Dynamics of an Almost Periodic Single-Species System with Harvesting Rate and Feedback Control on Time Scales

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Abstract—**This paper is concerned with a single-species model with nonlinear harvesting rate and feedback control on time scales, which modified from Ref. [11]. Based on the theory of calculus on time scales, by applying the methods used in Ref. [11], but improved, sufficient conditions which guarantee the permanence and the existence of a unique globally attractive positive almost periodic solution of the system are obtained. Finally, numerical simulations are presented to illustrate the feasibility and effectiveness of the results. The results in this paper improved and generalized the results derived in [11].**

Index Terms—**Permanence; Almost periodic solution; Global attractivity; Time scale.**

I. Introduction

THE theory of time scales, which has achieved much attention, was firstly introduced by S. Hilger in his PhD thesis in 1988 [1], in order to unify continuous and discrete analysis. The study of dynamic equations on time scales can combine the continuous and discrete situations; by choosing the time scale to be the set of real numbers, the general results yields a result for ordinary differential equations; and by choosing the time scale to be the set of integers, the same general results yields a result for difference equations. However, since there many other time scales than just the set of real numbers or the set of integers, one has more general results. In the past few years, many good results about the study of the systems on time scales are obtained; see, for example, [2-8].

Notice that ecosystems are often disturbed by outside continuous forces in the real world, such as seasonal effects and variations in weather conditions, food supplies, mating habits, etc., the assumption of almost periodicity of the parameters is a way of incorporating the almost periodicity of a temporally nonuniform environment with incommensurable periods (nonintegral multiples). Almost periodicity of different types of ecosystems received more recently researchers' special attention, see [9-11] and the references therein.

In [11], Hu and Lv study an almost periodic single-species system with feedback control on time scales as follows:

$$\begin{cases} x^{\Delta}(t) &= r(t)x(t)[1-\frac{x(t)}{a(t)+d(t)x(t)} \\ & \quad -b(t)x(\sigma(t))-c(t)y(t)], \\ y^{\Delta}(t) &= -\eta(t)y(t)+g(t)x(t), \end{cases} \tag{1}$$

where $t\in\mathbb{T}$, $\mathbb{T}$ is an almost time scale. All the coefficients $r(t),a(t),b(t),c(t),d(t),\eta(t),g(t)$ are continuous, almost periodic functions.

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However, in system (1), there is a term $\sigma(t)$ on the right side of the first equation. Let $\mathbb{T}$ be a special time scale, for example, $\mathbb{T}=\mathbb{Z}$, etc., then system (1) is not conform to ecology significance, that is, system (1) can't accurately describe the growth law of the species. Therefore, there is a need to establish a new dynamic model on time scales.

Motivated by the above, in this paper, we devote to studying the following single-species system with feedback control on time scales

$$\begin{cases} x^{\Delta}(t) &= x(t)[r(t)-a(t)x(t)-c(t)y(t)] \\ & \quad -\frac{b(t)x^2(t)}{d(t)+x^2(t)}, \\ y^{\Delta}(t) &= -\eta(t)y(t)+g(t)x(t), \end{cases} \tag{2}$$

where $t\in\mathbb{T}$, $\mathbb{T}$ is an almost time scale. $x(t)$ is the density of species x at $t\in\mathbb{T}$; $y(t)$ is control variables; $r(t)$ expresses the intrinsic growth rate of species x at t; $a(t)$ stands for the interspecific competing rate of species x at t; the form of $\frac{b(t)x^2(t)}{d(t)+x^2(t)}$ denotes harvesting rate invoking by Murray (see [12]). All the coefficients $r(t),a(t),b(t),c(t),d(t),\eta(t),g(t)$ are continuous, almost periodic functions.

For convenience, we introduce the notation

$$f^u=\sup_{t\in\mathbb{T}}f(t),\ f^l=\inf_{t\in\mathbb{T}}f(t),$$

where f is a positive and bounded function. Throughout this paper, we assume that the coefficients of the almost periodic system (2) satisfy

$$\min\{r^l,a^l,b^l,c^l,d^l,\eta^l,g^l\}>0,$$
$$\max\{r^u,a^u,b^u,c^u,d^u,\eta^u,g^u\}<+\infty.$$

The initial condition of system (2) in the form

$$x(t_0)=x_0,\ y(t_0)=y_0,\ t_0\in\mathbb{T},\ x_0>0,\ y_0>0. \tag{3}$$

The aim of this paper is, by applying the methods used in [12], but improved, to obtain sufficient conditions for the permanence and the existence of a unique globally attractive positive almost periodic solution of system (2).

In this paper, the time scale $\mathbb{T}$ considered is unbounded above, and for each interval $\mathbb{I}$ of $\mathbb{T}$, we denote by $\mathbb{I}_{\mathbb{T}}=\mathbb{I}\cap\mathbb{T}$.

II. Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ and the graininess $\mu:\mathbb{T}\to\mathbb{R}^+$ are defined, respectively, by

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\},\rho(t)=\sup\{s\in\mathbb{T}:s<t\},$$

$$\mu(t)=\sigma(t)-t.$$

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A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

The basic theories of calculus on time scales, one can see [13].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu p\, q, \quad \ominus p = -\frac{p}{1 + \mu p}, \quad p \ominus q = p \oplus (\ominus q).$$

Lemma 1. (see [13]) *If $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*
(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$;
(vi) $(e_p(t, s))^\Delta = p(t)e_p(t, s)$.

Lemma 2. (see [14]) *Assume that $a > 0, b > 0, -b \in \mathcal{R}^+$, and $y(t) > 0$, $t \in [t_0, +\infty)_\mathbb{T}$.*
(i) If $y^\Delta(t) \geq y(t)(b - ay(t))$, then $\liminf_{t\to+\infty} y(t) = \frac{b}{a}$.
(ii) If $y^\Delta(t) \leq y(t)(b - ay(t))$, then $\limsup_{t\to+\infty} y(t) = \frac{b}{a}$.

Let $\mathbb{T}$ be a time scale with at least two positive points, one of them being always one: $1 \in \mathbb{T}$, there exists at least one point $t \in \mathbb{T}$ such that $0 < t \neq 1$. Define the natural logarithm function on the time scale $\mathbb{T}$ by

$$L_\mathbb{T}(t) = \int_1^t \frac{1}{\tau}\Delta\tau, \ t \in \mathbb{T} \cap (0, +\infty).$$

Lemma 3. (see [15]) *Assume that $x : \mathbb{T} \to \mathbb{R}^+$ is strictly increasing and $\widetilde{\mathbb{T}} := x(\mathbb{T})$ is a time scale. If $x^\Delta(t)$ exists for $t \in \mathbb{T}^k$, then*

$$\frac{\Delta}{\Delta t} L_\mathbb{T}(x(t)) = \frac{x^\Delta(t)}{x(t)}.$$

Lemma 4. (see [13]) *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$, then $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$\begin{aligned}(fg)^\Delta(t) &= f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) \\ &= f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).\end{aligned}$$

Definition 1. (see [16]) *A time scale $\mathbb{T}$ is called an almost periodic time scale if*

$$\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\}.$$

Definition 2. (see [16]) *Let $\mathbb{T}$ be an almost periodic time scale. A function $f : \mathbb{T} \to \mathbb{R}$ is called an almost periodic function if the ε-translation set of f*

$$E\{\varepsilon, f\} = \{\tau \in \Pi : |f(t + \tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$, there exists at least a $\tau \in E\{\varepsilon, f\}$ such that

$$|f(t + \tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}.$$

τ is called the ε-translation number of f and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$.

Lemma 5. (see [11]) *Let $\mathbb{T}$ be an almost periodic time scale. If $f(t), g(t)$ are almost periodic functions, then, for any $\varepsilon > 0$, $E\{\varepsilon, f\} \cap E\{\varepsilon, g\}$ is a nonempty relatively dense set in $\mathbb{T}$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$, there exists at least a $\tau \in E\{\varepsilon, f\} \cap E\{\varepsilon, g\}$ such that*

$$|f(t + \tau) - f(t)| < \varepsilon, |g(t + \tau) - g(t)| < \varepsilon, \forall t \in \mathbb{T}.$$

III. Main results

Assume that the coefficients of (2) satisfy

(H_1) $r^l - c^u M_2 > 0$.

Lemma 6. *Let $(x(t), y(t))$ be any positive solution of system (2) with initial condition (3). If (H_1) hold, then system (2) is permanent, that is, any positive solution $(x(t), y(t))$ of system (2) satisfies*

$$m_1 \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M_1, \tag{4}$$

$$m_2 \leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq M_2, \tag{5}$$

especially if $m_1 \leq x_0 \leq M_1$, $m_2 \leq y_0 \leq M_2$, then

$$m_1 \leq x(t) \leq M_1, \ m_2 \leq y(t) \leq M_2, \ t \in [t_0, +\infty)_\mathbb{T},$$

where

$$M_1 = \frac{r^u}{a^l}, M_2 = \frac{g^u M_1}{\eta^l}, m_1 = \frac{r^l - c^u M_2}{a^u + \frac{b^u}{d^l}}, m_2 = \frac{g^l m_1}{\eta^u}.$$

Proof: Assume that $(x(t), y(t))$ be any positive solution of system (2) with initial condition (3). From the first equation of system (2), we have

$$x^\Delta(t) \leq x(t)[r^u - a^l x(t)]. \tag{6}$$

By Lemma 2, we can get

$$\limsup_{t\to+\infty} x(t) \leq \frac{r^u}{a^l} := M_1.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$x(t) < M_1 + \varepsilon, \ \forall t \in [T_1, +\infty]_{\mathbb{T}}.$$

From the second equation of system (2), when $t \in [T_1, +\infty)_{\mathbb{T}}$,

$$y^{\Delta}(t) < -\eta^l y(t) + g^u(M_1 + \varepsilon).$$

Let $\varepsilon \to 0$, then

$$y^{\Delta}(t) \le -\eta^l y(t) + g^u M_1. \quad (7)$$

By Lemma 2, we can get

$$\limsup_{t\to+\infty} y(t) = \frac{g^u M_1}{\eta^l} := M_2.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_2 > T_1$ such that

$$y(t) < M_2 + \varepsilon, \ \forall t \in [T_2, +\infty]_{\mathbb{T}}.$$

On the other hand, from the first equation of system (2), when $t \in [T_2, +\infty)_{\mathbb{T}}$,

$$x^{\Delta}(t) > x(t)[r^l - c^u(M_2 + \varepsilon) - (a^u + \frac{b^u}{d^l})x(t)].$$

Let $\varepsilon \to 0$, then

$$x^{\Delta}(t) \ge x(t)[r^l - c^u M_2 - (a^u + \frac{b^u}{d^l})x(t)]. \quad (8)$$

By Lemma 2, we can get

$$\liminf_{t\to+\infty} x(t) = \frac{r^l - c^u M_2}{a^u + \frac{b^u}{d^l}} := m_1.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_3 > T_2$ such that

$$x(t) > m_1 - \varepsilon, \ \forall t \in [T_3, +\infty]_{\mathbb{T}}.$$

From the second equation of system (2), when $t \in [T_3, +\infty)_{\mathbb{T}}$,

$$y^{\Delta}(t) > -\eta^u y(t) + g^l(m_1 - \varepsilon).$$

Let $\varepsilon \to 0$, then

$$y^{\Delta}(t) \ge -\eta^u y(t) + g^l m_1. \quad (9)$$

By Lemma 2, we can get

$$\liminf_{t\to+\infty} y(t) = \frac{g^l m_1}{\eta^u} := m_2.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_4 > T_3$ such that

$$y(t) > m_2 - \varepsilon, \ \forall t \in [T_4, +\infty]_{\mathbb{T}}.$$

In special case, if $m_1 \le x_0 \le M_1$, $m_2 \le y_0 \le M_2$, by Lemma 2, it follows from (6)-(9) that

$$m_1 \le x(t) \le M_1, \ m_2 \le y(t) \le M_2, \ t \in [t_0, +\infty)_{\mathbb{T}},$$

This completes the proof.

Let $S(\mathbb{T})$ be the set of all solutions $(x(t), y(t))$ of system (2) satisfying $m_1 \le x(t) \le M_1, \ m_2 \le y(t) \le M_2$ for all $t \in \mathbb{T}$.

Lemma 7. $S(\mathbb{T}) \ne \emptyset$.

Proof: By Lemma 6, we see that for any $t_0 \in \mathbb{T}$ with $m_1 \le x_0 \le M_1$, $m_2 \le y_0 \le M_2$, system (2) has a solution $(x(t), y(t))$ satisfying $m_1 \le x(t) \le M_1, \ m_2 \le y(t) \le M_2, t \in [t_0, +\infty)_{\mathbb{T}}$. Since $r(t)$, $a(t)$, $b(t)$, $c(t)$, $d(t)$, $\eta(t)$, $g(t)$ are almost periodic, it follows from Lemma 5 that there exists a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that $r(t + t_n) \to r(t), a(t + t_n) \to a(t), b(t + t_n) \to b(t), c(t + t_n) \to c(t), d(t + t_n) \to d(t), \eta(t + t_n) \to \eta(t), g(t + t_n) \to g(t)$ as $n \to +\infty$ uniformly on $\mathbb{T}$.

We claim that $\{x(t + t_n)\}$ and $\{y(t + t_n)\}$ are uniformly bounded and equi-continuous on any bounded interval in $\mathbb{T}$.

In fact, for any bounded interval $[\alpha, \beta]_{\mathbb{T}} \subset \mathbb{T}$, when n is large enough, $\alpha + t_n > t_0$, then $t + t_n > t_0, \forall t \in [\alpha, \beta]_{\mathbb{T}}$. So, $m_1 \le x(t + t_n) \le M_1, \ m_2 \le y(t + t_n) \le M_2$ for any $t \in [\alpha, \beta]_{\mathbb{T}}$, that is, $\{x(t+t_n)\}$ and $\{y(t+t_n)\}$ are uniformly bounded. On the other hand, $\forall t_1, t_2 \in [\alpha, \beta]_{\mathbb{T}}$, from the mean value theorem of differential calculus on time scales, we have

$$\begin{aligned} & |x(t_1 + t_n) - x(t_2 + t_n)| \\ \le \ & M_1(r^u + a^u M_1 + c^u M_2 + \frac{b^u M_1}{d^l})|t_1 - t_2|, \quad (10) \\ & |y(t_1 + t_n) - y(t_2 + t_n)| \\ \le \ & (\eta^u M_2 + g^u M_1)|t_1 - t_2|. \quad (11) \end{aligned}$$

The inequalities (10) and (11) show that $\{x(t + t_n)\}$ and $\{y(t + t_n)\}$ are equi-continuous on $[\alpha, \beta]_{\mathbb{T}}$. By the arbitrary of $[\alpha, \beta]_{\mathbb{T}}$, the conclusion is valid.

By Ascoli-Arzela theorem, there exists a subsequence of $\{t_n\}$, we still denote it as $\{t_n\}$, such that

$$x(t + t_n) \to p(t), y(t + t_n) \to q(t),$$

as $n \to +\infty$ uniformly in t on any bounded interval in $\mathbb{T}$.

Furthermore,

$$\begin{cases} x^{\Delta}(t + t_n) &= x(t + t_n)[r(t + t_n) \\ & \quad -a(t + t_n)x(t + t_n) \\ & \quad -c(t + t_n)y(t + t_n)] \\ & \quad -\frac{b(t+t_n)x^2(t+t_n)}{d(t+t_n)+x^2(t+t_n)}, \\ y^{\Delta}(t + t_n) &= -\eta(t + t_n)y(t + t_n) \\ & \quad +g(t)x(t + t_n). \end{cases}$$

Let $n \to +\infty$, then

$$\begin{cases} p^{\Delta}(t) &= p(t)[r(t) - a(t)p(t) - c(t)q(t)] \\ & \quad -\frac{b(t)p^2(t)}{d(t)+p^2(t)}, \\ q^{\Delta}(t) &= -\eta(t)q(t) + g(t)p(t). \end{cases}$$

It is clear that $(p(t), q(t))$ is a solution of system (2). Moreover,

$$m_1 \le p(t) \le M_1, \ m_2 \le q(t) \le M_2, \ \forall t \in \mathbb{T}.$$

This completes the proof.

Lemma 8. *In addition to the condition (H_1), assume further that the coefficients of system (2) satisfy the following conditions:*

(H_2) $a^l - g^u > 0$*;* $d^l - M_1^2 > 0$*;* $\eta^l - c^u > 0$*;*

(H_3) $0 < \gamma < \min\{m_1(a^l - g^u), \eta^l - c^u\}$ *and* $\lambda \in \mathcal{R}^+$.

Then system (2) is globally attractive.

Proof: Let $z_1(t) = (x_1(t), y_1(t))$ and $z_2(t) = (x_2(t), y_2(t))$ be any two positive solutions of system (2). It follows from (4)-(5) that for sufficient small positive constant ε_0 $(0 < \varepsilon_0 < \min\{m_1, m_2\})$, there exists a $T > 0$ such that

$$\begin{aligned} & m_1 - \varepsilon_0 < x_i(t) < M_1 + \varepsilon_0, && (12)\\ & m_2 - \varepsilon_0 < y_i(t) < M_2 + \varepsilon_0, && (13)\\ & t \in [T, +\infty)_{\mathbb{T}}, i = 1, 2. \end{aligned}$$

Since $x_i(t), i = 1, 2$ are positive, bounded and differentiable functions on $\mathbb{T}$, then there exists a positive, bounded and differentiable function $m(t), t \in \mathbb{T}$, such that $x_i(t)(1 + m(t)), i = 1, 2$ are strictly increasing on $\mathbb{T}$. By Lemma 3 and Lemma 4, we have

$$\begin{aligned} & \frac{\Delta}{\Delta t} L_{\mathbb{T}}(x_i(t)[1 + m(t)]) \\ = & \frac{x_i^{\Delta}(t)[1 + m(t)] + x_i(\sigma(t))m^{\Delta}(t)}{x_i(t)[1 + m(t)]} \\ = & \frac{x_i^{\Delta}(t)}{x_i(t)} + \frac{x_i(\sigma(t))m^{\Delta}(t)}{x_i(t)[1 + m(t)]},\ i = 1, 2. \end{aligned}$$

Here, we can choose a function $m(t)$ such that $\frac{|m^{\Delta}(t)|}{1+m(t)}$ is bounded on $\mathbb{T}$, that is, there exist two positive constants $\zeta > 0$ and $\xi > 0$ such that $0 < \zeta < \frac{|m^{\Delta}(t)|}{1+m(t)} < \xi$, $\forall t \in \mathbb{T}$.

Set

$$\begin{aligned} V(t) = & |e_{-\delta}(t, T)|(|L_{\mathbb{T}}(x_1(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(x_2(t)(1 + m(t)))| + |y_1(t) - y_2(t)|), \end{aligned}$$

where $\delta \geq 0$ is a constant (if $\mu(t) = 0$, then $\delta = 0$; if $\mu(t) > 0$, then $\delta > 0$). It follows from the mean value theorem of differential calculus on time scales for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} & \frac{1}{M_1 + \varepsilon_0}|x_1(t) - x_2(t)| \\ \leq & |L_{\mathbb{T}}(x_1(t)(1 + m(t))) - L_{\mathbb{T}}(x_2(t)(1 + m(t)))| \\ \leq & \frac{1}{m_1 - \varepsilon_0}|x_1(t) - x_2(t)|. \quad (14) \end{aligned}$$

Let $0 < \gamma < \min\{m_1(a^l - g^u), \eta^l - c^u\}$. We divide the proof into two cases.

Case I. If $\mu(t) > 0$, set $\delta > \max\{\frac{M_1\xi}{m_1}, \gamma\}$ and $1-\mu(t)\delta < 0$. Calculating the upper right derivatives of $V(t)$ along the solution of system (2), it follows from (12)-(14) and (H_2) that for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} & D^+V(t) \\ = & |e_{-\delta}(t, T)|\mathrm{sgn}(x_1(t) - x_2(t))\bigg[\frac{x_1^{\Delta}(t)}{x_1(t)} - \frac{x_2^{\Delta}(t)}{x_2(t)} \\ & +\frac{m^{\Delta}(t)}{1 + m(t)}\bigg(\frac{x_1(\sigma(t))}{x_1(t)} - \frac{x_2(\sigma(t))}{x_2(t)}\bigg)\bigg] \\ & -\delta|e_{-\delta}(t, T)||L_{\mathbb{T}}(x_1(\sigma(t))(1 + m(\sigma(t)))) \\ & -L_{\mathbb{T}}(x_2(\sigma(t))(1 + m(\sigma(t))))| \\ & +|e_{-\delta}(t, T)|\mathrm{sgn}(y_1(t) - y_2(t))(y_1^{\Delta}(t) - y_2^{\Delta}(t)) \\ & -\delta|e_{-\delta}(t, T)||y_1(\sigma(t)) - y_2(\sigma(t))| \\ \leq & -|e_{-\delta}(t, T)|\bigg[a(t) + \frac{b(t)(d(t) - x_1(t)x_2(t))}{(d(t) + x_1^2(t))(d(t) + x_2^2(t))} \\ & -g(t) \\ & +\frac{|m^{\Delta}(t)|}{1 + m(t)}\frac{x_1(\sigma(t))}{x_1(t)x_2(t)}\bigg]|x_1(t) - x_2(t)| \\ & -|e_{-\delta}(t, T)|\bigg[\frac{\delta}{M_1 + \varepsilon_0} - \frac{|m^{\Delta}(t)|}{1 + m(t)}\frac{1}{x_2(t)}\bigg] \\ & \times|x_1(\sigma(t)) - x_2(\sigma(t))| \\ & -|e_{-\delta}(t, T)|(\eta(t) - c(t))|y_1(t) - y_2(t)| \\ & -\delta|e_{-\delta}(t, T)||y_1(\sigma(t)) - y_2(\sigma(t))| \\ \leq & -|e_{-\delta}(t, T)|(a^l - g^u)|x_1(t) - x_2(t)| \\ & -|e_{-\delta}(t, T)|(\eta^l - c^u)|y_1(t) - y_2(t)| \\ \leq & -|e_{-\delta}(t, T)|\big[(m_1 - \varepsilon_0)(a^l - g^u) \\ & \times|L_{\mathbb{T}}(x_1(t)(1 + m(t))) - L_{\mathbb{T}}(x_2(t)(1 + m(t)))| \\ & +(\eta^l - c^u)|y_1(t) - y_2(t)|\big] \\ \leq & -\gamma|e_{-\delta}(t, T)|(|L_{\mathbb{T}}(x_1(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(x_2(t)(1 + m(t)))| + |y_1(t) - y_2(t)|) \\ = & -\gamma V(t). \quad (15) \end{aligned}$$

By the comparison theorem, (15) and (H_3), we have

$$\begin{aligned} V(t) \leq & e_{-\gamma}(t, T)V(T) \\ < & 2\bigg(\frac{M_1 + \varepsilon_0}{m_1 - \varepsilon_0} + M_2 + \varepsilon_0\bigg)e_{-\gamma}(t, T), \end{aligned}$$

that is,

$$\begin{aligned} & |e_{-\delta}(t, T)|(|L_{\mathbb{T}}(x_1(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(x_2(t)(1 + m(t)))| + |y_1(t) - y_2(t)|) \\ < & 2\bigg(\frac{M_1 + \varepsilon_0}{m_1 - \varepsilon_0} + M_2 + \varepsilon_0\bigg)|e_{-\gamma}(t, T)|, \end{aligned}$$

then

$$\begin{aligned} & \frac{1}{M_1 + \varepsilon_0}|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)| \\ < & 2\bigg(\frac{M_1 + \varepsilon_0}{m_1 - \varepsilon_0} + M_2 + \varepsilon_0\bigg)|e_{(-\gamma)\ominus(-\delta)}(t, T)|. \quad (16) \end{aligned}$$

Since $1 - \mu(t)\delta < 0$ and $0 < \gamma < \delta$, then $(-\gamma) \ominus (-\delta) < 0$. It follows from (16) that

$$\lim_{t\to+\infty}|x_1(t) - x_2(t)| = 0,\ \lim_{t\to+\infty}|y_1(t) - y_2(t)| = 0.$$

Case II. If $\mu(t) = 0$, set $\delta = 0$, then $\sigma(t) = t$ and $e_{-\delta}(t, T) = 1$. Calculating the upper right derivatives of $V(t)$ along the solution of system (2), it follows from (12)-(14) and (H_2) that for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} & D^+V(t) \\ = & \mathrm{sgn}(x_1(t) - x_2(t))\bigg(\frac{x_1^{\Delta}(t)}{x_1(t)} - \frac{x_2^{\Delta}(t)}{x_2(t)}\bigg) \\ & +\mathrm{sgn}(y_1(t) - y_2(t))(y_1^{\Delta}(t) - y_2^{\Delta}(t)) \\ \leq & -\bigg(a(t) + \frac{b(t)(d(t) - x_1(t)x_2(t))}{(d(t) + x_1^2(t))(d(t) + x_2^2(t))} - g(t)\bigg) \\ & \times|x_1(t) - x_2(t)| \\ & -(\eta(t) - c(t))|y_1(t) - y_2(t)| \\ \leq & -[(m_1 - \varepsilon_0)(a^l - g^u)|L_{\mathbb{T}}(x_1(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(x_2(t)(1 + m(t)))| + (\eta^l - c^u)|y_1(t) - y_2(t)|] \\ \leq & -\gamma(|L_{\mathbb{T}}(x_1(t)(1 + m(t))) - L_{\mathbb{T}}(x_2(t)(1 + m(t)))| \\ & +|y_1(t) - y_2(t)|) \\ = & -\gamma V(t), \quad (17) \end{aligned}$$

By the comparison theorem,(17) and (H_3), we have

$$\begin{aligned} V(t) &\le e_{-\gamma}(t,T)V(T) \\ &< 2\Big(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\Big)e_{-\gamma}(t,T), \end{aligned}$$

that is,

$$\begin{aligned} &|L_{\mathbb{T}}(x_1(t)(1+m(t)))-L_{\mathbb{T}}(x_2(t)(1+m(t)))| \\ &+|y_1(t)-y_2(t)| \\ &< 2\Big(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\Big)e_{-\gamma}(t,T), \end{aligned}$$

then

$$\begin{aligned} &\frac{1}{M_1+\varepsilon_0}|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)| \\ &< 2\Big(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\Big)e_{-\gamma}(t,T). \end{aligned} \tag{18}$$

It follows from (18) that

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|=0,\ \lim_{t\to+\infty}|y_1(t)-y_2(t)|=0.$$

From the above discussion, we can see that system (2) is globally attractive. This completes the proof.

Theorem 1. *Assume that the conditions $(H_1)-(H_3)$ hold, then system (2) has a unique globally attractive positive almost periodic solution.*

Proof: By Lemma 7, there exists a bounded positive solution $u(t)=(u_1(t),u_2(t))\in S(\mathbb{T})$, then there exists a sequence $\{t'_k\}$, $\{t'_k\}\to+\infty$ as $k\to+\infty$, such that $(u_1(t+t'_k),u_2(t+t'_k))$ is a solution of the following system:

$$\begin{cases} x^{\Delta}(t) &= x(t)[r(t+t'_k)-a(t+t'_k)x(t) \\ &\quad -c(t+t'_k)y(t)]-\frac{b(t+t'_k)x^2(t)}{d(t+t'_k)+x^2(t)}, \\ y^{\Delta}(t) &= -\eta(t+t'_k)y(t)+g(t+t'_k)x(t). \end{cases}$$

From the above discussion and Lemma 6, we have that not only $\{u_i(t+t'_k)\}, i=1,2$ but also $\{u_i^{\Delta}(t+t'_k)\}, i=1,2$ are uniformly bounded, thus $\{u_i(t+t'_k)\}, i=1,2$ are uniformly bounded and equi-continuous. By Ascoli-Arzela theorem, there exists a subsequence of $\{u_i(t+t_k)\}\subseteq\{u_i(t+t'_k)\}$ such that for any $\varepsilon>0$, there exists a $N(\varepsilon)>0$ with the property that if $m,k>N(\varepsilon)$ then

$$|u_i(t+t_m)-u_i(t+t_k)|<\varepsilon,\ i=1,2.$$

It shows that $u_i(t), i=1,2$ are asymptotically almost periodic functions, then $\{u_i(t+t_k)\}, i=1,2$ are the sum of an almost periodic function $q_i(t+t_k), i=1,2$ and a continuous function $p_i(t+t_k), i=1,2$ defined on $\mathbb{T}$, that is,

$$u_i(t+t_k)=p_i(t+t_k)+q_i(t+t_k),\ \forall t\in\mathbb{T},$$

where

$$\lim_{k\to+\infty}p_i(t+t_k)=0,\ \lim_{k\to+\infty}q_i(t+t_k)=q_i(t),$$

$q_i(t)$ is an almost periodic function. It means that $\lim_{k\to+\infty}u_i(t+t_k)=q_i(t), i=1,2.$

On the other hand

$$\begin{aligned} &\lim_{k\to+\infty}u_i^{\Delta}(t+t_k) \\ =\ &\lim_{k\to+\infty}\lim_{h\to0}\frac{u_i(t+t_k+h)-u_i(t+t_k)}{h} \\ =\ &\lim_{h\to0}\lim_{k\to+\infty}\frac{u_i(t+t_k+h)-u_i(t+t_k)}{h} \\ =\ &\lim_{h\to0}\frac{q_i(t+h)-q_i(t)}{h}. \end{aligned}$$

So, the limit $q_i(t), i=1,2$ exist.

Next, we shall prove that $(q_1(t),q_2(t))$ is an almost solution of system (2).

From the properties of almost periodic function, there exists a sequence $\{t_n\}$, $t_n\to+\infty$ as $n\to+\infty$, such that $r(t+t_n)\to r(t), a(t+t_n)\to a(t), b(t+t_n)\to b(t), c(t+t_n)\to c(t), d(t+t_n)\to d(t), \eta(t+t_n)\to\eta(t), g(t+t_n)\to g(t)$ as $n\to+\infty$ uniformly on $\mathbb{T}$.

It is easy to know that $u_i(t+t_n)\to q_i(t), i=1,2$ as $n\to+\infty$, then we have

$$\begin{aligned} q_1^{\Delta}(t) &= \lim_{n\to+\infty}u_1^{\Delta}(t+t_n) \\ &= \lim_{n\to+\infty}\Big[x(t+t_n)[r(t+t_n) \\ &\quad -a(t+t_n)x(t+t_n)-c(t+t_n)y(t+t_n)] \\ &\quad -\frac{b(t+t_n)x^2(t+t_n)}{d(t+t_n)+x^2(t+t_n)}\Big] \\ &= q_1(t)[r(t)-a(t)q_1(t)-c(t)q_2(t)] \\ &\quad -\frac{b(t)q_1^2(t)}{d(t)+q_1^2(t)}, \\ q_2^{\Delta}(t) &= \lim_{n\to+\infty}u_2^{\Delta}(t+t_n) \\ &= \lim_{n\to+\infty}[-\eta(t+t_n)u_2(t+t_n) \\ &\quad +g(t+t_n)x(t+t_n)] \\ &= -\eta(t)q_2(t)+g(t)q_1(t). \end{aligned}$$

This proves that $(q_1(t),q_2(t))$ is a positive almost periodic solution of system (2). Together with Lemma 8, system (2) has a unique globally attractive positive almost periodic solution. This completes the proof.

IV. Example and simulations

Consider the following system on time scales

$$\begin{cases} x^{\Delta}(t) &= x(t)[(0.8+0.2\sin\sqrt{2}t)-x(t) \\ &\quad -0.2y(t)]-\frac{x^2(t)}{(4.5+0.5\sin t)+x^2(t)}, \\ y^{\Delta}(t) &= -(0.4+0.1\cos\sqrt{3}t)y(t) \\ &\quad +(0.015+0.005\sin\sqrt{2}t)x(t). \end{cases} \tag{19}$$

By a direct calculation, we can get

$$\begin{aligned} &r^u=1, r^l=0.6, a^u=a^l=1, b^u=b^l=1, \\ &c^u=c^l=0.2, \eta^u=0.5, \eta^l=0.3, g^u=0.02, \\ &g^l=0.01, d^u=5, d^l=4, \\ &M_1=1, M_2=0.0667, m_1=0.4693, m_2=0.0094, \end{aligned}$$

then

(H_1) $r^l-c^uM_2=0.5867>0$;

(H_2) $a^l-g^u=0.9800>0$; $d^l-M_1^2=3.0000>0$; $\eta^l-c^u=0.1000>0$;

(H_3) $\min\{m_1(a^l - g^u), \eta^l - c^u\} = 0.1 > 0.$

Choose a $\lambda \in (0, 0.1)$ with $\lambda \in \mathcal{R}^+$, then conditions (H_1)-(H_3) hold. According to Theorem 1, system (19) has a unique globally attractive positive almost periodic solution. Dynamic simulations of system (19) with $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, see Figures 1 and 2, respectively.

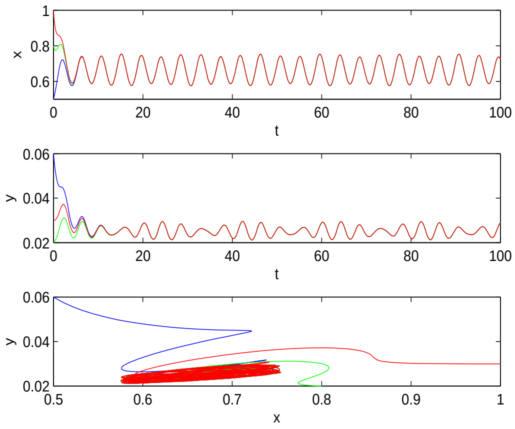

Fig. 1. $\mathbb{T} = \mathbb{R}$. Dynamics behavior of system (5.1) with initial condition $(x(0), y(0)) = \{[0.5, 0.06]; [0.8, 0.02]; [1, 0.03]\}$.

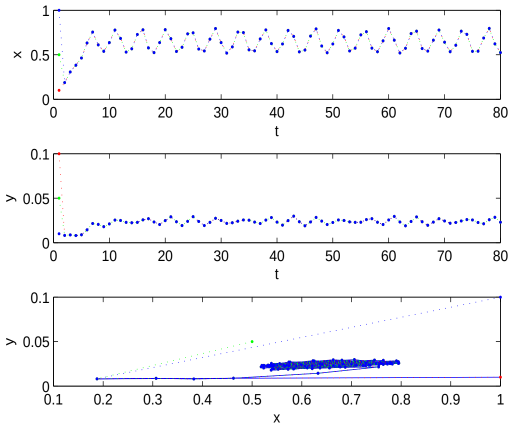

Fig. 2. $\mathbb{T} = \mathbb{Z}$. Dynamics behavior of system (5.1) with initial condition $(x(1), y(1)) = \{[0.1, 0.01]; [0.5, 0.05]; [1, 0.1]\}$.

V. Conclusion

This paper studied a single-species model with feedback control on time scales. Based on the theory of calculus on time scales, by using the properties of almost periodic functions and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the existence of a unique globally attractive positive almost periodic solution of the system are obtained.

This paper provided an effective method for the further study on the existence of almost periodic solution on time scales. Future work will include biological or epidemic dynamic systems modeling and analysis on time scales, one may see [19-22].

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