative scheme is defined as the sequence $\{Sx_n\}_{n=0}^{\infty}$ such that

$$Sx_{n+1}=(1-\alpha_n)Sx_n+\alpha_n Ty_n$$

$$Sy_n=(1-\beta_n)Sx_n+\beta_n Tx_n$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0, 1) such that $\sum_{n=0}^{\infty}\alpha_n=\infty$.

Recently, Sadib and Iqbal [1] introduced the modified Jungck-Mann iterative scheme and modified Jungck-Ishikawa iterative scheme. These schemes were employed to approximate the coincidence points of some pairs of generalized Zamfirescus contractive maps in any arbitrary Banach space with the assumption that one of the maps is subjective while the other is differentiable.

Let $S,T:Y\to X$ be two non-self mappings with $T(Y)\subseteq S(Y)$, S is onto and T is differentiable. Then for any $x_0\in Y$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ is

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$$Sx_{n+1}=(1-\alpha_n)Sx_n+\alpha_nT\theta x_n \quad (1)$$

and

$$Sx_{n+1}=(1-\alpha_n)Sx_n+\alpha_nT\theta y_n$$
$$Sy_n=(1-\beta_n)Sx_n+\beta_nT\theta x_n \quad (2)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0, 1),

$$T\theta x_n=\frac{-cT'x_nSx_n+Tx_n}{1-cTx_n}$$

and

$$T\theta y_n=\frac{-cT'y_nSy_n+Ty_n}{1-cTy_n}; \quad c\in\mathbb{R}.$$

Equations (1) and (2) are called modified Jungck-Mann and Jungck-Ishikawa iterative schemes respectively. It was also proved in the same reference that $T\theta$ coincides with S and T at their coincident point.

The concept of stability of a fixed point iteration procedure was initiated by Ostrowski [5]. Many researchers have systematically studied and improved on his result (see: Bosede and Rhoades [6], Harder and Hicks [7], Akewe and Okeke [8]). Olatinwo [9] introduced the following class of generalized contractive-like operators to obtain some stability results for the Jungck-Noor iterative scheme in an arbitrary Banach space.

For $S,T:Y\to X$ with $T(Y)\subseteq S(Y)$ where S(Y) is a complete subspace of X. There exists a real number $\delta\in[0,1)$ and a monotonic increasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ such that $\psi(0)=0$ and every $x,y\in Y$, we get

$$d(Tx,Ty)\le\delta d(Sx,Sy)+\psi(d(Sx,Tx))$$

In a Banach space setting, the above inequality becomes

$$\|Tx-Ty\|\le\delta\|Sx-Sy\|+\psi(\|Sx-Tx\|) \quad (3)$$

Lemma: [10] : Let δ be a real number satisfying $\delta\in[0,1)$ and $\{\in_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{n\to\infty}\in_n=0$. Then, for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $n_{n+1}=\delta u_n+\in_n,\ n\ge 0$, we have $\lim_{n\to\infty}u_n=0$.

III. CONVERGENCE RESULTS

In this section, we prove that the modified Jungck-Ishikawa iterative scheme converges strongly to the fixed point of the operators T and S.

Theorem I: Let (X, ||.||) be a complete normed space and Y be any arbitrary set. Suppose that $S,T:Y\to X$ are two maps with $T(Y)\subseteq S(Y)$, where S(Y) is a complete subspace of X, T is differentiable. Let z be coincidence point T and S (I.e Tz = Sz = p). Suppose S and T satisfy the contractive condition

$$\|Tx-Ty\|\le\delta\|Sx-Sy\|+\psi(\|Sx-Tx\|)$$

where $\delta\in[0,1)$ and $\psi(t)$ a monotonic increasing and continuous function. Then, for any $x_0\in Y$, the modified jungck-Ishikawa iteration $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Furthermore, if Y=X and S, T commute at p (i.e S and T are weakly compatiable) then p is the unique common fixed point of S and T.

Proof: Considering (2) and (3) coupled with the fact that Tz= Sz=p, we have

$$\begin{aligned}\|Sx_{n+1}-p\|&\le(1-\alpha_n)\|Sx_n-p\|+\alpha_n\|p-T\theta y_n\|\\&=(1-\alpha_n)\|Sx_n-p\|+\alpha_n\|p-T\theta y_n\|\\&\le(1-\alpha_n)\|Sx_n-p\|+\alpha_n(\delta\|Sp-Sy_n\| \quad (4)\\&\quad+\psi\|Sp-Tp\|))\\&=(1-\alpha_n)\|Sx_n-p\|+\alpha_n\delta\|Sp-Sy_n\|\end{aligned}$$

In view of (3) and (4) we obtain,

$$\begin{aligned}\|Sy_n-p\|&\le(1-\beta_n)\|Sx_n-p\|+\beta_n\|Tp-T\theta x_n\|\\&\le(1-\beta_n)\|Sx_n-p\|+\beta_n(\delta\|Sp-Sx_n\|\\&\quad+\psi\|Sp-Tp\|))\\&=(1-\beta_n)\|Sx_n-p\|+\beta_n\delta\|Sp-Sx_n\| \quad (5)\end{aligned}$$

Substituting (5) into (4) yields,

$$\begin{aligned}\|Sx_{n+1}-p\| &\le (1-\alpha_n)\|Sx_n-p\|+\alpha_n\delta(1-\beta_n)\\ &\quad \|Sx_n-p\|+\beta_n\delta\|Sp-Sx_n\|\\ &\le (1-\alpha_n)\|Sx_n-p\|+\alpha_n\delta\|Sx_n-p\|\\ &\quad -\alpha_n\beta_n\delta\|Sx_n-p\|+\alpha_n\beta_n\delta^2\|Sx_n-p\|\\ &\le (1-\alpha_n+\alpha_n\delta)\|Sx_n-p\|\\ &\quad -\alpha_n\beta_n\delta(1-\delta)\|Sx_n-p\|\\ &\le (1-\alpha_n(1-\delta)\|Sx_n-p\|\\ &\quad -\alpha_n\beta_n\delta(1-\delta)\|Sx_n-p\|\\ &\le (1-\alpha_n(1-\delta)\|Sx_n-p\|\\ &\le (1-(1-\delta)\|Sx_n-p\|\end{aligned}$$

Hence $Sx_n \to p$ since $0 \le \delta < 1$ for all n.

Next, we show that p is unique.

Suppose there exists another point of coincidence p_2 such that $Tz_1 = Sz_1 = T\theta z_1 = p_1$ and $Tz_2 = Sz_2 = T\theta z_2 = p_2$. Using (3), we have

$$\begin{aligned}\|p_1-p_2\| &= \|T\theta z_1 - T\theta z_2\|\\ &\le \delta\|Sz_1-Sz_2\|+\psi\|Sz_1-T\theta z_1\|\\ &= \delta\|Sz_1-Sz_2\|\end{aligned}$$

since $0 \le \delta < 1$, then $p_1 = p_2$ and so p is unique. If S and T are weakly compatible then TSz=STz and so Tp=Sp. Hence, p is a coincidence point of S and T. Thus, the coincidence point of S and T is unique and p=z. Hence $Sp = T\theta p = p$. Therefore, p is the unique common fixed point of S and T.

Corollary: Let (X, ||.||) be a complete normed space and Y be any arbitrary set. Suppose that $S,T: Y \to X$ are two maps with $T(Y) \subseteq S(Y)$, where S(Y) is a complete subspace of X, T is differentiable. Let z be coincidence point of T and S (I.e Tz = Sz = p). Suppose S and T satisfy the contractive condition

$$\|Tx-Ty\| \le \delta\|Sx-Sy\|+2\delta\|Sx-Tx\| \qquad (6)$$

where $\delta \in [0,1)$. Then, for any $x_0 \in Y$, the modified jungck-Ishikawa iteration $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Furthermore, if Y=X and S, T commute at p (i.e S and T are weakly compatiable) then p is the unique common fixed point of S and T.

Remarks: A weaker version of the corollary is the main result of Sadiq and Iqbal [1] where the convergence is to the coincidence point of S and T and S is assumed injective.

IV. Stability Results

In this section, we establish the stability of the modified Jungck-Ishikawa iterative scheme in a complete normed space.

Theorem II: Let (X, ||.||) be a complete normed space and Y be any arbitrary set. Suppose that $S,T: Y \to X$ are two maps with $T(Y) \subseteq S(Y)$, where S(Y) is a complete subspace of X, T is differentiable with a common fixed point of p satisfying the condition

$$\|Tx-Ty\| \le \delta\|Sx-Sy\|+\psi(\|Sx-Tx)$$

For each $x, y \in X$, $\delta \in [0,1)$ and $\psi(t)$ a monotonic increasing and continuous function. For arbitrary $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the modified jungck-Ishikawa iterative defined in (2). Then the modified jungck-Ishikawa iteration scheme is (S, T) stable.

Proof: We establish in Theorem I that the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined in (2) converges strongly to p. Let $\{Sx_n\}_{n=0}^{\infty}$ and $\{Sy_n\}_{n=0}^{\infty}$ be real sequence in Y. Let $\in_n = \|x_{n+1}-(1-\alpha_n)Sx_n-\alpha_n T\theta y_n\|,\ n \ge 0$ where

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n T\theta y_n$$

$$Sy_n = (1-\beta_n)Sx_n + \beta_n T\theta x_n$$

And let $\lim_{n\to\infty} \in_n = 0$. Then we shall prove that $\lim_{n\to\infty} Sx_n = p$ for mappings satisfying condition (3). That is,

$$\begin{aligned}\|Sx_{n+1}-p\| &\le \|Sx_{n+1}-(1-\alpha_n)Sx_n+\alpha_n T\theta y_n\|\\ &\quad +\|(1-\alpha_n)Sx_n-\alpha_n T\theta y_n\\ &\quad -(1-\alpha_n-\alpha_n)p\| \qquad (7)\\ &\le \in_n +(1-\alpha_n)\|Sx_n-p\|\\ &\quad +\alpha_n\|T\theta y_n-p\|\end{aligned}$$

Using condition (7) with $y_n = x$, we have

$$\begin{aligned}\|T\theta y_n-p\| &= \|T\theta y_n - Tp\|\\ &\le \delta\|Sy_n-p\|+\psi\|Sy_n-Tp\| \qquad (8)\\ &= \delta\|Sy_n-p\|\end{aligned}$$

Substituting (8) into (7) yields,

$$\begin{aligned}\|Sx_{n+1}-p\| &\le \in_n +(1-\alpha_n)\|Sx_n-p\|\\ &\quad +\alpha_n\delta\|Sy_n-p\|\end{aligned} \qquad (9)$$

and

$$\begin{aligned} \| Sy_n - p \| &\le \| (1-\beta_n)Sx_n + \beta_n (T\theta x_n - p) \| \\ &= \| (1-\beta_n)Sx_n + \beta_n T\theta x_n \\ &\quad -(1-\beta_n+\beta_n) p \| \\ &= \| (1-\beta_n)(Sx_n - p) + \beta_n (T\theta x_n - p) \| \\ &\le (1-\beta_n) \| Sx_n - p \| + \beta_n \| T\theta x_n - p \| \\ &\le (1-\beta_n) \| Sx_n - p \| + \beta_n \delta \| Sx_n - p \| \end{aligned} \tag{10}$$

Substituting (10) into (9) yields

$$\begin{aligned} \| Sx_{n+1} - p \| &\le \epsilon_n + (1-\alpha_n) \| Sx_n - p \| \\ &\quad + \alpha_n \delta (1-\beta_n) \| Sx_n - p \| \\ &\quad + \beta_n \delta \| Sx_n - p \| \\ &= \epsilon_n + (1-\alpha_n + \alpha_n \delta (1-(1-\delta)\beta_n)) \\ &\quad \| Sx_n - p \| \\ &= \epsilon_n + (1-\alpha_n) \| Sx_n - p \| \\ &\quad + \alpha_n \delta (1-\beta_n) \| Sx_n - p \| \\ &\quad + \alpha_n \beta_n \delta^2 \| Sx_n - p \| \\ &= \epsilon_n + (1-\alpha_n) \| Sx_n - p \| \\ &\quad + \alpha_n \delta (1-\beta_n + \beta_n \delta) \| Sx_n - p \| \\ &= \epsilon_n + (1-\alpha_n) \| Sx_n - p \| \\ &\quad + \alpha_n \delta (1-(1-\delta)\beta_n) \| Sx_n - p \| \\ &= \epsilon_n + (1-\alpha_n + \alpha_n \delta (1-(1-\delta)\beta_n)) \\ &\quad \| Sx_n - p \| \end{aligned}$$

Observe that

$$0 \le (1-\alpha_n + \alpha_n \delta (1-(1-\delta)\beta_n)) < 1 \tag{11}$$

Therefore, taking the limit of inequality (11) and using the Lemma we get $\lim_{n\to\infty} \| Sx_n - p \| = 0$. Thus $\lim_{n\to\infty} Sx_n = p$. Therefore, the modified Jungck-Ishikawa iterative scheme is (S, T)-stable.

Theorem III: Let (X, ||.||) be a complete normed space and Y be any arbitrary set. Suppose that $S, T : Y \to X$ are two maps with $T(Y) \subseteq S(Y)$, where S(Y) is a complete subspace of X, T is differentiable with a common fixed point of p satisfying the condition

$$\| Tx - Ty \| \le \delta \| Sx - Sy \| + \psi(\| Sx - Tx \|)$$

For each $x, y \in X$, $\delta \in [0,1)$ and $\psi(t)$ a monotonic increasing and continuous function. For arbitrary $x_0 \in Y$, let the modified jungck-Mann iterative $\{Sx_n\}_{n=0}^{\infty}$ defined in (1). Then the modified jungck-Ishikawa iteration scheme is (S, T) stable.

Proof: The proof of Theorem 4.2 follows from the proof of Theorem II, by letting $\beta_n = 0$ for each $n \in \mathbb{N}$.

V. CONCLUSION

The results showed that the modified Jungck-Ishikawa iterative scheme and modified Jungck-Mann iterative scheme converge strongly to the unique common fixed point of the operators S and T. We equally establish that, these iterative schemes are (S, T)-stable.

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