Rainbow Colorings on Pyramid Networks

Fu-Hsing Wang, *Member, IAENG*, and Cheng-Ju Hsu

***Abstract*—An edge coloring graph G is rainbow connected if every two vertices are connected by a rainbow path, i.e., a path with all edges of different colors. An edge coloring under which G is rainbow connected is a rainbow coloring. Rainbow connection number of G is the minimum number of colors needed under a rainbow coloring. In this paper, we propose a lower bound to the size of the rainbow connection number of pyramid networks. We believe that our techniques used for the lower bound is useful to prove lower bounds in the class of pyramid-like networks. In this paper, we also give a linear-time algorithm for constructing a rainbow coloring on pyramid networks and thus get an upper bound to the rainbow connection number of pyramid networks. The result shows that although the ratio of the upper bound and the lower bound are associated with a proportional increase in the dimension of the networks, the resulting ratios are still bounded.**

***Index Terms*—rainbow connection number, rainbow coloring, rainbow path, pyramid networks.**

I. Introduction

INTERCONNECTION networks have an enormous impact on the quality of communications between users and data transmissions. To address this issue, many research problems, including Hamiltonian connectivity [10], [11], $k-$path vertex cover [21], linear $k$arboricity [20], for interconnection networks were widely discussed. A powerful and analytical tool in studying interconnection networks is graph theory because interconnection network usually can be modeled as a simple graph whose vertices represent processing nodes of the system and edges represent communication links. Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges, respectively, of a graph G. The order of G is $|V(G)|$. An *edge coloring* of a graph is a function from its edge set to the set of natural numbers. A path between two vertices u and v is called a $u-v$ path. A $u-v$ path in an edge colored graph with no two edges sharing the same color is called a *rainbow* $u-v$ *path.* An edge-colored graph G is *rainbow connected* if any two vertices u and v are connected by a rainbow $u-v$ path. In this case, the edge coloring of G is called a *rainbow coloring* of G. The *rainbow connection number* of a connected graph G, denoted by $\chi_r(G)$, is the smallest number of colors that are needed to make G rainbow connected. A *rainbow* $k-$*coloring* of a graph is a rainbow coloring that uses k colors.

The problem of rainbow coloring in graphs was introduced by Chartrand et al. in [3] and has application in secure transfer of classified information between various agencies which may have other agencies as intermediaries by assigning passwords between agencies [8]. Every pair of agencies with one or more secure paths along with distinct passwords reveals the rainbow connection and is prohibitive to intruder [8]. The problem and its applications are intensively discussed in detail from the combinatorial perspective, with over 100 papers published (see good surveys [7], [13] and a book [14] for an overview).

The rainbow connection number and rainbow coloring have been studied from both the algorithmic and graph-theoretic points of view. Chakraborty et al. showed that computing the rainbow connection number of a general graph is NP-hard [2]. In fact, even deciding whether $\chi_r(G) = 2$ holds for a graph G is an NP-complete problem [2]. In [7], Eiben et al. gave an algorithm for deciding whether it is possible to obtain a rainbow coloring by saving a fixed number of colors. An easy observation that $\text{diam}(G) \leq \chi_r(G) \leq |V(G)| - 1$, where the diameter $\text{diam}(G)$ is the length of the longest shortest path in G. It is easy to verify that $\chi_r(G) = 1$ if and only if G is a complete graph, and $\chi_r(G) = |V(G)|-1$ if and only if G is a tree. In [1], Caro et al. provided sufficient conditions that guarantee $\chi_r(G) = 2$ and determined a threshold function for a random graph to have $\chi_r(G) = 2$. Also notice that $\chi_r(G) \leq |V(G)|-1$ for a general graph G, since one may color the edges of a given spanning tree with distinct colors (and color the remaining edges with one of the already used colors). Most recent research has been devoted to study the bounds of the rainbow connection numbers on random regular graphs [5], connected outerplanar graphs [6], triangular snake graphs [16], etc. Chartrand et al. computed the precise rainbow connection number for certain special graphs, e.g., Peterson graphs and complete multi-partite graphs [3].

We focus attention on the construction of rainbow colorings of a given pyramid network. Pyramid networks have powerful architecture for many applications such as image processing, visualization, and data mining [4]. The major advantage of pyramid networks for image processing systems is hierarchical abstracting and transferring of the data from different directions and forward them toward the apex of a pyramid network [18]. Its features include the fault-tolerate properties such as fault diameter, $\omega-$wide diameter [9]. Pyramid network also can be implemented with more efficient parallel algorithms than mesh connected networks for such problems as image processing and digital geometry [15], [17]. In this paper, we propose an efficient time algorithm for finding a rainbow path for any two vertices of a pyramid network. As far as we know, no rainbow path algorithm exists for pyramid networks.

The rest of the article is structured as follows. Section II gives the definition of pyramid networks. In Section III, we give a simple and general lower bound argument which yields lower bounds to the size of the rainbow connection numbers in any pyramid-like graphs. An upper bound for pyramid

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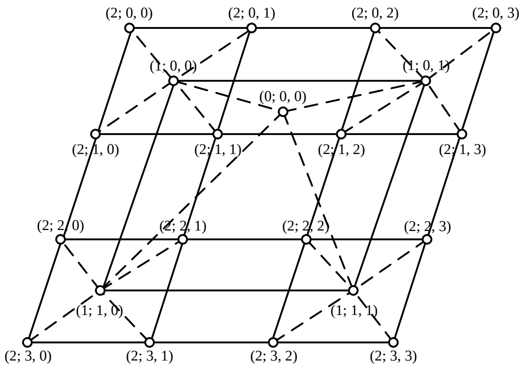

Fig. 1. A top view of the pyramid $\mathbb{P}_2$.

networks is presented in Section IV. Section V presents algorithms for finding a rainbow path on given two arbitrary vertices of a pyramid network. Finally, concluding remarks are given in the last section.

II. Preliminaries

A *square mesh* M_k of order $2^k \times 2^k$ has the vertex set $V(M_k) = \{(x, y) \,|\, 0 \leq x, y \leq 2^k - 1\}$ where any two vertices (x_1, y_1) and (x_2, y_2) are connected by an edge iff $|x_1 - x_2| + |y_1 - y_2| = 1$.

Let $\mathbb{P}_n$ be an n-*dimensional pyramid* with the vertex set $\bigcup_{k=0}^{n} V_k$, where $V_k = \{(k; x, y) \,|\, 0 \leq x, y \leq 2^k - 1\}$. We label the vertex v of V_k as $(k; x, y)$, where k, x and y are the layer number, row number and column number, respectively, of v. The subgraph induced by V_k is connected as an M_k and called the *layer* k of $\mathbb{P}_n$. For simplicity, we let M_k denote the subgraph induced by V_k. In M_k, a subgraph induced by the set of vertices with the same row number x (respectively, column number y) is called row x (respectively, column y). Vertex $(k; x, y)$ has exactly four *children* $(k+1; 2x, 2y)$, $(k+1; 2x, 2y+1)$, $(k+1; 2x+1, 2y)$, $(k+1; 2x+1, 2y+1)$ in V_{k+1} and a *parent vertex* $(k-1; \lfloor \frac{x}{2} \rfloor, \lfloor \frac{y}{2} \rfloor)$ in V_{k-1}, where $2 \leq k \leq n-1$. Let $p(v)$ denote the parent vertex of v. Every vertex on the shortest path from v to $(0; 0, 0)$ is an *ancestor* of v. An edge $[v, p(v)]$ incident on v and $p(v)$ is called a *layer edge*, while every edge of an M_k is called a *mesh edge*. Let L_k denote the set of layer edges between V_k and V_{k+1}. The *distance* $d_G(u, v)$ between two vertices u and v of G is the minimum length of the $u-v$ paths, where every vertex on a $u-v$ path is a vertex in G. Figure 1 depicts an example of the 2-layered pyramid $\mathbb{P}_2$. The vertex $(1; 0, 0)$ has four children $(2; 0, 0), (2; 0, 1), (2; 1, 1)$ and $(2; 1, 0)$. In contrast the parent vertex of $(2; 0, 0)$ is $(1; 0, 0)$. Both vertices $(0; 0, 0)$ and $(1; 0, 0)$ are ancestors of the vertex $(2; 0, 0)$. The dash lines indicate layer edges, while the solid lines are mesh edges. The layer edges connecting the vertex $(1; 0, 0)$ and its four children are in L_1. The distance $d_{M_2}((2; 0, 0), (2; 3, 3))$ is equal to 6, while $d_G((2; 0, 0), (2; 3, 3))$ is equal to 4.

III. Lower Bound

In this section we present a simple argument useful to prove lower bounds in the class of pyramid-like networks. For simplicity of notation, we let χ_r be $\chi_r(\mathbb{P}_n)$ in the remaining text of this section. Our proof is based on the following observation.

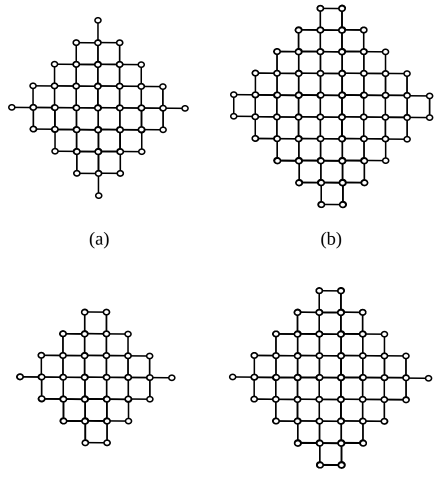

Fig. 2. Shapes of S_r for (a)$\chi_r = 8$; (b)$\chi_r = 10$; (c)$\chi_r = 7$; (d)$\chi_r = 9$.

Observation III.1. *For any two vertices u, v of distance greater than χ_r on $M_i, 2 \leq i \leq n$, every rainbow $u-v$ path contains $2a$ layer edges in L_{i-1}, where the integer $a \geq 1$.*

A maximal subgraph S_r of $M_i, 1 \leq i \leq n$, is called a *rainbow unit* if any two vertices in S_r are of distance less than or equal to χ_r. Two rainbow units S_{r_1} and S_{r_2} are said to be *disjoint* if $V(S_{r_1}) \cap V(S_{r_2}) = \emptyset$.

Lemma III.1. $|V(S_r)| \leq \frac{\chi_r^2 + 2\chi_r + 2}{2}$.

Proof: To be maximal, S_r has a shape as shown in Figure 2 depending on the parity of χ_r. For even χ_r, $\frac{\chi_r}{2}$ is either even or odd. If $\frac{\chi_r}{2}$ is even (refer to Figure 2(a)), then

$$\begin{aligned} |V(S_r)| &\leq (\frac{\chi_r}{2} + 1)^2 + 4(1 + 3 + \cdots + (\frac{\chi_r}{2} - 1)) \\ &= \frac{\chi_r^2 + 2\chi_r + 2}{2}. \end{aligned}$$

When $\frac{\chi_r}{2}$ is odd (see Figure 2(b)).

$$\begin{aligned} |V(S_r)| &\leq (\frac{\chi_r}{2} + 1)^2 + 4(2 + 4 + \cdots + (\frac{\chi_r}{2} - 1)) \\ &= \frac{\chi_r^2 + 2\chi_r}{2}. \end{aligned}$$

Consider odd χ_r. Figure 2(c) and Figure 2(d) depict the subcases even $\lceil \frac{\chi_r}{2} \rceil$ and odd $\lceil \frac{\chi_r}{2} \rceil$, respectively. If $\lceil \frac{\chi_r}{2} \rceil$ is even, then

$$\begin{aligned} |V(S_r)| \leq (\frac{\chi_r + 1}{2})(\frac{\chi_r + 3}{2}) + 2(1 + 3 + \cdots + \frac{\chi_r - 1}{2}) \\ + 2(2 + 4 + \cdots + \frac{\chi_r - 3}{2}) \\ = \frac{\chi_r^2 + 2\chi_r + 1}{2}. \end{aligned}$$

Otherwise $\lceil\frac{\chi_r}{2}\rceil$ is odd.

$$|V(S_r)| \le (\frac{\chi_r+1}{2})(\frac{\chi_r+3}{2}) + 2(1+3+\cdots+\frac{\chi_r-3}{2}) + 2(2+4+\cdots+\frac{\chi_r-1}{2}) = \frac{\chi_r^2+2\chi_r+1}{2}.$$

■

We now establish a lower bound of χ_r as follows:

Theorem III.1. *Any rainbow χ_r−coloring in $\mathbb{P}_n$ satisfies the inequality $\chi_r(\chi_r^2+2\chi_r+2) \ge \frac{8(4^n-1)}{3}$.*

Proof: For any two vertices u, v of distance greater than χ_r on $M_i, 1 \le i \le n$, every rainbow $u-v$ path, from Observation III.1, contains at least two layer edges in L_{i-1}. So there are at least $\frac{2^i \cdot 2^i}{|S_r|}$ disjoint rainbow units on $M_i, 1 \le i \le n$. And the layer edges incident on any two disjoint rainbow units must be assigned distinct colors. Then

$$\begin{aligned}\chi_r &\ge \frac{2^n\cdot 2^n}{|V(S_r)|} + \frac{2^{n-1}\cdot 2^{n-1}}{|V(S_r)|} + \cdots + \frac{2^1\cdot 2^1}{|V(S_r)|}\\ &= \frac{4^n}{|V(S_r)|} + \frac{4^{n-1}}{|V(S_r)|} + \cdots + \frac{4^1}{|V(S_r)|}\\ &= \frac{4^n}{|V(S_r)|}(1+\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^{n-1}})\\ &= \frac{4(4^n-1)}{3|V(S_r)|}.\end{aligned}$$

Furthermore, since, by Lemma III.1, $|V(S_r)| \le \frac{\chi_r^2+2\chi_r+2}{2}$, we have $\chi_r(\frac{\chi_r^2+2\chi_r+2}{2}) \ge \frac{4(4^n-1)}{3}$. Therefore, $\chi_r(\chi_r^2+2\chi_r+2) \ge \frac{8(4^n-1)}{3}$. ■

IV. Upper Bound

Wang and Hsu [19] gave the exact values of the rainbow connection number $\chi_r(\mathbb{P}_n)$ for $n \le 3$ as follows:

Theorem IV.1. *[19] $\chi_r(\mathbb{P}_1) = 2, \chi_r(\mathbb{P}_2) = 4$ and $\chi_r(\mathbb{P}_3) = 8$.*

In this paper, we further discuss $\chi_r(\mathbb{P}_n)$ for $n \ge 4$. Let φ be an edge coloring on $\mathbb{P}_n$ and $\varphi(e)$ be the color number assigned to the edge e. We also let $\varphi(E) = \bigcup_{e\in E}\varphi(e)$ for an edge set E. Let $\tau = \lceil\frac{2n+\log\frac{n}{3}-1}{3}\rceil$.

An $M_k, k > 2(n-\tau)$, can be partitioned into square submeshes of order $2^{k+\tau-n} \times 2^{k+\tau-n}$, and each square submesh is also called a *cluster*. Especially, every $M_k, 1 \le k \le 2(n-\tau)$, is regarded as a cluster. In a cluster, the edges of a row are assigned the color numbers in an ascending order from the leftmost edge to the rightmost edge of the row, while the edges of a column are assigned color numbers in an ascending order from the top edge to the bottom edge of the column under the edge coloring φ. In φ, all layer edges incident on a cluster are assigned the same color number. The formal definition of φ is as follows:

Definition IV.1. *Let φ be an edge coloring in $\mathbb{P}_n$ for $n \ge 4$.*

(1) $\varphi([(k;x,y),(k;x,y+1)]) =$

$$\begin{cases} y & \text{if } 1 \le k < 2(n-\tau),\\ (2\tau-n)4^{n-\tau}+y & \text{if } k = 2(n-\tau),\\ y \bmod 2^{k+\tau-n} & \text{if } 2(n-\tau) < k \le n,\end{cases}$$
where $0 \le x \le 2^k-1$ and $0 \le y \le 2^k-2$.

(2) $\varphi([(k;x,y),(k;x+1,y)]) =$

$$\begin{cases} x+(2^k-1) & \text{if } 1 \le k < 2(n-\tau),\\ (2\tau-n)4^{n-\tau}+(2^k-1)+x & \text{if } k = 2(n-\tau),\\ x \bmod 2^{k+\tau-n} + (2^{k+\tau-n}-1) & \text{if } 2(n-\tau) < k \le n,\end{cases}$$
where $0 \le x \le 2^k-2$ and $0 \le y \le 2^k-1$.

(3) $\varphi([(k;x,y),p(k;x,y)]) =$

$$\begin{cases} (2\tau-n)4^{n-\tau}+2(n-\tau)-k & \text{if } 1 \le k \le 2(n-\tau),\\ (n-k)4^{n-\tau}+\lfloor\frac{x}{2^{k+\tau-n}}\rfloor 2^{n-\tau} + \lfloor\frac{y}{2^{k+\tau-n}}\rfloor & \text{if } 2(n-\tau) < k \le n,\end{cases}$$
where $0 \le x, y \le 2^k-1$.

Definition IV.1(1) and (2) are used to assign color numbers to the edges of a row and a column, respectively, on a cluster. Definition IV.1(3) assigns color numbers to all layer edges. From Definition IV.1(2), the color numbers of all edges of a column in a cluster S on M_k, $1 \le k \le 2(n-\tau)$, are (2^k-1) more than the color numbers of all edges of any row in S. For $k > 2(n-\tau)$, the color numbers of all edges of a column in S are $(2^{k+\tau-n}-1)$ more than the color numbers of all edges of any row. To ensure that all layer edges incident on a cluster are assigned the same color number, we define $\varphi([u,p(u)]) = \varphi([v,p(v)])$ by dividing $2^{k+\tau-n}$ on the column indexes and row indexes for any two vertices u, v of a cluster (see Definition IV.1(3)).

Figure 3 depicts the edge coloring of layer k, where $2(n-\tau) < k \le n$. Layer k is partitioned into $2^{n-\tau}\times 2^{n-\tau}$ clusters, and each cluster in layer k is of order $2^{k+\tau-n}\times 2^{k+\tau-n}$. In each cluster of layer k, the edges of a row are assigned the color numbers in $\{0,1,\ldots,2^{k+\tau-n}-2\}$, while the edges of a column are assigned the color numbers in $\{2^{k+\tau-n}-1, 2^{k+\tau-n},\ldots,2^{k+\tau-n+1}-3\}$ under the edge coloring φ. The edge colorings on layers $1,2,\ldots,2(n-\tau)$ are similar to the edge coloring of a cluster on layer k.

Besides, we use Figure 4 as an example to illustrate the edge colorings on layers $3, 4$ and 5 of $\mathbb{P}_6$. Note that $n = 6, \tau = 4$ and $2(n-\tau) = 4$. In Figure 4, layers 6 and 5 are partitioned into 16 clusters. So, $|\varphi(E(L_5))| = |\varphi(E(L_4))| = 16$. Then $\varphi(E(L_5)) = \{0,1,\ldots,15\}$ and $\varphi(E(L_4)) = \{16,17,\ldots,31\}$. Since $(2\tau-n)4^{n-\tau} = 32$, the color numbers of the edges on layer 4 are in $\{32,33,\ldots,45\}$.

For a higher dimensional $\mathbb{P}_n$, we use Table I and Table II to demonstrate the range of values of $\varphi(E(\mathbb{P}_{10}))$ with $\tau = 7$ and $2(n-\tau) = 6$.

Lemma IV.1. *If S is a cluster of $\mathbb{P}_n$, then S is rainbow connected under the edge coloring φ.*

Proof: For any two distinct vertices $s, t \in V(S)$, we want to show that there is a rainbow $s-t$ path under φ. Let

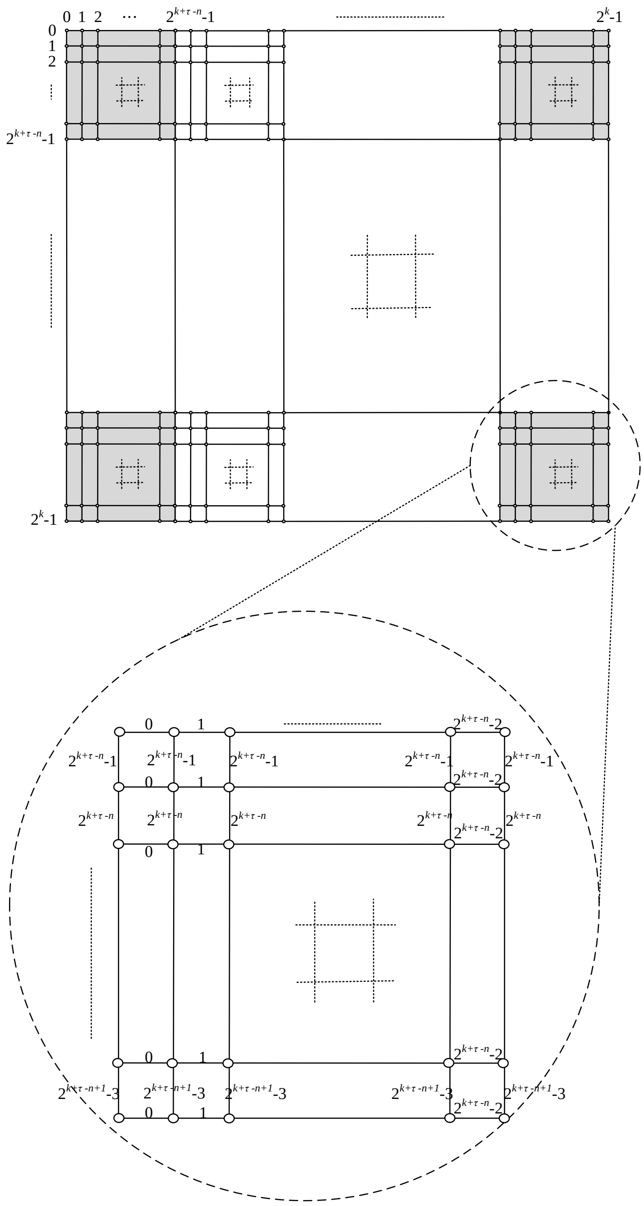

Fig. 3. The edge coloring of layer k, where $2(n-\tau) < k \leq n$. Every cluster is colored the same as shown in the bottom of the figure.

$s = (k; x_1, y_1), t = (k; x_2, y_2)$ and $v = (k; x_2, y_1)$, where $1 \leq k \leq n$ and $x_1 \leq x_2$. By Definition IV.1(2), the color numbers of the edges incident on the vertices in column y_1 are in an ascending order from the edge $[(k; x_1, y_1), (k; x_1 + 1, y_1)]$ to the edge $[(k; x_2 - 1, y_1), (k; x_2, y_1)]$. So there exists a rainbow $s - v$ path P_1. If $y_1 = y_2$, then v is t and hence completes the proof. Otherwise, by Definition IV.1(1), we also have a rainbow $v - t$ path P_2. If $1 \leq k \leq 2(n - \tau)$, the color numbers of the edges of P_1, by Definition IV.1(1)-(2), are $(2^k - 1)$ more than the color numbers of the edges of P_2. Thus the concatenation of P_1 and P_2 is a rainbow $a - t$ path. When $k > 2(n - \tau)$. The color numbers of the edges of P_1, by Definition IV.1(1)-(2), are $(2^{k+\tau-n} - 1)$ more than the color numbers of the edges of P_2. So the concatenation of P_1 and P_2 is a rainbow $s - t$ path. ■

We now show that φ is a rainbow coloring of a pyramid network.

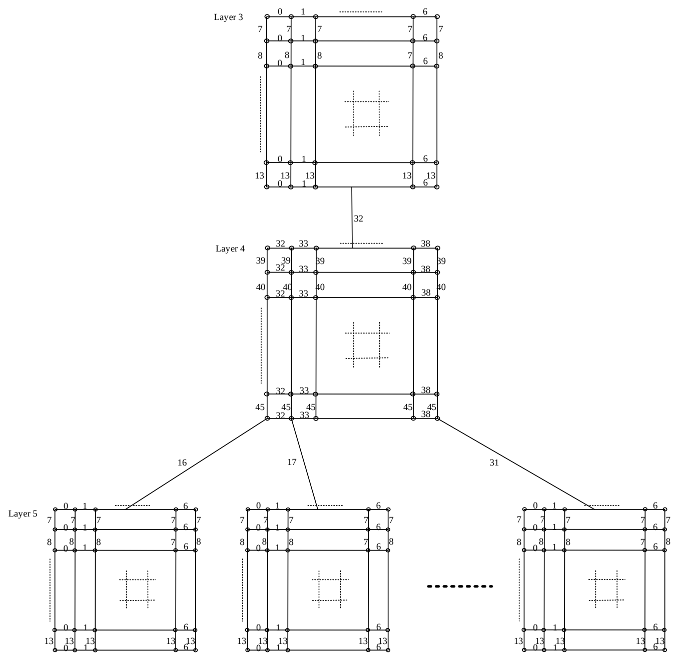

Fig. 4. The edge colorings on layers $3, 4$ and 5 of $\mathbb{P}_6$.

TABLE I
THE RANGE OF COLOR NUMBERS FOR THE MESH EDGES OF $\mathbb{P}_{10}$.

mesh edges	range of color numbers	
	row	column
M_1	0-0	1-1
M_2	0-2	3-5
M_3	0-6	7-13
M_4	0-14	15-29
M_5	0-30	31-60
M_6	$256 - 318$	$319 - 381$
M_7	$0 - 14$	$15 - 29$
M_8	$0 - 30$	$31 - 60$
M_9	$0 - 62$	$63 - 125$
M_{10}	$0 - 126$	$127 - 253$

Theorem IV.2. *The edge coloring φ is a rainbow coloring on $\mathbb{P}_n$.*

Proof: Let $s = (k_1; x_1, y_1), t = (k_2; x_2, y_2) \in V(\mathbb{P}_n)$, where $0 \leq k_1 \leq k_2 \leq n$. If s and t are on the same cluster, then, by Lemma IV.1, there is clearly a rainbow $s - t$ path. When s and t are on two different clusters, three cases are considered depending on the values of k_1 and k_2.

TABLE II
THE RANGE OF COLOR NUMBERS FOR THE LAYER EDGES OF $\mathbb{P}_{10}$.

layer edges	range of color numbers
L_0	$261 - 261$
L_1	$260 - 260$
L_2	$259 - 259$
L_3	$258 - 258$
L_4	$257 - 257$
L_5	$256 - 256$
L_6	$192 - 255$
L_7	$128 - 191$
L_8	$64 - 127$
L_9	$0 - 63$

Case 1. $k_1 = 0$.
Vertex s is indeed an ancestor of t. Let P be the $s - t$ path consisted only of layer edges. The edges of P, by Definition IV.1(3), are assigned distinct color numbers.

Case 2. $1 \leq k_1 < 2(n - \tau)$.
Let a_t be the ancestor of t in M_{k_1}. By Definition IV.1(3), we have a rainbow $t - a_t$ path P_1 consisting only of layer edges. Since M_{k_1} is in fact a cluster, by Lemma IV.1, we have a rainbow $s - a_t$ path P_2 consisting only of mesh edges. Clearly, the concatenation of the paths P_1 and P_2 construct a rainbow $s - t$ path.

Case 3. $2(n - \tau) \leq k_1 \leq n$.
Let $a_s, a_t \in V(M_{2(n-\tau)})$ and a_s and a_t be the ancestors of s and t, respectively. From Definition IV.1(3), we get a rainbow $s - a_s$ path P_1 and a rainbow $t - a_t$ path P_2 consisted only of layer edges. Notice that the color numbers of P_1 and P_2 are less than or equal to $(2\tau - n)4^{n-\tau} - 1$. Since $M_{2(n-\tau)}$ is in fact a cluster, by Lemma IV.1, we have a rainbow $a_s - a_t$ path P_3 and P_3 are consisting only of mesh edges. Because the color numbers of every mesh edge of P_3, from Definition IV.1(2), is greater than the color number of any layer edge in P_1 and P_2, it follows that the concatenation of the paths P_1, P_3 and P_2 constructs a rainbow $s - t$ path. ■

Clearly, the largest color number in $\varphi(E(\mathbb{P}_n))$ gives an upper bound to the size of the rainbow connection number on $\mathbb{P}_n$. According to Definition IV.1, the largest color number is assigned to the bottom edge of a column on layer $2(n-\tau)$ or layer n. Let $c_1 = (2\tau - n)4^{n-\tau} + (2^{k_1} - 1) + x_1$ be the color number assigned to the bottom edge of any column on layer $2(n - \tau)$, where $k_1 = 2(n - \tau)$ and $x_1 = 2^{k_1} - 2$. Let $c_2 = x_2 \bmod 2^{k_2+\tau-n} + (2^{k_2+\tau-n} - 1)$ be the color number assigned to the bottom edge of any column on layer n, where $x_2 = 2^{\tau} - 2$ and $k_2 = n$. Then

$$\begin{aligned} c_1 &= (2\tau - n)4^{n-\tau} + 2^{2(n-\tau)} + 2^{2(n-\tau)} - 3 \\ &= (2\tau - n + 2)4^{n-\tau} - 3 \end{aligned}$$

and

$$\begin{aligned} c_2 &= (2^{\tau} - 2) \bmod 2^{n+\tau-n} + 2^{n+\tau-n} - 1 \\ &= 2^{\tau+1} - 3. \end{aligned}$$

Let $max(c_1, c_2)$ denote the larger value of c_1 and c_2. The next theorem holds.

Theorem IV.3. $\chi_r(\mathbb{P}_n) \leq max((2\tau - n + 2)4^{n-\tau} - 3, 2^{\tau+1} - 3)$, *where* $\tau = \lceil \frac{2n + log \frac{n}{3} - 1}{3} \rceil$.

V. Rainbow Path Construction

According to the rainbow coloring φ, we further design the algorithm **Rainbow Path** for finding a rainbow $s - t$ path for any two distinct vertices s and t of $\mathbb{P}_n$. Algorithm **Rainbow Path** is with time complexity $O(n)$ because the amount of operations is bounded by the length of a rainbow path.

Function **Path-on-a-Cluster**(u, v)

Input: Vertices $u = (k; x_1, y_1), v = (k; x_2, y_2)$.
Output: A rainbow $u - v$ path P.
begin
The mesh edges connecting the starting vertex $(k; x_1, y_1)$ to the destination vertex $(k; x_1, y_2)$ in row x_1 form the subpath P_1;
The mesh edges connecting the starting vertex $(k; x_1, y_2)$ to the destination vertex $(k; x_2, y_2)$ in column y_2 form the subpath P_2;
return $P = P_1 + P_2$;
end

Function **Path-Connecting-Layers**(u, v)

Input: Vertices $u = (k_1; x_1, y_1), v = (k_2; x_2, y_2)$.
Output: A rainbow $u - v$ path P.
begin
Iteratively add layer edges $[w = (k_2 - k; \lfloor \frac{x_2}{2^k} \rfloor, \lfloor \frac{y_2}{2^k} \rfloor), p(w)]$ to P, for each $k = 0, 1, \ldots, k_2 - k_1 - 1$;
return P;
end

VI. Concluding Remarks

In this paper, we establish a lower bound and an upper bound to the size of the rainbow connection number in an n-dimensional pyramid network $\mathbb{P}_n$. To the best of our knowledge, this is the first result for constructing rainbow coloring in $\mathbb{P}_n$. The ratio of the bounds is considered as a performance metric in our algorithm. The resulting values are shown in Table III. The data are calculated on different scenarios to see the lower bound and the upper bound for different scales of pyramid networks. In Table III, the row "diam($\mathbb{P}_n$)" provides a trivial lower bound to $\chi_r(\mathbb{P}_n)$, where n is the dimension of the given network. The results of an improved lower bound for $\chi_r(\mathbb{P}_n)$ (from Theorem III.1) are given in the row "Our lower bound". The row "Our upper bound" reveals the largest color number that assigned to the edges of $E(\mathbb{P}_n)$ under the edge coloring φ. The row "ratio 1" shows that $\chi_r(\mathbb{P}_n)$ increase sharply on the growing n in spite of the tiny diameter. The row "ratio 2" demonstrates the proximity of our lower bound and upper bound. Although the ratio of the upper bound and the lower bound are associated with a proportional increase in the dimension of the networks, the resulting values of ratio 2 are still bounded by 4.58 even when the given network is of dimension 80.

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Algorithm **Rainbow Path**

Input: Given vertices $s = (k_1; x_1, y_1), t = (k_2; x_2, y_2)$, where $k_1 \leq k_2$.
Output: A rainbow $s - t$ path P.
begin
  if s and t are in the same cluster
    P =Path-on-a-Cluster(s, t);
  else
    Case $k_1 = 0$:
      P =Path-Connecting-Layers(s, t);
    Case $1 \leq k_1 < 2(n - \tau)$:
      $a_t = (k_1; \lfloor \frac{x_2}{2^{k_2 - k_1}} \rfloor, \lfloor \frac{y_2}{2^{k_2 - k_1}} \rfloor)$;
      P_1 =Path-Connecting-Layers(a_t, t);
      P_2 =Path-on-a-Cluster(s, a_t);
      $P = P_1 + P_2$;
    Case $2(n - \tau) \leq k_1 \leq n$:
      $a_s = (2(n - \tau); \lfloor \frac{x_1}{2^{k_1 - 2(n-\tau)}} \rfloor, \lfloor \frac{y_1}{2^{k_1 - 2(n-\tau)}} \rfloor)$;
      $a_t = (2(n - \tau); \lfloor \frac{x_2}{2^{k_2 - 2(n-\tau)}} \rfloor, \lfloor \frac{y_2}{2^{k_2 - 2(n-\tau)}} \rfloor)$;
      P_1 =Path-Connecting-Layers(a_s, s);
      P_2 =Path-Connecting-Layers(a_t, t);
      P_3 =Path-on-a-Cluster(a_s, a_t);
      $P = P_1 + P_2 + P_3$;
end

TABLE III
THE TABLE OF RATIOS ON THE BOUNDS OF χ_r.

	n (dimension)			
	10	30	50	80
(a) diam($\mathbb{P}_n$)	20	60	100	160
(b) Our lower bound	141	1454084	15007998106	1.57E+16
(c) Our upper bound	381	4194301	68719476733	7.21E+16
ratio 1 = (b) / (a)	7.05	24234.73	1.50E+08	9.84E+13
ratio 2 = (c) / (b)	2.70	2.88	4.58	4.58

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