Dynamical Behavior of First-Order Nonlinear Fuzzy Difference Equation

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Abstract—**We investigate the existence, the uniqueness and the asymptotic behavior of the positive solution to first-order nonlinear fuzzy difference equations which read as:**

$$x_{n+1} = A + Bx_n e^{-Cx_n}, n = 0, 1, \cdots,$$

where (x_n) is a sequence of positive fuzzy numbers, A, B, C and the initial value x_0 are positive fuzzy numbers. Some conditions are obtained for the existence of positive fuzzy solution and stability of positive equilibrium. Finally an illustrative example is given to show the effectiveness of the obtained results.

Index Terms—**nonlinear fuzzy difference equation, asymptotic behavior, boundedness.**

I. Introduction

It is well known that difference equations appear naturally as discrete analogous of numerical solutions of differential equation, and that delay differential equation having many applications in economics, biology, computer science, control engineering and so on. The study of discrete dynamical systems described by difference equations or difference equations systems has received great attention in mathematical literature. In particular, the existence, persistence, boundedness, local asymptotic stability, and global character of positive solutions of difference equations have been discussed in recent years [1-13]. For example, EI-Metwally et al. [1] investigated the asymptotic behavior of population model

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}, n = 0, 1, \cdots,$$

where β is the population growth rate and α is the population immigration rate.

In 2013, Ibrahim and Zhang [2] studied the rate of convergence of a solution that converges to the equilibrium (0,0) for a system of two high-order nonlinear difference equations

$$x_{n+1} = \frac{x_{n-k}}{q + \prod_{i=0}^{k} y_{n-i}}, \quad y_{n+1} = \frac{y_{n-k}}{q + \prod_{i=0}^{k} x_{n-i}},$$

where $p, q \in (0, \infty), x_i \in (0, \infty), y_i \in (0, \infty)$ and $i = 0, 1, \cdots, k$.

Papaschinopoulos and Schinas [3] investigated the global behavior for the following two nonlinear difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0, 1, \cdots,$$

where A is a positive real number, p, q are positive integers, and $x_{-p}, \cdots, x_0, y_{-q}, \cdots, y_0$ are positive real numbers.

Although difference equations and a systems of difference equations are very simple in their forms, it is extremely difficult to understand the behavior of their solutions. Moreover, these models inherently process uncertainty or vagueness. In order to consider these uncertain factors, fuzzy set theory is a powerful tool for modeling uncertainty and processing vague or subjective information in a mathematical models [14,15]. More precisely, using of fuzzy difference equations is a natural way to model the discrete dynamical systems with embedded uncertainty.

Fuzzy difference equations is a difference equation where parameters and initial values are fuzzy numbers, and its solutions is sequences of fuzzy numbers. Due to the applicability of fuzzy difference equation to the analysis of phenomena where imprecision is inherent, this class of difference equations are very important topic from theoretical point of view and for applications. Recently there has been an increasing interest in the study of fuzzy difference equations (see[16-28]).

Motivated by the discussion above, we investigate the dynamical behaviors of fuzzy difference equations. We specialize our study in this paper to the following family of first-order nonlinear fuzzy difference equations

$$x_{n+1} = A + Bx_n e^{-Cx_n}; n = 0, 1, \cdots, \tag{1}$$

where (x_n) is a sequence of positive fuzzy numbers, the parameters A, B, C and the initial value x_0 are positive fuzzy numbers.

II. Preliminary and some definitions

For the seek of completeness, we give some basic definitions on fuzziness, necessary for the sequel.

Definition 2.1[18] A fuzzy number is a function if $A : R \to [0, 1]$ satisfying conditions (i)-(iv) written below:

(i) A is normal, i. e., there exists an $x \in R$ such that $A(x) = 1$;

(ii) A is fuzzy convex, i. e., for all $t \in [0, 1]$ and $x_1, x_2 \in R$ such that

$$A(tx_1 + (1 - t)x_2) \geq \min\{A(x_1), A(x_2)\};$$

(iii) A is upper semi-continuous on R;

(iv) The support of A, supp$A = \overline{\{x : A(x) > 0\}}$ is compact.

Definition 2.2 [18] An α-cut, $[A]_\alpha$, is a crisp set which contains all the elements of the universal set X that have a membership function at least to the degree of α and can be expressed as $[A]_\alpha = \{x \in X : A(x) \geq \alpha\}$, namely, $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$.

It is obvious that if A is a positive real number then A is a fuzzy numbers and that $[A]_\alpha = [A, A], \alpha \in (0, 1]$. Then

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we say that A is a trivial fuzzy number.

Definition 2.3 [18] A fuzzy number A is completely determined by any pair $A=(A_l,A_r)$ of functions $A_l(\alpha),A_r(\alpha):[0,1]\to R$ satisfying the following conditions:
(i) $A_l(\alpha)$ is bounded, nondecreasing and left continuous function for all $\alpha\in(0,1]$;
(ii) $A_r(\alpha)$ is bounded, nonincreasing and left continuous function for all $\alpha\in(0,1]$;
(iii) $A_l(\alpha)$ and $A_r(\alpha)$ are right continuous for $\alpha=0$;
(iv) For all $\alpha\in(0,1]$, $A_l(\alpha)\le A_r(\alpha)$.

For every $A=(A_l,A_r), B=(B_l,B_r)$ and $k>0$, we define addition and multiplication as follows:
(i) $(A+B)_l(\alpha)=A_l(\alpha)+B_r(\alpha),\ (A+B)_r(\alpha)=A_r(\alpha)+B_r(\alpha)$;
(ii) $(kA)_l(\alpha)=kA_l(\alpha),(kA)_r(\alpha)=kA_r(\alpha)$.

The collection of all fuzzy numbers with addition and multiplication defined by (i)-(ii) is denoted by E^1.

Definition 2.4 [18] The distance between two arbitrary fuzzy numbers A and B is defined as follows:

$$D(A,B)=\sup_{\alpha\in[0,1]}\max\{|A_l(\alpha)-B_l(\alpha)|,|A_r(\alpha)-B_r(\alpha)|\}. \tag{2}$$

It is clear that (E^1,D) is a complete metric space.
The fuzzy analog of the boundedness and persistence (see [18,19]) is as follows.

Definition 2.5 A sequence of positive fuzzy numbers (x_n) persists (resp. is bounded) if there exists a positive real number M (resp. N) such that

$$\text{supp}x_n\subset[M,\infty)(\text{resp. supp}x_n\subset(0,N]),n=1,2,\cdots,$$

Definition 2.6 A sequence of positive fuzzy numbers (x_n) is bounded and persists if there exist positive real numbers $M,N>0$ such that

$$\text{supp}x_n\subset[M,N],\quad n=1,2,\cdots.$$

Definition 2.7 A positive fuzzy number x is a positive equilibrium of (1), if

$$x=A+Bxe^{-Cx}.$$

Let (x_n) be a sequence of positive fuzzy numbers and x is a positive fuzzy number, Suppose that

$$[x_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}],\ n=0,1,2,\cdots,\quad \alpha\in(0,1], \tag{3}$$

and

$$[x]_\alpha=[L_\alpha,R_\alpha],\quad \alpha\in(0,1]. \tag{4}$$

Definition 2.8 A sequence (x_n) converges to x with respect to D as $n\to\infty$ if $\lim_{n\to\infty}D(x_n,x)=0$.

Definition 2.9 (i) The positive equilibrium x of (1) is stable if for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that for every positive solution x_n of (1), which satisfies $D(x_0,x)\le\delta$, we have $D(x_n,x)\le\varepsilon$ for all $n>0$.
(ii) The positive equilibrium x of (1) is asymptotically stable, if it is stable and every positive solution x_n of (1) converges to the positive equilibrium x of (1) with respect to D as $n\to\infty$.

III. Main results

Lemma 3.1[29] Let $g:R^+\times R^+\times R^+\to R^+$ be continuous, A,B,C are fuzzy numbers. Then

$$[g(A,B,C)]_\alpha=g([A]_\alpha,[B]_\alpha,[C]_\alpha).\quad \alpha\in(0,1]. \tag{5}$$

Lemma 3.2[19] Let $u\in E^1$, $[u]_\alpha=[u_l(\alpha),u_r(\alpha)]$, $\alpha\in(0,1]$, then $u_l(\alpha)$ and $u_r(\alpha)$ can be regarded as functions on (0,1], which satisfy
(i) $u_l(\alpha)$ is non-decreasing and left continuous;
(ii)$u_r(\alpha)$ is non-increasing and left continuous;
(iii)$u_l(1)\le u_r(1)$.
Conversely for any functions $a(\alpha)$ and $b(\alpha)$ defined on $(0,1]$ which satisfy (i)-(iii) in the above, there exists a unique $u\in E^1$ such that $[u]_\alpha=[a(\alpha),b(\alpha)]$ for any $\alpha\in(0,1]$.

Theorem 3.1 Consider the fuzzy difference equation (1), where A,B and C are positive fuzzy numbers. Then, for any positive fuzzy number x_0 , there exists a unique positive solution x_n of (1) with initial condition x_0.

Proof. We organize our proof along the lines of the proof of Proposition 2.1 [18]. Assume that there exists a sequence of positive fuzzy numbers (x_n) satisfing (1) with initial conditions x_0. Consider the α-cuts,$\alpha\in(0,1],n=0,1,2,\cdots$,

$$\begin{cases}[x_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}], & [A]_\alpha=[A_{l,\alpha},A_{r,\alpha}],\\ [B]_\alpha=[B_{l,\alpha},B_{r,\alpha}], & [C]_\alpha=[C_{l,\alpha},C_{r,\alpha}].\end{cases} \tag{6}$$

It follows from (1), (6) and Lemma 3.1 that

$$\begin{aligned}[x_{n+1}]_\alpha &= [L_{n+1,\alpha},R_{n+1,\alpha}]=[A+Bx_ne^{-Cx_n}]_\alpha\\ &= [A]_\alpha+[B]_\alpha[x_n]_\alpha[e^{-Cx_n}]_\alpha\\ &= [A_{l,\alpha},A_{r,\alpha}]\\ &\quad +[B_{l,\alpha},B_{r,\alpha}][L_{n,\alpha},R_{n,\alpha}]\times[e^{-Cx_n}]_\alpha\\ &= [A_{l,\alpha}+B_{l,\alpha}L_{n,\alpha}e^{-C_{r,\alpha}R_{n,\alpha}},\\ &\quad A_{r,\alpha}+B_{r,\alpha}R_{n,\alpha}e^{-C_{l,\alpha}L_{n,\alpha}}],\end{aligned}$$

from which we have that for $n=0,1,2,\cdots,\alpha\in(0,1]$

$$\begin{cases}L_{n+1,\alpha}=A_{l,\alpha}+B_{l,\alpha}L_{n,\alpha}e^{-C_{r,\alpha}R_{n,\alpha}},\\ R_{n+1,\alpha}=A_{r,\alpha}+B_{r,\alpha}R_{n,\alpha}e^{-C_{l,\alpha}L_{n,\alpha}}.\end{cases} \tag{7}$$

Then it is obvious that, for any initial values condition $(L_{0,\alpha},R_{0,\alpha}),\alpha\in(0,1]$, there exists a unique solution $(L_{n,\alpha},R_{n,\alpha})$. Now we prove that $[L_{n,\alpha},R_{n,\alpha}],\alpha\in(0,1]$, where $(L_{n,\alpha},R_{n,\alpha})$ is the solution of system (7) with the initial condition $(L_{0,\alpha},R_{0,\alpha})$, determines the solution x_n of (1) with the initial conditions x_0 such that

$$[x_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}],\alpha\in(0,1],n=0,1,2,\cdots, \tag{8}$$

where A,B,C and x_0 are positive fuzzy numbers, for any $\alpha_1,\alpha_2\in(0,1],\alpha_1\le\alpha_2$, we have

$$\begin{cases}0<A_{l,\alpha_1}\le A_{l,\alpha_2}\le A_{r,\alpha_2}\le A_{r,\alpha_1}\\ 0<B_{l,\alpha_1}\le B_{l,\alpha_2}\le B_{r,\alpha_2}\le B_{r,\alpha_1}\\ 0<C_{l,\alpha_1}\le C_{l,\alpha_2}\le C_{r,\alpha_2}\le C_{r,\alpha_1}\\ 0<L_{0,\alpha_1}\le L_{0,\alpha_2}\le R_{0,\alpha_2}\le R_{0,\alpha_1}.\end{cases} \tag{9}$$

We recall that

$$L_{n,\alpha_1} \le L_{n,\alpha_2} \le R_{n,\alpha_2} \le R_{n,\alpha_1}, n = 0, 1, 2, \cdots. \quad (10)$$

By induction, we get from (9) that (10) holds for $n = 0, 1$. Assume now that (10) is true for $n \le h, h \in \{1, 2, \cdots\}$. Then we get from (6), (8) and (9) that, for $n \le h$,

$$\begin{aligned} L_{h+1,\alpha_1} &= A_{l,\alpha_1} + B_{l,\alpha_1} L_{h,\alpha_1} e^{-C_{r,\alpha_1} R_{n,\alpha_1}} \\ &\le A_{l,\alpha_2} + B_{l,\alpha_2} L_{h,\alpha_2} e^{-C_{r,\alpha_2} R_{n,\alpha_2}} = L_{h+1,\alpha_2} \\ &= A_{l,\alpha_2} + B_{l,\alpha_2} L_{h,\alpha_2} e^{-C_{r,\alpha_2} R_{n,\alpha_2}} \\ &\le A_{r,\alpha_2} + B_{r,\alpha_2} R_{h,\alpha_2} e^{-C_{l,\alpha_2} L_{n,\alpha_2}} = R_{h+1,\alpha_2} \\ &= A_{r,\alpha_2} + B_{r,\alpha_2} R_{h,\alpha_2} e^{-C_{l,\alpha_2} L_{n,\alpha_2}} \\ &\le A_{r,\alpha_1} + B_{r,\alpha_1} R_{h,\alpha_1} e^{-C_{l,\alpha_1} L_{n,\alpha_1}} = R_{h+1,\alpha_1}. \end{aligned}$$

Therefore (10) is satisfied. Moreover from (7) we have, for $\forall \alpha \in (0, 1]$

$$\begin{cases} L_{1,\alpha} = A_{l,\alpha} + B_{l,\alpha} L_{0,\alpha} e^{-C_{r,\alpha} R_{0,\alpha}}, \\ \\ R_{1,\alpha} = A_{r,\alpha} + B_{r,\alpha} R_{0,\alpha} e^{-C_{l,\alpha} L_{0,\alpha}}, \end{cases} \quad (11)$$

where A, B, C and x_0 are positive fuzzy numbers, then we have that $A_{l,\alpha}, A_{r,\alpha}, B_{l,\alpha}, B_{r,\alpha}, C_{l,\alpha}, C_{r,\alpha}, L_{0,\alpha}, R_{0,\alpha}$ are left continuous. So from (11) we have that $L_{1,\alpha}, R_{1,\alpha}$ are also left continuous. By induction we can get that $L_{n,\alpha}, R_{n,\alpha}, n = 1, 2, \cdots$ are left continuous.

Now we prove that the support of x_n, suppx_n $= \overline{\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]}$ is compact. It is enough to show that $\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]$ is bounded. Let $n = 1$, Since A, B, C and x_0 are positive fuzzy numbers, there exist constants $M_A > 0, M_B > 0, M_C > 0, M_0 > 0, N_A > 0, N_B > 0, N_C > 0, N_0 > 0$ such that, for all $\alpha \in (0, 1]$,

$$\begin{cases} [A_{l,\alpha}, A_{r,\alpha}] \subset [M_A, N_A], \quad [B_{l,\alpha}, B_{r,\alpha}] \subset [M_B, N_B], \\ \\ [C_{l,\alpha}, C_{r,\alpha}] \subset [M_C, N_C], \quad [L_{0,\alpha}, R_{0,\alpha}] \subset [M_0, N_0]. \end{cases} \quad (12)$$

So, from (11) and (12) we have

$$\begin{aligned} [L_{1,\alpha}, R_{1,\alpha}] \subset \big[& M_A + M_B M_0 e^{-N_c N_0}, \\ & N_A + N_B N_0 e^{-M_c M_0} \big]. \end{aligned} \quad (13)$$

From which we can get that, for all $\alpha \in (0, 1]$.

$$\begin{aligned} \bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset \big[& M_A + M_B M_0 e^{-N_c N_0}, \\ & N_A + N_B N_0 e^{-M_c M_0} \big]. \end{aligned} \quad (14)$$

Therefore (14) implies that

$$\overline{\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}]} \text{ is compact,}$$

and

$$\overline{\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}]} \subset (0, \infty).$$

Deducing inductively, we can follow that $\overline{\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]}$ is compact, and

$$\overline{\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]} \subset (0, \infty), \quad n = 1, 2, \cdots. \quad (15)$$

Therefore, (9), (15) and since $L_{n,\alpha}, R_{n,\alpha}$ are left continuous, we have that $[L_{n,\alpha}, R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers (x_n) such that (10) holds .

Next, we will prove that, for $\forall \alpha \in (0, 1]$, x_n is a unique solution of (1) with the initial condition x_0. Assume that there exists another solution $\overline{x}_n$ of (1) with the initial conditions x_0. Then from arguing as above we can easily prove that

$$[\overline{x}_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0, 1], \quad n = 0, 1, 2, \cdots. \quad (16)$$

Then from (8) and (16), we have $[x_n]_\alpha = [\overline{x}_n]_\alpha$, $\alpha \in (0, 1], n = 0, 1, 2, \cdots$, namely, $x_n = \overline{x}_n, n = 0, 1, \cdots$. Thus the proof of Theorem 3.1 is completed.

Lemma 3.3 (Comparison principle) Let $a \in [0, \infty), b \in [0, \infty)$ and let $\{X_n\}_{n=0}^{\infty}, \{Y_n\}_{n=0}^{\infty}$ be sequences of real numbers, such that $X_0 \le Y_0$, and for $n = 0, 1 \cdots$

$$\begin{cases} X_{n+1} \le a X_n + b, \\ \\ Y_{n+1} = a Y_n + b, \end{cases}$$

Then $X_n \le Y_n$, for all $n \ge 0$.

Lemma 3.4 Consider the difference equation system, for $n = 0, 1, \cdots$,

$$\begin{cases} y_{n+1} = a_1 + b_1 y_n e^{-c_1 z_n}, \\ \\ z_{n+1} = a_2 + b_2 z_n e^{-c_2 y_n}, \end{cases} \quad (17)$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ and the initial values y_0, z_0 are positive real numbers. If

$$b_1 < e^{c_1 a_2}, \quad b_2 < e^{c_2 a_1}. \quad (18)$$

Then solution of (17) is bounded and persists.

Proof. Assume that $\{(y_n, z_n)\}_{n=0}^{\infty}$ is a positive solution of (17). Then it follows from (17) that, for $n \ge 0$, we have

$$\begin{cases} y_{n+1} = a_1 + b_1 y_n e^{-c_1 z_n} > a_1, \\ \\ z_{n+1} = a_2 + b_2 z_n e^{-c_2 y_n} > a_2. \end{cases} \quad (19)$$

Thus

$$\begin{cases} y_{n+1} = a_1 + b_1 y_n e^{-c_1 z_n} < a_1 + b_1 y_n e^{-c_1 a_2}, \\ \\ z_{n+1} = a_2 + b_2 z_n e^{-c_2 y_n} < a_2 + b_2 z_n e^{-c_2 a_1}. \end{cases} \quad (20)$$

Now we consider the following system of difference equations, for $n = 1, 2, \cdots$,

$$Y_{n+1} = a_1 + b_1 Y_n e^{-c_1 a_2}, \quad Z_{n+1} = a_2 + b_2 Z_n e^{-c_2 a_1},$$

with initial conditions $y_0 \le Y_0, z_0 \le Z_0$, and it follows from Lemma 3.3 that

$$y_n \le Y_n, \quad z_n \le Z_n, n \ge 0.$$

So

$$\lim_{n\to\infty} Y_n = \frac{a_1}{1 - b_1 e^{-c_1 a_2}}, \quad \lim_{n\to\infty} Z_n = \frac{a_2}{1 - b_2 e^{-c_2 a_1}},$$

and

$$\lim_{n\to\infty} \sup y_n \le \frac{a_1}{1 - b_1 e^{-c_1 a_2}}, \lim_{n\to\infty} \sup z_n \le \frac{a_2}{1 - b_2 e^{-c_2 a_1}}.$$

Therefore $\{(y_n, z_n)\}_{n=0}^{\infty}$ is bounded and persists, the proof is completed.

Theorem 3.2 Consider the fuzzy difference equation (1), where A, B, C and initial value x_0 are positive fuzzy numbers. If for $\forall \alpha \in (0.1]$,

$$B_{r,\alpha} < e^{C_{l,\alpha} A_{l,\alpha}}. \tag{21}$$

Then every positive solution of (1) is bounded and persists.
Proof. Assume that x_n is a positive solution of equation (1) such that (8) holds. From (7), we have, for $n = 1, 2, \cdots, \alpha \in (0, 1]$,

$$A_{l,\alpha} \leq L_{n,\alpha}, A_{r,\alpha} \leq R_{n,\alpha} \tag{22}$$

From Lemma 3.3 and (20), we can get, for $n = 1, 2, \cdots$,

$$\begin{cases} L_{n,\alpha} \leq \frac{A_{l,\alpha}}{1-B_{l,\alpha}e^{-C_{r,\alpha}A_{r,\alpha}}}, \\ R_{n,\alpha} \leq \frac{A_{r,\alpha}}{1-B_{r,\alpha}e^{-C_{l,\alpha}A_{l,\alpha}}}. \end{cases} \tag{23}$$

Hence from (12), (20) and (21) we can get, for $n \geq 1, \alpha \in (0, 1]$,

$$[L_{n,\alpha}, R_{n,\alpha}] \subset \left[M_A, \frac{N_A}{1 - N_B e^{-M_C M_A}} \right]. \tag{24}$$

From which it is obvious that, for $n \geq 1$.

$$\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}] \subset \left[M_A, \frac{N_A}{1 - N_B e^{-M_C M_A}} \right], \tag{25}$$

from which we get

$$\overline{\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]} \subset \left[M_A, \frac{N_A}{1 - N_B e^{-M_C M_A}} \right]. \tag{26}$$

The positive solution of (1) is bounded and persists.

Lemma 3.5 Consider the difference equations system (17), where $a_i, b_i, c_i (i = 1, 2)$ are positive real numbers. If (18) hold, then there exists a unique positive equilibrium $(\overline{y}, \overline{z})$ such that

$$a_1 < \overline{y} < \frac{a_1}{1 - b_1 e^{-c_1 a_2}}, \quad a_2 < \overline{z} < \frac{a_2}{1 - b_2 e^{-c_2 a_1}}. \tag{27}$$

Proof. Let (y, z) be the solution of the following system.

$$\begin{cases} y = a_1 + b_1 y e^{-c_1 z}, \\ z = a_2 + b_2 z e^{-c_2 y}. \end{cases} \tag{28}$$

Set

$$f(y) = a_1 + b_1 y e^{-\frac{a_2 c_1}{1 - b_2 e^{-c_2 y}}} - y. \tag{29}$$

Then

$$f(a_1) = b_1 a_1 e^{-\frac{c_1 a_2}{1 - b_2 e^{-c_2 a_1}}} > 0, \quad \lim_{y \to \infty} f(y) = -\infty \tag{30}$$

and

$$\begin{aligned} f'(y) &= b_1 e^{-\frac{a_2 c_1}{1 - b_2 e^{-c_2 y}}} \\ &\quad - b_1 y \frac{c_1 a_2 b_2 c_2 e^{-c_2 y}}{(1 - b_2 e^{-c_2 y})^2} e^{-\frac{a_2 c_1}{1 - b_2 e^{-c_2 y}}} - 1 \\ &< 0. \end{aligned} \tag{31}$$

It follows from (30) and (31) that (28) has exactly one solution $\overline{y} > a_1$. Similarly, we can obtain that $\overline{z} > a_2$.

From (28), we have

$$\overline{y} < \frac{a_1}{1 - b_1 e^{-c_1 a_2}}, \quad \overline{z} < \frac{a_2}{1 - b_2 e^{-c_2 a_1}}.$$

The proof is completed.

Theorem 3.3 Consider the difference equation system (17), where $a_i, b_i, c_i (i = 1, 2)$ are positive real numbers and the initial values y_0, z_0 are positive real numbers. Assume that (18) and that the following condition holds

$$\begin{aligned} \max \Bigg\{ & b_1 e^{-c_1 a_2} \left(1 + \frac{a_1 c_1}{1 - b_1 e^{-c_1 a_2}}\right), \\ & b_2 e^{-c_2 a_1} \left(1 + \frac{a_2 c_2}{1 - b_2 e^{-c_2 a_1}}\right) \Bigg\} < 1, \end{aligned} \tag{32}$$

Then the positive equilibrium $(\overline{y}, \overline{z})$ of system (17) is globally asymptotically stable.
Proof. We first prove that the positive equilibrium point $(\overline{y}, \overline{z})$ is locally asymptotically stable. We can easily obtain that the linearized system of (17) about the positive equilibrium $(\overline{y}, \overline{z})$ is

$$\Phi_{n+1} = D \Phi_n,$$

where

$$D = (d_{ij})_{2 \times 2} = \begin{pmatrix} b_1 e^{-c_1 \overline{z}} & b_1 c_1 \overline{y} e^{-c_1 \overline{z}} \\ -b_2 c_1 \overline{z} e^{-c_2 \overline{y}} & b_2 e^{-c_2 \overline{y}} \end{pmatrix}.$$

The norm of this matrix is

$$\|D\| = \max \left\{ b_1 e^{-c_1 \overline{z}} (1 + c_1 \overline{y}), b_2 e^{-c_2 \overline{y}} (1 + c_2 \overline{z}) \right\}.$$

From (32), we get that

$$\begin{aligned} \|D\| \leq \max \Bigg\{ & b_1 e^{-c_1 a_2} \left(1 + \frac{a_1 c_1}{1 - b_1 e^{-c_1 a_2}}\right), \\ & b_2 e^{-c_2 a_1} \left(1 + \frac{a_2 c_2}{1 - b_2 e^{-c_2 a_1}}\right) \Bigg\} < 1. \end{aligned}$$

Therefore, since $|\lambda_i| < \|D\| < 1$, $\lambda_i (i = 1, 2)$ are the eigenvalues of D, we have all eigenvalues of D lie inside the unit disk. This implies that $(\overline{y}, \overline{z})$ is locally asymptotically stable.
Let (y_n, z_n) be a positive solution of (17). We prove that

$$\lim_{n \to \infty} y_n = \overline{y}, \quad \lim_{n \to \infty} z_n = \overline{z}. \tag{33}$$

Using Lemma 3.3, we get

$$\begin{cases} \Gamma_1 = \lim_{n \to \infty} \sup y_n < \infty, & \Gamma_2 = \lim_{n \to \infty} \sup z_n < \infty, \\ \gamma_1 = \lim_{n \to \infty} \inf y_n > 0, & \gamma_2 = \lim_{n \to \infty} \inf z_n > 0. \end{cases} \tag{34}$$

Then from (17) and (34), we have

$$\begin{cases} \Gamma_1 \leq \frac{a_1}{1 - b_1 e^{-c_1 \gamma_2}}, & \gamma_1 \geq \frac{a_1}{1 - b_1 e^{-c_1 \Gamma_2}}, \\ \Gamma_2 \leq \frac{a_2}{1 - b_2 e^{-c_2 \gamma_1}}, & \gamma_2 \geq \frac{a_2}{1 - b_2 e^{-c_c \Gamma_1}}. \end{cases} \tag{35}$$

From (35), it follows that

$$\begin{cases} \Gamma_1 \gamma_2 \leq \frac{a_1 \gamma_2}{1 - b_1 e^{-c_1 \gamma_2}}, & \Gamma_2 \gamma_1 \leq \frac{a_2 \gamma_1}{1 - b_2 e^{-c_2 \gamma_1}}, \\ \Gamma_2 \gamma_1 \geq \frac{a_1 \Gamma_2}{1 - b_1 e^{-c_1 \Gamma_2}}, & \Gamma_1 \gamma_2 \geq \frac{a_2 \Gamma_1}{1 - b_2 e^{-c_2 \Gamma_1}}. \end{cases}$$

From which we can get

$$\begin{cases} \frac{a_2\Gamma_1}{1-b_2e^{-c_2\Gamma_1}} \le \Gamma_1\gamma_2 \le \frac{a_1\gamma_2}{1-b_1e^{-c_1\gamma_2}}, \\ \frac{a_1\Gamma_2}{1-b_1e^{-c_1\Gamma_2}} \le \Gamma_2\gamma_1 \le \frac{a_2\gamma_1}{1-b_2e^{-c_2\gamma_1}}, \end{cases} \tag{36}$$

Set

$$g(y) = \frac{a_2 y}{1-b_2e^{-c_2 y}}, \quad f(z) = \frac{a_1 z}{1-b_1e^{-c_1 z}}, \tag{37}$$

and

$$y \in \left(a_1, \frac{a_1}{1-b_1e^{-c_1a_2}}\right), \quad z \in \left(a_2, \frac{a_2}{1-b_2e^{-c_2a_1}}\right).$$

Then form (37) , we can get that

$$\begin{cases} g^{'}(y) = \frac{a_2[1-b_2e^{-c_2y}(1+c_2y)]}{(1-b_2e^{-c_2y})^2}, \\ f^{'}(z) = \frac{a_1[1-b_1e^{-c_1z}(1+c_1z)]}{(1-b_1e^{-c_1z})^2}, \end{cases} \tag{38}$$

Since

$$y \in \left(a_1, \frac{a_1}{1-b_1e^{-c_1a_2}}\right) \quad z \in \left(a_2, \frac{a_2}{1-b_2e^{-c_2a_1}}\right),$$

we can get

$$\begin{cases} 1 - b_2e^{-c_2y}(1+c_2y) > 1-b_2e^{-c_2a_1} \\ \quad \times(1+\frac{a_2}{1-b_2e^{-c_2a_1}}) > 0, \\ 1 - b_1e^{-c_1z}(1+c_1z) > 1-b_1e^{-c_1a_2} \\ \quad \times(1+\frac{a_1}{1-b_1e^{-c_1a_2}}) > 0. \end{cases} \tag{39}$$

Therefore from (38) and (39), we can get

$$g'(y) > 0, \;\; f'(z) > 0,$$

for

$$y \in \left(a_1, \frac{a_1}{1-b_1e^{-c_1a_2}}\right), z \in \left(a_2, \frac{a_2}{1-b_2e^{-c_2a_1}}\right).$$

Hence, g, f are monotone increasing function, together with (36), it implies that $\Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2$ and we get in fine that $\lim_{n\to\infty} y_n = \overline{y}, \; \lim_{n\to\infty} z_n = \overline{z}$. The proof of Theorem 3.3 is completed.

Remark 3.1 Condition (32) of Theorem 3.3 is different from that of [4], These condition guarantee the existence of the unique positive equilibrium of (17) to be globally asymptotically stable.

Theorem 3.4 Consider the fuzzy difference equation (1), where A, B, C and the initial value are positive fuzzy numbers, If (21) hold, then the following statements are true.
(i) The system (1) has a unique positive equilibrium.
(ii)The every positive solution x_n of (1) converges to the unique positive equilibrium x with respect to D as $n \to \infty$.
Proof. (i) We consider the system

$$\begin{cases} L_\alpha = A_{l,\alpha} + B_{l,\alpha}L_\alpha e^{-C_{r,\alpha}R_\alpha}, \\ R_\alpha = A_{r,\alpha} + B_{r,\alpha}R_\alpha e^{-C_{l,\alpha}L_\alpha}. \end{cases} \tag{40}$$

Obviously, (40) has a unique solution (L_α, R_α).
Assume that x_n is a positive solution of (1), so that $[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0,1], n = 0,1,2\cdots$. Then applying Theorem 3.3, we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \;\; \lim_{n\to\infty} R_{n,\alpha} = R_\alpha. \tag{41}$$

From (22) and (40), we have, for $0 < \alpha_1 \le \alpha_2 \le 1$,

$$0 < L_{\alpha_1} \le L_{\alpha_2} \le R_{\alpha_2} \le R_{\alpha_1} \tag{42}$$

Since $A_{l,\alpha}, A_{r,\alpha}, B_{l,\alpha}, B_{r,\alpha}$ are left continuous, it follows from (40) that L_α, R_α are also left continuous. From (12) and (40) we have

$$\begin{cases} L_\alpha \ge A_{l,\alpha} \ge M_A, \\ R_\alpha = \frac{A_{r,\alpha}}{1-B_{r,\alpha}e^{-C_{l,\alpha}L_\alpha}} \le \frac{N_A}{1-N_Be^{-M_AM_C}}. \end{cases} \tag{43}$$

Therefore (43) implies that

$$[L_\alpha, R_\alpha] \subset \left[M_A, \frac{N_A}{1-N_Be^{-M_AM_C}}\right].$$

From that it is clear that

$$\begin{cases} \bigcup_{\alpha\in(0,1]}[L_\alpha, R_\alpha] \text{ is compact} \\ \bigcup_{\alpha\in(0,1]}[L_\alpha, R_\alpha] \subset (0,\infty). \end{cases} \tag{44}$$

From Lemma 3.2, (40), (42) and (44), it follows that L_α, R_α, determine a fuzzy number x such that

$$x = A + Bxe^{-Cx}, [x] = [L_\alpha, R_\alpha], \alpha \in (0,1],$$

and so x is a positive equilibrium of (1).

Let $\overline{x}$ is another positive equilibrium of (1), then there exist function $\overline{L_\alpha} : (0,1] \to (0,\infty)$, $\overline{R_\alpha} : (0,1] \to (0,\infty)$, such that

$$\overline{x} = A + B\overline{x}e^{-C\overline{x}}, \;\; [\overline{x}]_\alpha = [\overline{L_\alpha}, \overline{R_\alpha}], \alpha \in (0,1].$$

From which we get

$$\overline{L_\alpha} = A_{l,\alpha} + B_{l,\alpha}\overline{L_\alpha}e^{-C_{r,\alpha}\overline{R_\alpha}}, \;\; \overline{R_\alpha} = A_{r,\alpha} + B_{r,\alpha}\overline{R_\alpha}e^{-C_{l,\alpha}\overline{L_\alpha}}.$$

So $L_\alpha = \overline{L_\alpha}, R_\alpha = \overline{R_\alpha}, \alpha \in (0,1]$, we can get $x = \overline{x}$. This completes the proof of (i).
(ii) From (41), we can get

$$\begin{aligned} & \lim_{n\to\infty} D(x_n, x) \\ = & \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \{\max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\}\} \\ = & \; 0 \end{aligned} \tag{45}$$

From which it is clear that every positive solution x_n of (1) converges the unique equilibrium x with respect to D as $n \to \infty$.

IV. Numerical example

To illustrate our result, we give an example to show effectiveness our results obtained .

Example 4.1 Consider the following fuzzy difference equation

$$x_{n+1} = A + Bx_ne^{-Cx_n}, n = 0,1,\cdots, \tag{46}$$

where A, B, C are fuzzy numbers such that

$$A(x) = \begin{cases} 10x - 2, & 0.2 \le x \le 0.3, \\ -5x + 2.5, & 0.3 \le x \le 0.5, \end{cases} \tag{47}$$

and

$$B(x) = \begin{cases} 10x - 4, & 0.4 \le x \le 0.5, \\ -10x + 6, & 0.5 \le x \le 0.6, \end{cases} \tag{48}$$

and

$$C(x) = \begin{cases} 5x - 3, & 0.6 \le x \le 0.8, \\ -5x + 5, & 0.8 \le x \le 1, \end{cases} \tag{49}$$

we take the initial value x_0, such that

$$x_0 = \begin{cases} 10x - 3, & 0.3 \le x \le 0.4, \\ -5x + 3, & 0.4 \le x \le 0.6, \end{cases} \tag{50}$$

from (47), (48), (49), for $\alpha \in (0, 1]$, we have

$$\begin{cases} [A]_\alpha & = & [0.1\alpha + 0.2, 0.5 - 0.2\alpha], \\ [B]_\alpha & = & [0.1\alpha + 0.4, 0.6 - 0.1\alpha], \\ [C]_\alpha & = & [0.6 + 0.2\alpha, 1 - 0.2\alpha]. \end{cases} \tag{51}$$

And

$$\begin{cases} \overline{\bigcup_{\alpha \in (0,1]} [A]_\alpha} = [0.2, 0.5], \\ \overline{\bigcup_{\alpha \in (0,1]} [B]_\alpha} = [0.4, 0.6], \\ \overline{\bigcup_{\alpha \in (0,1]} [C]_\alpha} = [0.6, 1] \end{cases} \tag{52}$$

Moreover from (50), for $\alpha \in (0, 1]$, we have

$$[x]_0 = [0.1\alpha + 0.3, 0.6 - 0.2\alpha] \tag{53}$$

and so

$$\overline{\bigcup_{\alpha \in (0,1]} [x_0]_\alpha} = [0.3, 0.6] \tag{54}$$

From (46), it results in a coupled system of difference equation with parameter $\alpha \in (0, 1]$,

$$\begin{cases} L_{n+1,\alpha} = 0.1\alpha + 0.2 + (0.1\alpha + 0.4) L_{n,\alpha} e^{-(1-0.2\alpha) R_{n,\alpha}}, \\ R_{n+1,\alpha} = 0.5 - 0.2\alpha + (0.6 - 0.1\alpha) R_{n,\alpha} e^{-(0.6+0.2\alpha) L_{n,\alpha}} \end{cases} \tag{55}$$

and, $B_{r,\alpha} < e^{A_{l,\alpha} C_{l,\alpha}}$, for every $\alpha \in (0, 1]$, the condition (21) is satisfied, so from the Theorem 3.3 we get that every positive solution x_n of (55) is bounded and persists, and in addition, (46) has a unique positive equilibrium

$$\overline{x} = (0.233, 0.46, 1.046).$$

Every positive solution x_n of (46) converges to the unique positive equilibrium $\overline{x}$ with respect to D as $n \to \infty$ (see Fig.1-Fig.5).

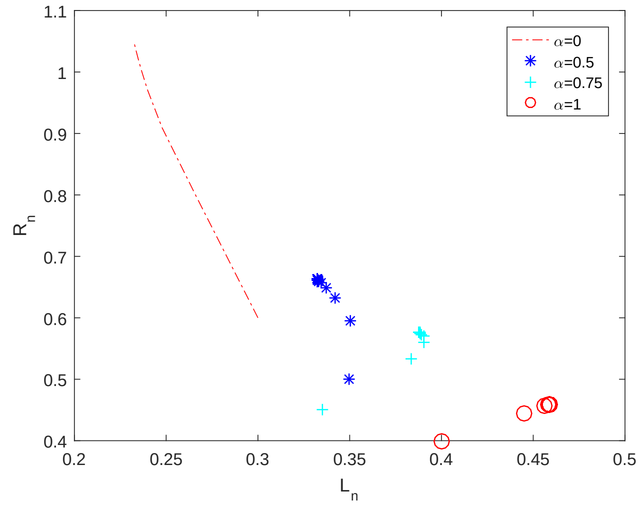

Fig. 1. The dynamical behavior of (55).

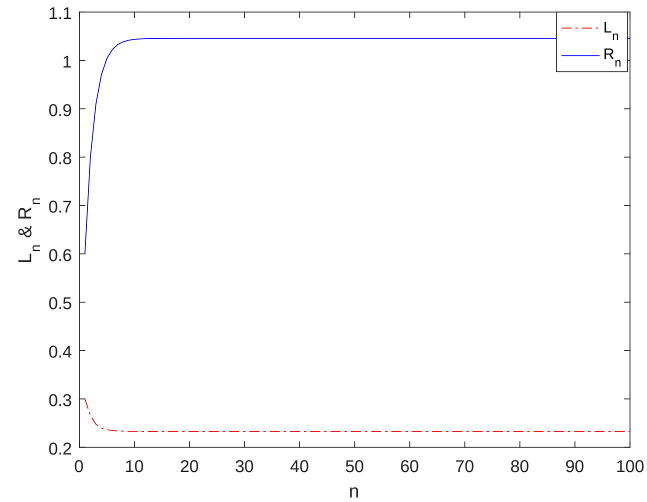

Fig. 2. The solution of system (55) at $\alpha = 0$.

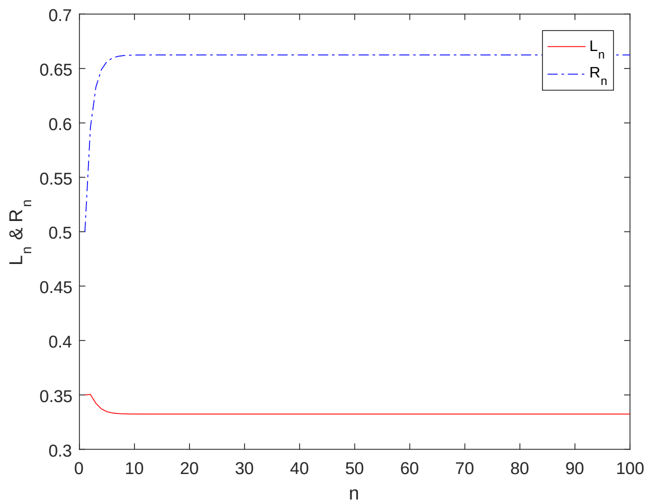

Fig. 3. The solution of system (55) at $\alpha = 0.5$.

V. Conclusion

In this paper, the first-order nonlinear fuzzy difference

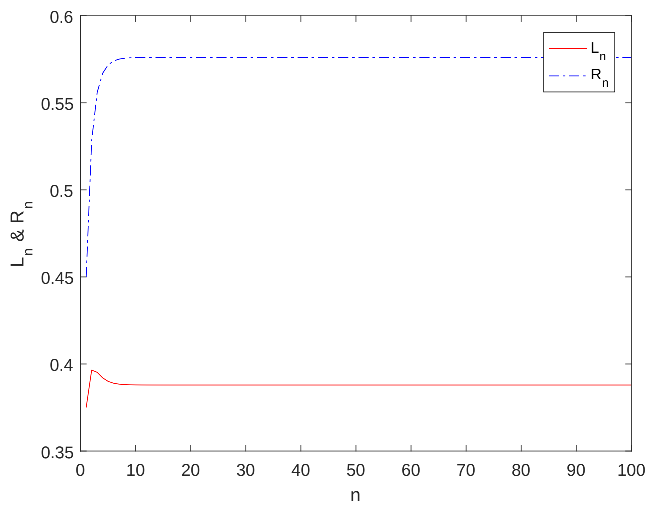

Fig. 4. The solution of system (55) at $\alpha = 0.75$.

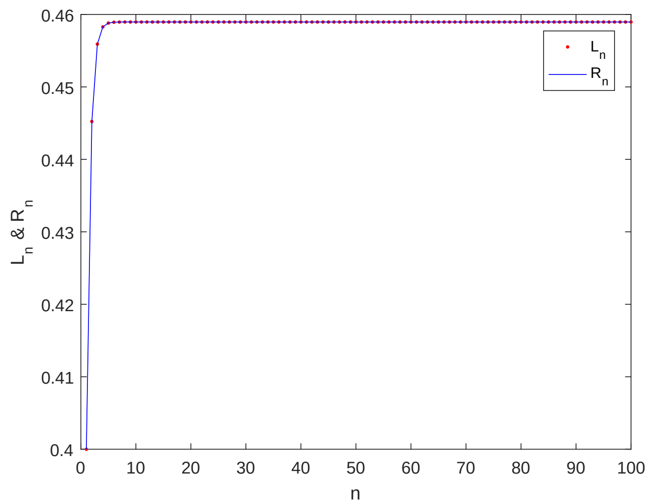

Fig. 5. The solution of system (55) at $\alpha = 1$.

equation

$$x_{n+1} = A + Bx_n e^{-Cx_n}, n = 0, 1, \cdots$$

is discussed. Firstly, the existence of the positive solution to this equation is proved. Secondly, we found that under condition $B_{r,\alpha} < e^{A_{l,\alpha}C_{l,\alpha}}, \alpha \in (0, 1]$, the positive solutions of first-order nonlinear fuzzy difference equation are bounded and persistent, and there exists an unique positive equilibrium $\overline{x}$ such that every positive solution converges it. Finally, a numerical example is given to illustrate our results obtained.

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