On Anti-periodic Solutions for FCNNs with Mixed Delays and Impulsive Effects

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Abstract—**This paper is concerned with anti-periodic solutions for fuzzy cellular neural networks (FCNNs) with mixed delays and impulsive effects. Using differential inequality, and constructing some suitable Lyapunov functional, some conditions are established for the existence and global exponential stability of anti-periodic solutions of FCNNs with mixed delays and impulsive effects. These results are new and complementary to previously known references. Moreover an example is given to illustrate results established.**

Index Terms—**exponential stability, anti-periodic solutions, fuzzy cellular neural networks, mixed delays, impulsive effects.**

I. Introduction

CELLULAR neural networks (CNNs) first introduced by Chua and Yang [1], [2] have attracted much attention in recent years. This is mostly because they have the wide range of promising applications fields such as associated memory, parallel computing, pattern recognition, signal processing and optimization problems. CNNs are described by the basic circuit units and these units are called cells. Each unit processes several input signals and produces an output signal which is received by other units connected to it including itself. In the implementation of a signal or influence traveling through neural networks, time delays do exist and influence dynamical behavior of a working network. Recently many results on the problem of global stability of equilibrium points and periodic solutions of neural networks have been reported (see [3], [4], [5], [6], [7], [8], [9]). Besides delay effects, it has been observed that many evolutionary processes, including those related to neural networks, may exhibit impulsive effects. In these evolutionary processes, the solutions of system are not continuous but present jumps which could cause instability of dynamical system. Consequently, many neural networks with impulses have been studied extensively, and a great deal of literatures are focused these problems on the existence and stability of an equilibrium point and periodic solutions (see, for example [10], [11], [12], [13], [14], [15], [16], [17], [18]).

It is well-known that Yang and Yang [19], [20] first introduced another type cellular neural networks model called fuzzy cellular neural networks (FCNNs). These models combined fuzzy operations (fuzzy AND and fuzzy OR) with cellular neural networks. However, it is worth noting that T-S fuzzy neural networks are different from FCNNs [21]. T-S fuzzy neural networks are based a set of fuzzy rules to describe nonlinear system. Recently researchers have found that FCNNs are useful in image processing, and some results have been reported on stability and periodicity of FCNNs (see [22], [23], [24], [25]).

In applied sciences, the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations. The signal transmission process of neural networks can often be described as an anti-periodic process. In recent years the anti-periodic problem of neural networks has been studied by many authors (see [26], [27], [28], [29], [30], [31], [32], [33] and references therein). Shao [26] studied the existence and exponential stability of the anti-periodic solutions of recurrent neural networks with time-varying and continuous distributed delays. Shi and Dong [27], applying inequality technique and Lyapunov functional theory, studied the existence and global exponential stability of anti-periodic solution for delayed Hopfield neural networks with impulsive effects. However, to the best of our knowledge, few authors have considered the problem of anti-periodic solutions for FCNNs with time-varying delays or distributed delays and impulsive effects. Zhang, Yang and Liu [34] studied the existence and global exponential stability of anti-periodic solutions for fuzzy Cohen-Grossberg neural networks with impulsive effects on time scales.

Motivated by the above discussion, it is worth continuing the investigation of existence and stability of anti-periodic solutions for FCNNs with mixed delays and impulsive effects. This paper is concerned with the following model

$$\begin{cases} x_i'(t) &= -a_i(t)x_i(t) + \sum_{j=1}^n d_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) \\ & \quad + \bigwedge_{j=1}^n \alpha_{ij}(t)\int_{-\infty}^t k_{ij}(t-s) g_j(x_j(s))ds \\ & \quad + \bigvee_{j=1}^n \beta_{ij}(t)\int_{-\infty}^t k_{ij}(t-s) g_j(x_j(s))ds \\ & \quad + E_i(t)],\ t\ge 0, t\neq t_k \\ x_i(t_k^+) &= (1+I_{ik})x_i(t_k), k=1,2,\cdots, \\ x_i(t) &= \varphi_i(t), t\in[-\tau,0], i=1,2,\cdots,n. \end{cases} \tag{1}$$

where n corresponds to the number of units in a neural network. $x_i(t)$ is the activations of the i-th neuron at the time t. $a_i(t), d_{ij}(t), \alpha_{ij}(t), \beta_{ij}(t), E_i(t), f_j(t), g_j(t), \tau_{ij}(t)$ are continuous functions on R. $a_i(t)$ represents the amplification function and $a_i(t)>0$. $d_{ij}(t)$ denotes the synaptic connection weight of the unit j on the unit i at time t. $\alpha_{ij}(t)$ and $\beta_{ij}(t)$ are elements of fuzzy feedback MIN template and fuzzy feedback MAX template, respectively. $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operation, respectively. $E_i(t)$ denotes the i-th component of an external input source introduced from outside the network to the ith cell. $\tau_{ij}(t)$ is time-varying delay satisfying $0\le\tau_{ij}(t)\le\tau$, τ is a positive

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constant. $f_j(\cdot)$ and $g_j(\cdot)$ are the activation functions. The delay kernel $k_{ij} : R^+ \to R^+$ are real valued nonnegative continuous functions satisfying $\int_0^\infty k_{ij}(s)ds = k_{ij}^+$. k_{ij}^+ is a positive constant.

The rest of this paper is structured as follows. In next section, we introduce some definitions and lemmas. In Sect. 3, applying differential inequality, constructing suitable Lyapunov functional, we shall derive new sufficient conditions for the global exponential stability of anti-periodic solutions of system (1). An example is given to demonstrate the effectiveness of our results in Sect. 4. Finally a general conclusion is drawn in Sect. 5.

II. Preliminaries

For the sake of convenience, we introduce some notations

$$\overline{d}_{ij} = \sup_{t\in R}|d_{ij}(t)|, \overline{\alpha}_{ij} = \sup_{t\in R}|\alpha_{ij}(t)|, \overline{\beta}_{ij} = \sup_{t\in R}|\beta_{ij}(t)|.$$

$$\overline{E}_i = \sup_{t\in R}|E_i(t)|, a_i^- = \min_{t\in R}|a_i(t)|, \tau = \sup_{t\in R}\max_{1\le i,j\le n}\{\tau_{ij}(t)\}.$$

Throughout this paper, we make the following assumptions

(**A1**) For $i, j = 1, 2, \cdots, n, k = 1, 2, \cdots,$ $d_{ij}, \alpha_{ij}, \beta_{ij}, E_i : R \to R, c_i, \tau_{ij} : R \to R^+$ are continuous functions, and there exist $\omega > 0$ such that for $v \in R$

$$a_i(t+\omega) = a_i(t), d_{ij}(t+\omega)f_j(-v) = -d_{ij}(t)f_j(v),$$

$$E_i(t+\omega) = -E_i(t), \tau_{ij}(t+\omega) = \tau_{ij}(t),$$

$$\alpha_{ij}(t+\omega)\int_{-\infty}^{t+\omega} k_{ij}(t-s+\omega)g_j(v_j)ds = -\alpha_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)g_j(v_j)ds$$

$$\beta_{ij}(t+\omega)\int_{-\infty}^{t+\omega} k_{ij}(t-s+\omega)g_j(v_j)ds = -\beta_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)g_j(v_j)ds$$

(**A2**) The sequence of times $\{t_k\}(k \in N)$ satisfies $t_k < t_{k+1}$, $\lim_{k\to+\infty} t_k = +\infty$, and $-2 \le I_{ik} \le 0$ for $i \in \{1, 2, \cdots, n\}, k \in N$.

(**A3**) For $i, j = 1, 2, \cdots, n, k = 1, 2, \cdots,$ there exists a positive integer q such that

$$I_{i(k+q)} = I_{ik},\ t_{k+q} = t_k + q.$$

(**A4**) $f_j(\cdot), g_j(\cdot) \in C(R, R)$, and there exist positive numbers $M_f, M_g, \mu_j, \nu_j (j = 1, 2, \cdots, n)$ such that, for $u, v \in R$,

$$f_j(0) = 0,\quad |f_j(u)| \le M_f,\quad |f_j(u) - f_j(v)| \le \mu_j|u - v|,$$

$$g_j(0) = 0,\quad |g_j(u)| \le M_g,\quad |g_j(u) - g_j(v)| \le \nu_j|u - v|.$$

Remark 2.1 In assumption (A4), the activation functions $f_j, g_j, j = 1, 2, \cdots, n,$ are typically assumed to be bounded and Lipchtiz continuous and need not to be differentiable.

Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in R^n$, where T denotes the transposition. We define $|x| = (|x_1|, |x_2|, \cdots, |x_n|)^T$ and $\|x\| = \max_{1\le i\le n}|x_i|$. Obviously, the solution $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ of (1) has components $x_i(t)$ piece-wise continuous on $(-\tau, +\infty)$, $x(t)$ is differentiable on the open intervals (t_{k-1}, t_k) and $x(t_k^+)$ exists.

Definition 2.1 *A solution $x(t)$ of system (1) is said to be ω anti-periodic solution, if*

$$x(t+\omega) = -x(t),\quad t \ne t_k.$$

$$x(t_k+\omega)^+ = -x(t_k^+),\quad k = 1, 2, \cdots,$$

and the smallest positive number ω is called ω anti-periodic of function $x(t)$.

Definition 2.2 *Let $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T$ be an anti-periodic solution of (1) with initial value $\varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \cdots, \varphi_n^*(t))^T$. If there exist constants $\lambda > 0, M > 1$ such that for every solution $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ with an initial value $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T$,*

$$|x_i(t) - x_i^*(t)| \le M\|\varphi - \varphi^*\|e^{-\lambda t}, \text{for all } t > 0.$$

where $\|\varphi - \varphi^*\| = \sup_{-\tau\le s\le 0}\max_{1\le i\le n}|\varphi_(s) - \varphi_i^*(s)|$. Then $x(t)$ is said to be globally exponentially stable.

Lemma 2.1 [19] *Let u and v be two states of system (1), then we have*

$$\left|\bigwedge_{j=1}^n \alpha_{ij}(t)g_j(u) - \bigwedge_{j=1}^n \alpha_{ij}(t)g_j(v)\right| \le \sum_{j=1}^n |\alpha_{ij}(t)||g_j(u) - g_j(v)|,$$

and

$$\left|\bigvee_{j=1}^n \beta_{ij}(t)g_j(u) - \bigvee_{j=1}^n \beta_{ij}(t)g_j(v)\right| \le \sum_{j=1}^n |\beta_{ij}(t)||g_j(u) - g_j(v)|.$$

Lemma 2.2 *Let $(A1) - (A4)$ hold, Suppose that $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is a solution of system (1) with initial conditions*

$$x_i(s) = \varphi_i(s), |\varphi_i(s)| < \eta,\ s \in [-\tau, 0], \tag{2}$$

where $i = 1, 2, \cdots, n$. Then

$$|x_i(t)| < \eta, |\varphi_i(s)| < \eta,\ t \ge 0, \tag{3}$$

where

$$\eta > \frac{\Theta}{a_i^-}, \Theta = \sum_{j=1}^n \overline{a}_{ij}M_f + \sum_{j=1}^n(\overline{\alpha}_{ij} + \overline{\beta}_{ij})k_{ij}^+M_g + \overline{E}_i. \tag{4}$$

Proof. For any given initial condition, assumption (A4) guarantees the existence and uniqueness of $x(t)$, the solution to (1) in $[-\tau, +\infty)$.

By way of contradiction, suppose that (4) does not hold. Notice that $x_i(t_k^+) = (1 + I_{ik})x_i(t_k)$ and assumption (A2), then

$$|x_i(t_k^+)| = |(1 + I_{ik})x_i(t_k)| \le |x_i(t_k)|.$$

If $|x_i(t_k^+)| \ge \eta$, then $|x_i(t_k)| \ge \eta$. Thus we may assume that there must exist $i \in \{1, 2, \cdots, n\}$ and $t^* \in (t_k, t_{k+1}]$ such that for all $t \in (-\tau, t^*)$,

$$|x_i(t^*)| = \eta,\quad |x_j(t^*)| < \eta. \tag{5}$$

where $j = 1, 2, \cdots, n$. By directly computing the upper right derivative of $|x_i(t)|$, together with the assumptions (4), (5), (A4) and Lemma 2.1, we get that

$$\begin{aligned}
0 \leq\ & D^+|x_i(t^*)| \leq -a_i(t^*)x_i(t^*) \\
& + \Bigg| \sum_{j=1}^{n} d_{ij}(t^*) f_j(t^* - \tau_{ij}(t^*)) \\
& + \bigwedge_{j=1}^{n} \alpha_{ij}(t^*) \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(x_j(s)) ds \\
& + \bigvee_{j=1}^{n} \beta_{ij}(t^*) \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(x_j(s)) ds + E_i(t^*) \Bigg| \\
\leq\ & -a_i(t^*)x_i(t^*) \\
& + \left| \sum_{j=1}^{n} d_{ij}(t^*)[f_j(t^* - \tau_{ij}(t^*)) - f_j(0)] \right| \\
& + \Bigg| \bigwedge_{j=1}^{n} \alpha_{ij}(t^*) \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(x_j(s)) ds \\
& - \bigwedge_{j=1}^{n} \alpha_{ij}(t^*) \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(0) ds \Bigg| \\
& + \Bigg| \bigvee_{j=1}^{n} \beta_{ij}(t^*) \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(x_j(s)) ds \\
& - \bigvee_{j=1}^{n} \beta_{ij}(t^*) \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(0) ds \Bigg| + |E_i(t^*)| \\
\leq\ & -a_i(t^*)x_i(t^*) \\
& + \sum_{j=1}^{n} |d_{ij}(t^*)| |f_j(t^* - \tau_{ij}(t^*)) - f_j(0)| \\
& + \sum_{j=1}^{n} |\alpha_{ij}(t^*)| \Bigg| \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(x_j(s)) ds \\
& - \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(0) ds \Bigg| \\
& + \sum_{j=1}^{n} |\beta_{ij}(t^*)| \Bigg| \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(x_j(s)) ds \\
& - \int_{-\infty}^{t^*} k_{ij}(t^* - s) g_j(0) ds \Bigg| + |E_i(t^*)| \\
\leq\ & -a_i(t^*)x_i(t^*) \\
& + \sum_{j=1}^{n} |d_{ij}(t^*)| |f_j(t^* - \tau_{ij}(t^*))| + |E_i(t^*)| \\
& + \sum_{j=1}^{n} |\alpha_{ij}(t^*)| \int_{-\infty}^{t^*} |k_{ij}(t^* - s)| |g_j(x_j(s))| ds \\
& + \sum_{j=1}^{n} |\beta_{ij}(t^*)| \int_{-\infty}^{t^*} |k_{ij}(t^* - s)| |g_j(x_j(s))| ds \\
\leq\ & -a^- \eta + \sum_{j=1}^{n} \overline{d}_{ij} M_f + \sum_{j=1}^{n} (\overline{\alpha}_{ij} + \overline{\beta}_{ij}) k_{ij}^+ M_g + \overline{E}_i \\
<\ & 0
\end{aligned}$$

which is a contradiction and implies that (4) holds. This completes the proof.

III. Main Result

In this section, we derive some sufficient conditions of existence and global exponential stability of anti periodic solution of system (1).

Theorem 3.1 *Assume that* $(A1)-(A4)$ *hold, if the following assumption is satisfied*

(A5): There exist constants $\gamma > 0, \lambda > 0, i, j = 1, 2, \cdots, n$, *such that*

$$(\lambda - a_i^-) + \sum_{j=1}^{n} (\overline{d}_{ij} \mu_j + (\overline{\alpha}_{ij} + \overline{\beta}_{ij}) k_{ij}^+ \nu_j) < -\gamma < 0 \quad (6)$$

Let $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T$ *be a solution of (1) with initial value* $\varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \cdots, \varphi_n^*(t))^T$, *and* $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ *be a solution of (1) with initial value* $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T$. *Then* $x^*(t)$ *is said to be globally exponentially stable.*

Proof. Let $y_i(t) = x_i(t) - x_i^*(t), i = 1, 2, \cdots, n$. Then

$$\begin{cases}
y_i'(t) = -a_i(t)(x_i(t) - x_i^*(t)) \\
\quad + \sum_{j=1}^{n} d_{ij}(t)(f_j(x_j(t - \tau_{ij}(t))) \\
\quad - f_j(x_j^*(t - \tau_{ij}(t)))) \\
\quad + \bigwedge_{j=1}^{n} \alpha_{ij}(t) \int_{-\infty}^{t} k_{ij}(t - s) \\
\quad \times (g_j(x_j(s)) - g_j(x_j^*(s))) ds \\
\quad + \bigvee_{j=1}^{n} \beta_{ij}(t) \int_{-\infty}^{t} k_{ij}(t - s) \\
\quad \times (g_j(x_j(s)) - g_j(x_j^*(s))) ds \\
y_i(t_k^+) = (1 + I_{ik}) y_i(t_k), k = 1, 2, \cdots,
\end{cases} \quad (7)$$

Define a Lyapunov functional as

$$V_i(t) = |y_i(t)| e^{\lambda t}, \quad i = 1, 2, \cdots, n. \quad (8)$$

It follows from (6), (7) and (8) that

$$\begin{aligned}
& D^+ V_i(t) \\
=\ & D^+(|y_i(t)|) e^{\lambda t} + \lambda |y_i(t)| e^{\lambda t} \\
\leq\ & (\lambda - a_i^-) |y_i(t)| e^{\lambda t} \\
+\ & \Bigg[\sum_{j=1}^{n} |d_{ij}(t)| |f_j(x_j(t - \tau_{ij}(t))) - f_j(x_j^*(t - \tau_{ij}(t)))| \\
+\ & \sum_{j=1}^{n} |\alpha_{ij}(t)| \int_{-\infty}^{t} |k_{ij}(t - s)| |g_j(x_j(s)) - g_j(x_j^*(s))| ds \\
+\ & \sum_{j=1}^{n} |\beta_{ij}(t)| \int_{-\infty}^{t} |k_{ij}(t - s)| |g_j(x_j(s)) - g_j(x_j^*(s))| ds \Bigg] \\
\leq\ & (\lambda - a_i^-) |y_i(t)| e^{\lambda t} + \sum_{j=1}^{n} \overline{d}_{ij} \mu_j |y_j(t - \tau_{ij}(t))| e^{\lambda t} \\
+\ & \sum_{j=1}^{n} (\overline{\alpha}_{ij} + \overline{\beta}_{ij}) k_{ij}^+ \nu_j |y_j(t)| e^{\lambda t}, t \neq t_k. \quad (9)
\end{aligned}$$

and

$$\begin{aligned} V_i(t_k^+) &= |y_i(t_k^+)|e^{\lambda t_k} = |x_i(t_k^+) - x_i^*(t_k^+)|e^{\lambda t_k} \\ &= |(1+I_{ik})y_i(t_k)|e^{\lambda t_k} \end{aligned} \quad (10)$$

where $i = 1, 2, \cdots, n$. Let $M > 0$ denote an arbitrary real number and set

$$\|\varphi - \varphi^*\| = \sup_{-\tau \le s \le 0} \max_{1 \le j \le n} |\varphi_j(s) - \varphi_j^*(s)| > 0.$$

By (8), we have for all $t \in (-\infty, 0], i = 1, 2, \cdots, n,$

$$V_i(t) = |y_i(t)|e^{\lambda t} < M\|\varphi - \varphi^*\|.$$

Thus we can claim that, for all $t \in (-\infty, t_1], i = 1, 2, \cdots, n,$

$$V_i(t) = |y_i(t)|e^{\lambda t} < M\|\varphi - \varphi^*\|. \quad (11)$$

Otherwise, there must exist $i \in \{1, 2, \cdots, n\}$ and $\delta_0 \in (-\tau, t_1]$ such that

$$V_i(\delta_0) = M\|\varphi - \varphi^*\|, \ V_j(t) < M\|\varphi - \varphi^*\|, \quad (12)$$

for all $t \in [-\tau, \tau_0), j = 1, 2, \cdots, n$. Combining (9), (10) with (11), we have

$$\begin{aligned} 0 &\le D^+V_i(\tau_0) \le (\lambda - a_i^-)|y_i(\tau_0)|e^{\lambda\tau_0} \\ &\quad + \sum_{j=1}^n \overline{d}_{ij}\mu_j|y_j(\tau_0 - \tau_{ij}(\tau_0))|e^{\lambda\tau_0} \\ &\quad + \sum_{j=1}^n (\overline{\alpha}_{ij} + \overline{\beta}_{ij})k_{ij}^+\nu_j|y_j(\tau_0)|e^{\lambda\tau_0} \\ &= (\lambda - a_i^-)|y_i(\tau_0)|e^{\lambda\tau_0} \\ &\quad + \sum_{j=1}^n \overline{d}_{ij}\mu_j|y_j(\tau_0 - \tau_{ij}(\tau_0))|e^{\lambda(\tau_0 - \tau_{ij}(\tau_0))}e^{\lambda\tau_{ij}(\tau_0)} \\ &\quad + \sum_{j=1}^n (\overline{\alpha}_{ij} + \overline{\beta}_{ij})k_{ij}^+\nu_j|y_j(\tau_0)|e^{\lambda\tau_0} \\ &\le (\lambda - a_i^-)M\|\varphi - \varphi^*\| + \sum_{j=1}^n \overline{d}_{ij}\mu_j M\|\varphi - \varphi^*\|e^{\lambda\tau} \\ &\quad + \sum_{j=1}^n (\overline{\alpha}_{ij} + \overline{\beta}_{ij})k_{ij}^+\nu_j M\|\varphi - \varphi^*\| \\ &= \left[(\lambda - a_i^-) + \sum_{j=1}^n \overline{d}_{ij}\mu_j \right. \\ &\quad \left. + \sum_{j=1}^n (\overline{\alpha}_{ij} + \overline{\beta}_{ij})k_{ij}^+\nu_j\right] M\|\varphi - \varphi^*\|. \end{aligned}$$

Then

$$(\lambda - a_i^-) + \sum_{j=1}^n \overline{d}_{ij}\mu_j + \sum_{j=1}^n (\overline{\alpha}_{ij} + \overline{\beta}_{ij})k_{ij}^+\nu_j > 0.$$

Which is contradiction with (A5). So (11) holds true. From (11), we have

$$V_i(t_1) = |y_i(t_1)|e^{\lambda t_1} < M\|\varphi - \varphi^*\|, i = 1, 2, \cdots,$$

and $i = 1, 2, \cdots,$

$$V_i(t_1^+) = |1 + I_{i1}||y_i(t_1)|e^{\lambda t_1} \le |y_i(t_1)|e^{\lambda t_1} < M\|\varphi - \varphi^*\|.$$

Therefore, for $t \in [t_1, t_2]$, we can repeat the above procedure and have

$$V_i(t) = |y_i(t)|e^{\lambda t_1} < M\|\varphi - \varphi^*\|, t \in [t_1, t_2], i = 1, 2, \cdots.$$

Similarly, it follows that

$$V_i(t) = |y_i(t)|e^{\lambda t_1} < M\|\varphi - \varphi^*\|, t > 0, i = 1, 2, \cdots.$$

Namely,

$$|x_i(t) - x_i^*(t)| = |y_i(t)| < M\|\varphi - \varphi^*\|, t > 0.$$

Now the proof is completed.

Theorem 3.2 *Assume that $(A1)-(A5)$ hold, then system (1) has exactly one ω-anti-periodic solution which is globally exponentially stable.*

Proof. Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ be a solution of system (1) with initial conditions

$$x_i(s) = \varphi_i(s), \ |\varphi_i(s)| < \eta, \ s \in [-\tau, 0], i = 1, 2, \cdots, n. \quad (13)$$

From Lemma 2.2, it follows that the solution $x(t)$ is bounded and

$$|x_i(t)| < \eta, t \in R, i = 1, 2, \cdots, n. \quad (14)$$

For any natural number p, it follows from system (1) that

$$\begin{aligned} &((-1)^{p+1}x_i(t + (p+1)\omega))' \\ = \ &(-1)^{p+1}\{-a_i(t + (p+1)\omega)x_i(t + (p+1)\omega) \\ &+ \sum_{j=1}^n d_{ij}(t + (p+1)\omega) \\ &\times f_j(x_j(t + (p+1)\omega - \tau_{ij}(t + (p+1)\omega))) \\ &+ \bigwedge_{j=1}^n \alpha_{ij}(t + (p+1)\omega) \\ &\times \int_{-\infty}^{t+(p+1)\omega} k_{ij}(t + (p+1)\omega - s)g_j(x_j(s))ds \\ &+ \bigvee_{j=1}^n \beta_{ij}(t + (p+1)\omega) \\ &\times \int_{-\infty}^{t+(p+1)\omega} k_{ij}(t + (p+1)\omega - s)g_j(x_j(s))ds \\ &+ E_i(t + (p+1)\omega)\} \\ = \ &-a_i(t)(-1)^{p+1}x_i(t + (p+1)\omega) \\ &+ \sum_{j=1}^n d_{ij}(t)f_j((-1)^{p+1}x_j(t + (p+1)\omega - \tau_{ij}(t))) \\ &+ \bigwedge_{j=1}^n \alpha_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)g_j(x_j(s))ds \\ &+ \bigvee_{j=1}^n \beta_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)g_j(x_j(s))ds \\ &+ E_i(t), t \ne t_k, \end{aligned} \quad (15)$$

and

$$\begin{aligned} &(-1)^{p+1}x_i((t_k + (p+1)\omega)^+) \\ = \ &(-1)^{p+1}(1 + I_{i(k+(p+1)q)})x_i(t + (p+1)\omega) \\ = \ &(-1)^{p+1}(1 + I_{ik})x_i(t + (p+1)\omega) \\ = \ &(1 + I_{ik})((-1)^{p+1}x_i(t + (p+1)\omega), \end{aligned} \quad (16)$$

where $i=1,2,\cdots,n, k=1,2,\cdots$. Thus $(-1)^{p+1}x(t+(p+1)\omega)$ is the solution of system (1). From Theorem 3.1, there exists a constant $M>1$ such that

$$\begin{aligned} & |(-1)^{p+1}x_i(t+(p+1)\omega)-(-1)^p x_i(t+p\omega)| \\ \leq\ & Me^{-\lambda(t+p\omega)}\sup_{-\infty\leq s\leq 0}\max_{1\leq i\leq n}|x_i(s+\omega)+x_i(s)| \\ \leq\ & 2e^{-\lambda(t+p\omega)}M\eta, \end{aligned} \tag{17}$$

and

$$\begin{aligned} & |(-1)^{p+1}x_i((t_k+(p+1)\omega)^+) \\ & -(-1)^p x_i((t_k+p\omega)^+)| \\ =\ & |x_i((t_k+(p+1)\omega)^+)+x_i((t_k+p\omega)^+)| \\ =\ & |1+I_{ik}||x_i(t_k+(p+1)\omega)+x_i(t_k+p\omega)| \\ \leq\ & 2M\eta e^{-\lambda(t_k+p\omega)}, \end{aligned} \tag{18}$$

where $k\in N, i=1,2,\cdots,n$. Therefore, for any natural number q, we have

$$\begin{aligned} & (-1)^{q+1}x_i(t+(q+1)\omega) \\ =\ & x_i(t)+\sum_{k=0}^{q}[(-1)^{k+1}x_i(t+(k+1)\omega) \\ & -(-1)^k x_i(t+k\omega)], t\neq t_k. \end{aligned} \tag{19}$$

It follows that

$$\begin{aligned} & |(-1)^{q+1}x_i(t+(q+1)\omega)| \\ \leq\ & |x_i(t)|+\sum_{k=0}^{q}|(-1)^{k+1}x_i(t+(k+1)\omega) \\ & -(-1)^k x_i(t+k\omega)|, t\neq t_k, \end{aligned} \tag{20}$$

and

$$\begin{aligned} & |(-1)^{q+1}x_i((t_k+(q+1)\omega)^+)| \\ =\ & |(1+I_{ik})(-1)^{q+1}x_i(t_k+(q+1)\omega)| \\ \leq\ & |(-1)^{q+1}x_i(t_k+(q+1)\omega)|, \end{aligned} \tag{21}$$

where $i=1,2,\cdots,n$. It follows from (17)-(21) that $(-1)^{q+1}x_i(t+(q+1)\omega)$ is a fundamental sequence on any compact set of R. Obviously, $\{(-1)^q x(t+q\omega)\}$ uniformly converges to a piece-wise continuous function $x^*(t)=(x_1^*(t),x_2^*(t),\cdots,x_n^*(t))^T$ on compact set of R.

Now we show that $x^*(t)$ is an ω-anti-periodic solution of system (1). Since

$$\begin{aligned} x^*(t+\omega) &= \lim_{q\to\infty}(-1)^q x(t+\omega+q\omega) \\ &= -\lim_{q+1\to\infty}(-1)^{q+1}x(t+(q+1)\omega) \\ &= -x^*(t), t\neq t_k, \end{aligned}$$

and

$$\begin{aligned} x^*((t+\omega)^+) &= \lim_{q\to\infty}(-1)^q x((t+\omega+q\omega)^+) \\ &= -\lim_{q+1\to\infty}(-1)^{q+1}x((t+(q+1)\omega)^+) \\ &= -x^*(t_k^+), k=1,2,\cdots. \end{aligned}$$

Namely, $x^*(t)$ is ω-anti-periodic.

Next we show that $x^*(t)$ is a solution of system (1). Noting that the right-hand side of (1) is piece-wise continuous. (15) and (16) imply that $\{((-1)^{p+1}x(t+(q+1)\omega))'\}$ uniformly converges to a piece-wise continuous function on any compact subset of R. Let $p\to\infty$ on both sides of (15) and (16), we can obtain

$$\begin{cases} \dot{x}_i^*(t) = -a_i(t)x_i^*(t) \\ \quad +\sum_{j=1}^n d_{ij}(t)f_j(x_j^*(t-\tau_{ij}(t))) \\ \quad +\bigwedge_{j=1}^n \alpha_{ij}(t)\int_{-\infty}^t k_{ij}(t-s)g_j(x_j^*(s))ds \\ \quad +\bigvee_{j=1}^n \beta_{ij}(t)\int_{-\infty}^t k_{ij}(t-s)g_j(x_j^*(s))ds \\ \quad +E_i(t)], t\geq 0, t\neq t_k \\ x_i^*(t_k^+) = (1+I_{ik})x_i^*(t_k), k=1,2,\cdots,\ i=1,2,\cdots,n. \end{cases} \tag{22}$$

Thus $x^*(t)$ is a solution of system (1). Applying Theorem 3.1, we can obtain that $x^*(t)$ is globally exponentially stable. The proof of Theorem 3.2 is completed.

Remark 3.1 In compared with the results published, the assumptions (A1)-(A5) which can assure the existence and exponential stability of system (1), have relation to the parameters of system and impulsive operators. The results published [24,25,26,32] can not be applied in this paper. Therefore the results obtained are new and complementary to previously known publication.

IV. An illustrative example

In this section, an example is given to show effectiveness of results obtained.

Example 4.1 Consider the following FCNNs with mixed delay and impulsive effects.

$$\begin{cases} x_i'(t) = -a_i(t)x_i(t)+\sum_{j=1}^2 d_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ \quad +\bigwedge_{j=1}^2 \alpha_{ij}(t)\int_{-\infty}^t k_{ij}(t-s)g_j(x_j(s))ds \\ \quad +\bigvee_{j=1}^2 \beta_{ij}(t)\int_{-\infty}^t k_{ij}(t-s)g_j(x_j(s))ds \\ \quad +E_i(t), t\neq \frac{k\pi}{2}, k=1,2,\cdots, \\ x_i(t_k^+)) = (1+I_{ik})x_i(t_k), i=1,2, \end{cases} \tag{23}$$

where $a_1(t)=2+|\sin t|, a_2(t)=2.4+|\cos t|, f_j(x)=g_j(x)=\frac{1}{2}(|x+1|-|x-1|)(j=1,2), k_{ij}(s)=1, k_{ij}^+=1, \tau_{ij}(t)=0.5|\sin t|$.

$$(d_{ij}(t))_{2\times 2}=\begin{pmatrix} 1/4|\cos t| & 1/8|\sin t| \\ 1/6|\sin t| & 1/3|\cos t| \end{pmatrix},$$

$$(\alpha_{ij}(t))_{2\times 2}=\begin{pmatrix} 1/8|\sin t| & 1/6|\cos t| \\ 1/6|\cos t| & 1/8|\sin t| \end{pmatrix}$$

$$(\beta_{ij}(t))_{2\times 2}=\begin{pmatrix} 1/16|\cos t| & 1/4|\sin t| \\ 1/4|\sin t| & 1/16|\cos t| \end{pmatrix},$$

$$(E_i(t))_{2\times 1}=\begin{pmatrix} 1/4\sin t \\ 1/3\cos t \end{pmatrix}$$

then, we can easily check that $\mu_j=\nu_j=1, a_1^-=2, a_2^-=2.2$, and

$$(\overline{d}_{ij})_{2\times 2}=\begin{pmatrix} 1/4 & 1/8 \\ 1/6 & 1/3 \end{pmatrix},\quad (\overline{\alpha}_{ij})_{2\times 2}=\begin{pmatrix} 1/8 & 1/6 \\ 1/6 & 1/8 \end{pmatrix}$$

$$(\overline{\beta}_{ij})_{2\times 2} = \begin{pmatrix} 1/16 & 1/4 \\ 1/4 & 1/16 \end{pmatrix}$$

Let $\gamma = 0.1, \lambda = 0.8$. Then

$$\begin{aligned} & (\lambda - a_1^-) + \sum_{j=1}^{2} (\overline{d}_{1j}\mu_j + (\overline{\alpha}_{1j} + \overline{\beta}_{1j})k_{ij}^+\nu_j) \\ = & (0.8-2) + (\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{6} + \frac{1}{16} + \frac{1}{4}) \\ = & -0.18 < -0.1 \end{aligned}$$

$$\begin{aligned} & (\lambda - a_2^-) + \sum_{j=1}^{2} (\overline{d}_{2j}\mu_j + (\overline{\alpha}_{2j} + \overline{\beta}_{2j})k_{ij}^+\nu_j) \\ = & (0.6-2.2) + (\frac{1}{8} + \frac{1}{3} + \frac{1}{6} + \frac{1}{8} + \frac{1}{4} + \frac{1}{16}) \\ = & -0.34 < -0.1 \end{aligned}$$

It is easy to conclude that system (23) satisfies all condition of Theorem 3.2, Thus system (23) has exactly one π-anti-periodic solutions which is globally exponentially stable (see Fig.1).

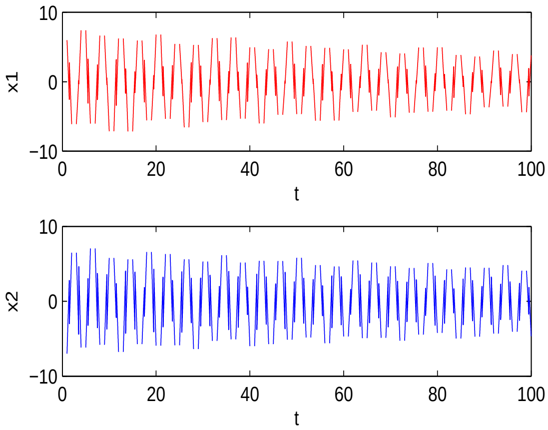

Fig.1: Numerical solution $(x_1(t), x_2(t))$ of systems (23) with initial value $(6, -7)$.

V. Conclusion

In this paper, the existence and globally exponential stability of anti-periodic solution for fuzzy cellular neural networks with mixed delays and impulsive effects are considered. With the aid of differential inequality techniques, some sufficient conditions set up here are easily verified and these conditions are correlated with parameters of the system (1). The obtained criteria can be applied to design globally exponential stability of anti-periodic fuzzy cellular neural networks.

References

[1] L. O. Chua and L. Yang, " Cellular neural networks: Theory", *IEEE Trans. Circuits Syst. I*, vol. 35, no. 10, pp. 1257-1272, 1988.
[2] L. O. Chua and L. Yang, "Cellular neural networks:Application", *IEEE Trans. Circ. Syst.I*, vol. 35, no. 10, pp. 1273-1290, 1988.
[3] Z. Q. Zhang, W. B. Liu, and D. M. Zhou, " Global asymptotic stability to a generalized Cohen-Grossberg BAM neural networks of neutral type delays", *Neural Netw.*, vol. 25, no. 1, pp. 94-105, 2012.
[4] Q. Zhang, X. Wei, and J. Xu, "Delay-dependent exponential stability of cellular neural networks with time-varying delays", *Chaos Solitons Fractals* , vol. 23, no. 4, pp. 1363-1369, 2005.
[5] Y. Q. Liu and W. S. Tang, "Existence and exponential stability of periodic solution for BAM neural networks with periodic coefficients and delays." *Neurocomputing*, vol. 69, no.(16-18), pp. 2152–2160, 2006.
[6] S.A. Hussain, M. Emran, M. Salman, and U. Shakeel,"Positioning a Mobile Subscriber in a Cellular Network System based on Signal Strength", *IAENG International Journal of Computer Science*, vol. 34, no. 2, pp. 245-250, 2007.
[7] L. Yang and Y. Li, "Existence and exponential stability of periodic solution for stochastic Hopfield neural networks on time scales", *Neurocomputing*,vol. 167, pp. 543-550, 2015.
[8] Y. Wu, T. Li, and Y. Wu, "Improved exponential stability criteria for recurrent neural networks with time-varying discrete and distributed delays", *International Journal of Automation and Computing*, vol. 7, no. 2, pp. 199-204, 2010.
[9] W. Wu, "Global exponential stability of a unique almost periodic solution for neutral-type cellular neural networks with distributed delays", *Journal of Applied Mathematics*, vol. 2014(10), pp. 1-8, 2014.
[10] Y. Xia, J. Cao, and S. Cheng, "Global exponential stability of delayed cellular neural networks with impulses", *Neurocomputing*,vol. 70, no.13, pp. 2495-2501, 2007.
[11] Q. Song and J. Zhang, "Global exponential stability of impulsive Cohen-Grossberg neural network with time-varying delays", *Nonlinear Analysis RWA*, vol. 9, no. 2, pp. 500-510, 2008.
[12] J. Zhao, "A Note on Hopfield neural network stability", *IAENG International Journal of Computer Science*, vol. 42, no.4, pp. 332-336, 2015.
[13] Q. Zhou, "Global exponential stability of BAM neural networks with distributed delays and impulses", *Nonlinear Analysis:RWA*, vol. 10, no.1, pp. 144-153, 2009.
[14] Y. Wang, W. Xiong, Q. Zhou, B. Xiao, and Y. Yu, " Global exponential stability of cellular neural networks with mixed delays and impulses," *Chaos Solitons & Fractals*, vol. 34, no. 3, pp. 896-902, 2007.
[15] Y. Li and L. Lu, " Global exponential stability and existence of periodic solutions of Hopfield-type neural networks with impulses," *Phys. Lett. A*,vol. 333, no. (1-2), pp. 62-71, 2004.
[16] Y. Yang and J. Cao, " Stability and periodicity in delayed cellular neural networks with impulsive effects", *Nonolinear Analysis: RWA*,vol. 8, no.1, pp. 362-374, 2007.
[17] Y. Li and J. Wang, "An analysis on the global exponential stability and the existence of periodic solutions for non-autonomous hybird BAM neural networks with distributed delays and impulsies", *Comput. Math. Appl.* vol. 56, no. 9, pp. 2256-2267, 2008.
[18] D. Li, X. Wang, and D. Xu, "Existence and global exponential stability of periodic solution for impulsive stochastic neural networks with delays," *Nonlinear Analysis Hybrid Systems*, vol. 6, no. 3, pp. 847-858, 2012.
[19] T. Yang and L. Yang, "The global stability of fuzzy cellular neural networks." *IEEE Trans. Circ. Syst. I* vol. 43, no.10, pp. 880–883, 1996.
[20] T. Yang, L. Yang, C. Wu, and L. O. Chua, "Fuzzy cellular neural networks: theory." *Proc IEEE Int Workshop Cellular Neural Networks Appl.* 1996, pp. 181–186.
[21] A. Martellim, E. Fugassa, A. Voci, and G. Brambilla,"Asymptotic stability criteria for T-S fuzzy neural networks with discrete interval and distributed time-varying delays", *Neural Computing and Applications*, vol. 21,no. 1, pp.357-367, 2012.
[22] X. Li, R. Rakkiyappan, and P. Balasubramaniam, "Existence and global stability analysis of equilibrium of fuzzy cellular neural networks with time delay in the leakage term under impulsive perturbations", *Journal of the Franklin Institute*, vol. 348, no. 2, pp. 135-155, 2011.
[23] Q. Zhang and R. Xiang, " Global asymptotic stability of fuzzy cellular neural networks with time-varying delays", *Phy. Lett. A*, vol. 372, no. 22, pp. 3971–3977, 2008.
[24] G. Yang, " New results on the stability of fuzzy cellular neural networks with time-varying leakage delays", *Neural Computing and Applications*,vol. 25, no. (7-8), pp 1709-1715, 2014.
[25] J. Liu, Q. Zhang, and Z. Luo"Dynamical analysis of fuzzy cellular neural networks with periodic coefficients and time-varying delays", *IAENG International Journal of Applied Mathematics*, vol.46, no. 3, pp. 298-304, 2016.
[26] J. Shao, "An anti-periodic solution for a class of recurrent neural networks", *J. Comput. Appl. Math.* vol. 228, no. 1, pp. 231-237, 2009.
[27] P. Shi and L. Dong, "Existence and exponential stability of anti-periodic solutions of Hopfield neural networks with impulses", *Appl. Math. Comput.* vol. 216, no.2, pp. 623-630, 2010.
[28] L. Pan and J. Cao, "Anti-periodic solution for delayed cellular neural networks with impulsive effects", *Nonlinear Analysis: Real World Applications* vol. 12, no. 6, pp. 3014-3027, 2011.
[29] A. Abdurahman and H. Jiang, "The existence and stability of the anti-periodic solution for delayed Cohen-Grossberg neural networks with impulsive effects", *Neurocomputing*, vol. 149, Part A, pp. 22-28, 2015.
[30] X. R. Wei, and Z.P. Qiu, "Anti-periodic solutions for BAM neural networks with time delays", *Appl. Mathe. Comput.*, vol. 221, pp. 221-229, 2013.

[31] Y. Liu, Y.Q. Yang, T. Liang, and L. Li, "Existence and global exponential stability of anti-periodic solutions for competitive neural networks with delays in the leakage terms on time scales", *Neurocomputing*, vol. 133, no. 8, PP. 471-482, 2014.

[32] Y.K. Li, L. Yang, and W.Q. Wu, "Anti-periodic solutions for a class of Cohen-Grossberg neural networks with time-varying on time scales", *Int. J. Syst. Sci*. vol. 42, no. 7, pp. 1127-1132, 2011.

[33] Y.K. Li, and J. Shu, "Anti-periodic solutions to impulsive shunting inhibitory cellular neural networks with distributed delays on time scales", *Commun. Nonlinear Sci. Numer. Simul.* vol. 16, no. 8, pp. 3326-3336, 2011.

[34] Q. Zhang, L. Yang, and J. Liu, Existence and stability of anti-periodic solutions for impulsive fuzzy Cohen-Grossberge neural networks on time scales, *Math. Slovaca*,vol. 64, no. 1, pp. 119-138, 2014.

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