Applications of Hesitant Fuzzy Sets to Completely Regular Semigroups

P. *Yairayong* *

Abstract— **In this paper, we focus on combining the theories of hesitant fuzzy sets on semigroups and establishing a new framework for hesitant fuzzy sets on semigroups. The aim of this manuscript is to apply hesitant fuzzy set for dealing with several kinds of theories in completely regular semigroups. We introduce the notions of hesitant fuzzy semiprime sets and hesitant fuzzy idempotent sets on semigroups and some properties of them are obtained. We also characterized left (right) regular, completely regular, intra-regular, regular, left (right) quasi-regular, quasi-regular intra-regular, left quasi-regular and semisimple semigroups by the properties of their hesitant fuzzy ideals. At the end we prove that every hesitant fuzzy two-sided ideal on a semigroup S is a hesitant fuzzy interior ideal if and only if S is a semisimple semigroup.**

***Index Terms*—hesitant fuzzy set, semigroup, completely regular, hesitant fuzzy two-sided ideal, hesitant fuzzy interior ideal.**

I. Introduction

In 1986, Drazin [1] studied some operations of completely regular semigroups and showed that the $\mathfrak{I}$ is a partial order on a given finite semigroup S if and only if S is completely regular. The structure of completely regular semigroups has been recently studied by many authors, for example, by Petrich, Trotter, Yang, He and others, see [2-14, 40]. In 1996, Ajan and Pastijn [15] showed that every finitely presented group is the greatest group homomorphic image of a finitely presented completely regular semigroup. In 2000, Gould and Smith [16] characterized semigroups that are straight left orders in completely regular semigroups. In 2002, Mashevitzky [17] showed that the variety V of completely regular semigroups that contains a nontrivial semilattice has finitary or unitary unification type if and only if V consists of strong semilattices of rectangular groups of finitary or unitary unification type. In 2009, Dolinka [18] studied some operations of finite completely regular semigroups and monoids. He also proved that the finite completely regular semigroup has a sub-log-exponential free spectrum if and only if it is locally orthodox and has nilpotent subgroups. In 2010, Khan et al. [19] characterized left simple, right simple, and completely regular ordered semigroups in terms of intuitionistic fuzzy quasi-ideals. In 2012, Khan et al. [20] discussed some characterizations of regular, left (resp. right) regular, completely regular and weakly regular ordered semigroups. In 2014, Maity and Ghosh [21] showed that the semiring R is quasi completely regular if and only if R is an idempotent semiring of quasi skew-rings.

In 2008, Kehayopulu and Tsingelis [22] has studied the right regular, left regular, regular and intra-regular ordered semigroups. They proved that the ordered semigroup S is intra-regular if and only $\mu \subseteq \mu \circ \mu$ for each fuzzy subset μ of S. Several examples and interesting properties of intra-regular semigroups (LA-semigroup, ordered semigroup) can be found in [23-27]. In 2009, Kehayopulu and Tsingelis [28] discussed some characterizations of intra-regular, both regular and intra-regular ordered semigroups in terms of fuzzy sets. In 2012, Khan et. al. [29] proved that the ordered semigroup S is simple if and only if it is $(\in, \in \vee q)$-fuzzy simple and characterized intra-regular ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy ideals. In 2013, Li and Feng [30] studied the (λ, μ)-fuzzy ideals and the (λ, μ)-fuzzy interior ideals in intra-regular ordered semigroups and investigated some of their properties. In 2015, Kar et. al. [31] introduced the concept of interval-valued fuzzy hyperideals of a semihypergroup and characterized regular semihypergroups, intra-regular semihypergroups by using interval-valued fuzzy hyperideals of semihypergroups. In 2016, Yousafzai et. al. [32] characterized intra-regular ordered LA-semigroups in terms of interval valued fuzzy ideals and interval valued fuzzy bi-ideals. In 2017, Ali et. al. [33] defined $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy soft right ideals in AG-groupoids and characterized intra-regular AG-groupoids. In 2018, Khan et. al. [34] introduced the concept of $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideals and $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy biideals in AG-groupoidsand and characterized intra-regular AG-groupoids.

In 2015, Jun et. al. [35] also discussed the relationships between hesitant fuzzy generalized bi-ideals and hesitant fuzzy semigroups and characterized (hesitant fuzzy) generalized bi-ideals and hesitant fuzzy bi-ideals. They introduced the notions of hesitant fuzzy (generalized) bi-ideals in semigroups. Ali et. al. [36] defined hesitant fuzzy sets, hesitant fuzzy product, characteristic hesitant fuzzy set, hesitant fuzzy AG-groupoids, hesitant fuzzy left ideal, hesitant fuzzy bi-ideal, hesitant fuzzy interior ideal and hesitant fuzzy quasi-ideal in AG-groupoids and investigated some of their properties. In 2016, Jun et. al. [37] also discussed the relationships between hesitant fuzzy semigroups with a frontier and a hesitant fuzzy semigroups.They showed that the hesitant union of two hesitant fuzzy semigroups with

Manuscript received November 22, 2018; revised January 14, 2020.

P. Yairayong* is with the Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand (e-mail: pairote0027@hotmail.com).

two frontiers may not be a hesitant fuzzy semigroup with two frontiers. In 2017, Jun et. al. [38] introduced the concept of (ε, δ)-hesitant fuzzy ideals, (ε, δ)-hesitant fuzzy bi-ideals and (ε, δ)-hesitant fuzzy interior ideals in semigroups and investigated some of their properties. In 2018, Abbasi et. al. [39] introduced the idea of hesitant fuzzy left (resp., right and two-sided) ideals, hesitant fuzzy bi-ideals, and hesitant fuzzy interior ideals in Γ-semigroups and investigated some of their properties.

The aim of this manuscript is to apply hesitant fuzzy set for dealing with several kinds of theories in completely regular semigroups. We introduce the notions of hesitant fuzzy semiprime sets and hesitant fuzzy idempotent sets on semigroups and some properties of them are obtained. We also characterized left (right) regular, completely regular, intra-regular, regular, left (right) quasi-regular, quasi-regular intra-regular, left quasi-regular and semisimple semigroups by the properties of their hesitant fuzzy ideals. At the end we prove that every hesitant fuzzy two-sided ideal on a semigroup S is a hesitant fuzzy interior ideal if and only if S is a semisimple semigroup.

II. Preliminaries

In this section, we recollect some preliminaries and prerequisites of semigroups and hesitant fuzzy sets which we shall apply in the results of the main section.

For a hesitant fuzzy set H on S and $x, y \in S,$ we use the notations $\mathrm{H}_x = \mathrm{H}(x)$ and $\mathrm{H}_x^y = \mathrm{H}_x \cap \mathrm{H}_y.$ It is obvious that $\mathrm{H}_x^y = \mathrm{H}_y^x.$

For every a subset A of a semigroup $S,$ define a map

$$(\mathrm{C}_A)_x = \begin{cases} [0,1] & ; x \in A \\ \varnothing & ;\text{otherwise.} \end{cases}$$

Then, C_A is a hesitant fuzzy set on S and is called **characteristic hesitant fuzzy set** on $S.$ The hesitant fuzzy set C_S is called **identity hesitant fuzzy set** on $S.$

Recall that a hesitant fuzzy set H on a semigroup S is called a **hesitant fuzzy semigroup** on S if $\mathrm{H}_x^y \subseteq \mathrm{H}_{xy}$ for all $x, y \in S.$ Recall that a hesitant fuzzy set H on a semigroup S is called a **hesitant fuzzy left (resp., right) ideal** on S if

$$\mathrm{H}_{xy} \supseteq \mathrm{H}_y \left(\mathrm{H}_{xy} \supseteq \mathrm{H}_x\right)$$

for all $x, y \in S.$ If a hesitant fuzzy set H on a semigroup S is both a hesitant fuzzy left ideal and a hesitant fuzzy right ideal on $S,$ we say that H is a **hesitant fuzzy ideal**} on $S.$ Recall that a hesitant fuzzy semigroup H on a semigroup S is called a **hesitant fuzzy bi-ideal** on S if $\mathrm{H}_{xyz} \supseteq \mathrm{H}_x^z$ for all $x, y, z \in S.$ If $\mathrm{H}(x) = [0,1]$ for all $x \in S,$ then it is easy to see that H is a hesitant fuzzy semigroup on a semigroup $S.$ We denote such type of hesitant fuzzy semigroup S by $\Upsilon.$ It is obvious that $\mathrm{C}_S = \Upsilon.$ The **hesitant fuzzy product** of two hesitant fuzzy sets F and G on a semigroup S is defined to be a hesitant fuzzy set $\mathrm{F} \odot \mathrm{G}$ on S which is given by

$$(\mathrm{F} \odot \mathrm{G})_x = \begin{cases} \bigcup\limits_{x=yz} \mathrm{F}(y) \cap \mathrm{G}(z) & ; \exists y, z \in S, \text{such that } x \in y \circ z \\ \varnothing & ;\text{otherwise.} \end{cases}$$

We denote by $\mathbf{H}(S)$ the set of all hesitant fuzzy sets on $S.$ Let S be a semigroups and $\mathrm{H}, \mathrm{F} \in \mathbf{H}(S).$ Then, H is called the **subset** of F, denoted by $\mathrm{H} \prec \mathrm{F}$ if $\mathrm{H}_x \subseteq \mathrm{F}_x$ for all $x \in S.$ For a non empty family of a hesitant fuzzy sets $\{\mathrm{H}_\alpha : \alpha \in \beta\},$ on a semigroup $S,$ the hesitant fuzzy sets $\text{ò}_{\alpha \in \beta} \mathrm{H}_\alpha$ and $\text{ó}_{\alpha \in \beta} \mathrm{H}_\alpha$ are defined as follows:

$$\left(\text{ò}_{\alpha \in \beta} \mathrm{H}_\alpha\right)_x = \bigcup_{\alpha \in \beta} (\mathrm{H}_\alpha)_x$$

and

$$\left(\text{ó}_{\alpha \in \beta} \mathrm{H}_\alpha\right)_x = \bigcap_{\alpha \in \beta} (\mathrm{H}_\alpha)_x.$$

If β is a finite set, say $\beta = \{1, 2, 3, \ldots, n\},$ then clearly $\text{ò}_{\alpha \in \beta} \mathrm{H}_\alpha = \mathrm{H}_1 \,\text{ò}\, \mathrm{H}_2 \,\text{ò}\, \ldots \text{ò}\, \mathrm{H}_n$ and

$$\text{ó}_{\alpha \in \beta} \mathrm{H}_\alpha = \mathrm{H}_1 \,\text{ó}\, \mathrm{H}_2 \,\text{ó}\, \ldots \text{ó}\, \mathrm{H}_n.$$

Recall that a hesitant fuzzy set H on a semigroup S is called a **hesitant fuzzy quasi-ideal** on S if

$$\left(\mathrm{H} \odot \Upsilon\right) \text{ó} \left(\Upsilon \odot \mathrm{H}\right) \prec \mathrm{H}.$$

As is easily seen, any hesitant fuzzy left ideal and any hesitant fuzzy right ideal on S is a hesitant fuzzy quasi-ideal on S and any hesitant fuzzy quasi-ideal on S is a hesitant fuzzy bi-ideal on $S.$ The converse of those properties do not hold in general. Recall that a hesitant fuzzy set H on a semigroup S is called a **hesitant fuzzy generalized bi-ideal** on S if

$$\mathrm{H}_{xyz} \supseteq \mathrm{H}_x^z$$

for all $x, y, z \in S.$ It is clear that every hesitant fuzzy bi-ideal on a semigroup S is a hesitant fuzzy generalized bi-ideal on $S,$ but the converse of this statement does not hold in general. Recall that a hesitant fuzzy set H on a semigroup S is called a **hesitant fuzzy interior ideal** on S if $\mathrm{H}_{xyz} \supseteq \mathrm{H}_y$ for all $x, y, z \in S.$

III. Hesitant fuzzy sets on completely regular semigroups

In this section we define the concept of hesitant fuzzy semiprime sets and then by using this idea we characterize the left (right) regular semigroup in terms of hesitant fuzzy left (right) ideals. Also we prove that every hesitant fuzzy set of a semigroup S is a hesitant fuzzy semiprime set if and only if the characteristic hesitant fuzzy set C_P is a hesitant fuzzy semiprime set.

Recall that a semigroup S is called **left (right) regular** if for each element x of S, there exists an element s in S such that $x = sx^2(x = x^2 s)$.

Now we characterize a left (right) regular semigroup in terms of hesitant fuzzy left (right) ideals.

Theorem 3.1: Let S be a semigroup. Then the following conditions are equivalent.

1) S is a left (right) regular semigroup.

2) For every hesitant fuzzy left (right) ideal H on S, $\mathrm{H}_x = \mathrm{H}_{x^2}$ for all $x \in S$.

Proof: First assume that S is a left regular semigroup. Let H be any hesitant fuzzy left ideal on S and x any element of S. Since S is left regular, there exists an element $s \in S$ such that $x = sx^2$. By assumption,

$$\mathrm{H}_x = \mathrm{H}_{sx^2} \supseteq \mathrm{H}_{x^2} \supseteq \mathrm{H}_x,$$

which means that $\mathrm{H}_x = \mathrm{H}_{x^2}$.

Conversely, assume that 2) holds. Let x be any element of S. It is easy to see that, the characteristic hesitant fuzzy set $\mathrm{C}_{L[x^2]}$ of the left ideal $L[x^2] = x^2 \cup Sx^2$ on S is a hesitant fuzzy left ideal on S. Then we have, $\left(\mathrm{C}_{L[x^2]}\right)_x = \left(\mathrm{C}_{L[x^2]}\right)_{x^2} = [0,1]$, since $x^2 \in L[x^2]$. It follows that, $x \in L[x^2] = x^2 \cup Sx^2$, there exists an element $s \in S$ such that $x = sx^2$. Consequently, S is left regular.

Now we introduce the notion of hesitant fuzzy semiprime sets on a semigroup.

Definition 3.2: A hesitant fuzzy set H on a semigroup S is called a **hesitant fuzzy semiprime** on S if $\mathrm{H}_x \supseteq \mathrm{H}_{x^2}$ for all $x \in S$.

Next, we proved that every hesitant fuzzy set of a semigroup S is a hesitant fuzzy semiprime set if and only if the characteristic hesitant fuzzy set C_P is a hesitant fuzzy semiprime set.

Theorem 3.3: Let P be a non empty subset of a semigroup S. Then the following conditions are equivalent.

1) P is a semiprime subset of S.

2) The characteristic hesitant fuzzy set C_P of P is a hesitant fuzzy semiprime set on S.

Proof: First assume that P is a semiprime subset of S. Let x be any element of S. If $x^2 \in P$, then, since P is a semiprime subset, $x \in P$. This implies that, $(\mathrm{C}_P)_x = [0,1] = (\mathrm{C}_P)_{x^2}$. If $x^2 \notin P$, then

$$(\mathrm{C}_P)_x \supseteq \varnothing = (\mathrm{C}_P)_{x^2}.$$

In any case, we have $(\mathrm{C}_P)_x \supseteq (\mathrm{C}_P)_{x^2}$ for all $x \in S$. Consequently, C_P is a hesitant fuzzy semiprime set on S.

Conversely, assume that C_P is a hesitant fuzzy semiprime set on S. Let x be any element of S such that $x^2 \in P$. By assumption, $(\mathrm{C}_P)_x \supseteq (\mathrm{C}_P)_{x^2} = [0,1]$, which implies that $(\mathrm{C}_P)_x = [0,1]$. Then we have, $x \in P$. Consequently, P is a semiprime subset of S.

The following theorem is a characterization theorem of hesitant fuzzy semiprime sets on a semigroup.

Theorem 3.4: Let H be a hesitant fuzzy semigroup on a semigroup S. Then the following conditions are equivalent.

1) H is a hesitant fuzzy semiprime set on S.

2) $\mathrm{H}_x = \mathrm{H}_{x^2}$ for all $x \in S$.

Proof: First assume that H is a hesitant fuzzy semiprime set on S. Let x be any element of S. By assumption, $\mathrm{H}_x \supseteq \mathrm{H}_{x^2} \supseteq \mathrm{H}_x^x = \mathrm{H}_x$. Consequently, $\mathrm{H}_x = \mathrm{H}_{x^2}$.

It is obvious that 2) implies 1).

Recall that an element x of a semigroup S is called **completely regular** if there exists an element s in S such that $x = xsx$ and $sx = xs$. A semigroup S is called **completely regular** if every element of S is completely regular.

Now we give characterizations of a completely regular semigroup in terms of hesitant fuzzy sets.

Theorem 3.5: Let S be a semigroup. Then the following conditions are equivalent.

1) S is a completely regular semigroup.

2) Every hesitant fuzzy quasi-ideal on S is a hesitant fuzzy semiprime set on S.

3) $\mathrm{H}_x = \mathrm{H}_{x^2}$ for every hesitant fuzzy generalized bi-ideal H on S and for all $x \in S$.

Proof: 1)$\Rightarrow$ 3). Assume that S is a completely regular semigroup. Let H be any hesitant fuzzy generalized bi-ideal of S and any element of S. By assumption, there exists an element $s \in S$ such that $x = x^2 s x^2$. This implies that,

$$\begin{aligned} \mathrm{H}_x &= \mathrm{H}_{x^2 s x^2} \\ &\supseteq \mathrm{H}_{x^2}^{x^2} \\ &= \mathrm{H}_{x^2} \\ &= \mathrm{H}_{x(x^2 s x^2)} \\ &\supseteq \mathrm{H}_x^x \\ &= \mathrm{H}_x. \end{aligned}$$

Consequently, $\mathrm{H}_x = \mathrm{H}_{x^2}$.

It is obvious that, 3)$\Rightarrow$ 2).

2)$\Rightarrow$1). Assume that 2) holds. Let x be any element of S. Since $Q[x^2]=x^2\cup\left(x^2S\cap Sx^2\right)$ is a quasi-ideal of S, we have $\mathrm{C}_{Q[x^2]}$ is a hesitant fuzzy quasi-ideal on S. By assumption, $\mathrm{C}_{Q[x^2]}$ is a hesitant fuzzy semiprime set on S, which implies that $\left(\mathrm{C}_{Q[x^2]}\right)_x\supseteq\left(\mathrm{C}_{Q[x^2]}\right)_{x^2}=[0,1]$. Therefore, $x\in Q[x^2]=x^2\cup\left(x^2S\cap Sx^2\right)$ and hence, $x=x^2$ or $x\in x^2S\cap Sx^2$. Consequently, S is a completely regular semigroup.

From Theorem 3.5 we can easily obtain the following theorem.

Theorem 3.6: Let S be a semigroup. Then the following conditions are equivalent.

1) Every hesitant fuzzy quasi-ideal on S is a hesitant fuzzy semiprime set on S.

2) Every hesitant fuzzy bi-ideal on S is a hesitant fuzzy semiprime set on S.

3) Every hesitant fuzzy generalized bi-ideal on S is a hesitant fuzzy semiprime set on S.

4) $\mathrm{H}_x=\mathrm{H}_{x^2}$ for every hesitant fuzzy quasi-ideal H on S and for all $x\in S$.

5) $\mathrm{H}_x=\mathrm{H}_{x^2}$ for every hesitant fuzzy bi-ideal H on S and for all $x\in S$.

6) $\mathrm{H}_x=\mathrm{H}_{x^2}$ for every hesitant fuzzy generalized bi-ideal H on S and for all $x\in S$.

IV. Hesitant Fuzzy Sets on Intra-Regular Semigroups

In this section we define the concept of hesitant fuzzy idempotent sets on semigroups and then by using this idea we characterize the intra-regular semigroups in terms of hesitant fuzzy quasi-ideals, hesitant fuzzy bi-ideals, and hesitant fuzzy generalized bi-ideals.

Recall that a semigroup S is called **intra-regular** if for each element x of S, there exists elements s_1 and s_2 in S such that $x=s_1x^2s_2$.

Now, we proved that every hesitant fuzzy two-sided ideal on a semigroup S is a hesitant fuzzy interior ideal if and only if S is an intra-regular semigroup.

Theorem 4.1: Let H be a hesitant fuzzy set on an intra-regular semigroup S. Then the following conditions are equivalent.

1) H is a hesitant fuzzy two-sided ideal on S.

2) H is a hesitant fuzzy interior ideal on S.

Proof: It is clear that 1) implies 2). Assume that H is a hesitant fuzzy interior ideal on S. Let x and y be any elements of S. Since S is an intra-regular semigroup, there exist elements s_1,s_2,s_3 and s_4 of S such that $x=s_1x^2s_2$ and $y=s_3y^2s_4$. Then, since H is a hesitant fuzzy interior ideal on S, we have

$$\mathrm{H}_{xy}=\mathrm{H}_{(s_1x^2s_2)y}=\mathrm{H}_{(s_1x)x(s_2y)}\supseteq\mathrm{H}_x$$

and $\mathrm{H}_{xy}=\mathrm{H}_{x(s_3y^2s_4)}=\mathrm{H}_{(xs_3)y(ys_4)}\supseteq\mathrm{H}_y$. Consequently, H is a hesitant fuzzy two-sided ideal on S.

Now we characterize an intra-regular semigroup in terms of hesitant fuzzy two-sided ideals and hesitant fuzzy interior ideals.

Theorem 4.2: Let S be a semigroup. Then the following conditions are equivalent.

1) S is an intra-regular semigroup.

2) $\mathrm{H}_x=\mathrm{H}_{x^2}$ for every hesitant fuzzy two-sided ideal H on S and for all $x\in S$.

3) $\mathrm{H}_x=\mathrm{H}_{x^2}$ for every hesitant fuzzy interior ideal H on S and for all $x\in S$.

Proof: 1)$\Rightarrow$ 3). Assume that S is an intra-regular semigroup. Let H be any hesitant fuzzy interior ideal on S and x any element of S. By assumption, there exist elements s_1 and s_2 of S such that $x=s_1x^2s_2$. Then we have,

$$\begin{aligned}\mathrm{H}_x &= \mathrm{H}_{s_1x^2s_2}\\ &\supseteq \mathrm{H}_{x^2}\\ &= \mathrm{H}_{x(s_1x^2s_2)}\\ &= \mathrm{H}_{(xs_1)x(xs_2)}\\ &\supseteq \mathrm{H}_x,\end{aligned}$$

which implies that $\mathrm{H}_x=\mathrm{H}_{x^2}$.

It is obvious that, 2)$\Rightarrow$3).

2)$\Rightarrow$1). Assume that 2) holds. Let x be any element of S. Since $J[x^2]=x^2\cup x^2S\cup Sx^2\cup Sx^2S$ is a two-sided ideal of S, we have $\mathrm{C}_{J[x^2]}$ is a hesitant fuzzy two-sided ideal on S. Then we have,

$$\left(\mathrm{C}_{J[x^2]}\right)_x\supseteq\left(\mathrm{C}_{J[x^2]}\right)_{x^2}=[0,1],$$

which implies that

$$x\in J[x^2]=x^2\cup x^2S\cup Sx^2\cup Sx^2S.$$

Hence, $x=x^2$ or $x\in x^2S$ or $x\in Sx^2$ or $x\in Sx^2S$. Consequently, S is an intra-regular semigroup.

In the sequel, the following results hold. The proof of them is straightforward, so we omit it.

Theorem 4.3: Let S be a semigroup. Then the following conditions are equivalent.

1) Every hesitant fuzzy two-sided ideal on S is a hesitant fuzzy semiprime set on S.

2) Every hesitant fuzzy interior ideal on S is a hesitant fuzzy semiprime set on S.

3) $\mathrm{H}_x = \mathrm{H}_{x^2}$ for every hesitant fuzzy two-sided ideal H on S and for all $x \in S$.

4) $\mathrm{H}_x = \mathrm{H}_{x^2}$ for every hesitant fuzzy interior ideal H on S and for all $x \in S$.

The following theorem is a characterization theorem of hesitant fuzzy interior ideals on a semigroup.

Theorem 4.4: Let H be a hesitant fuzzy set on an intra-regular semigroup S. Then the following conditions are equivalent.

1) H is a hesitant fuzzy interior ideal on S.

2) $\mathrm{H}_{xy} = \mathrm{H}_{yx}$ for every $x, y \in S$.

Proof: First assume that H is a hesitant fuzzy interior ideal on S. Let x and y be any elements of S. Then by Theorem 4.3,

$$\begin{aligned} \mathrm{H}_{xy} &= \mathrm{H}_{(xy)^2} \\ &= \mathrm{H}_{x(yx)y} \\ &\supseteq \mathrm{H}_{yx} \\ &= \mathrm{H}_{(yx)^2} \\ &= \mathrm{H}_{y(xy)x} \\ &\supseteq \mathrm{H}_{xy}, \end{aligned}$$

which implies that, $\mathrm{H}_{xy} = \mathrm{H}_{yx}$.

It is dear that 2) implies 1).

By the above theorem we have the following theorem.

Proposition 4.5: Let H be a hesitant fuzzy semiprime set on a semigroup S. If H is a hesitant fuzzy interior ideal on S, then $\mathrm{H}_{x^n} \supseteq \mathrm{H}_{x^{n+1}}$ for alt positive integers n.

Proof: Let n be any positive integer and x any element of S. By assumption,

$$\begin{aligned} \mathrm{H}_{x^n} &\supseteq \mathrm{H}_{x^{2n}} \\ &\supseteq \mathrm{H}_{x^{4n}} \\ &= \mathrm{H}_{x^{3n-2}x^{n+1}x} \\ &\supseteq \mathrm{H}_{x^{n+1}}, \end{aligned}$$

which implies that, $\mathrm{H}_{x^n} \supseteq \mathrm{H}_{x^{n+1}}$.

The following theorem is a crucial result in this paper.

Theorem 4.6: Let S be a semigroup. Then the following conditions are equivalent.

1) S is an intra-regular semigroup.

2) $\mathrm{H} \text{ ó } \mathrm{F} \prec \mathrm{H} \odot \mathrm{F}$ for every hesitant fuzzy left ideal H and every hesitant fuzzy right ideal F on S.

Proof: First assume that S is an intra-regular semigroup. Let H and F be any hesitant fuzzy left ideal and any hesitant fuzzy right ideal on S, respectively. Next, let x be any element of S. By assumption, there exist elements s_1 and s_2 of S such that $x = s_1 x^2 s_2$. Then we have,

$$\begin{aligned} (\mathrm{H} \odot \mathrm{F})_x &= \bigcup_{x=yz} \mathrm{H}_y \cap \mathrm{F}_z \\ &\supseteq \mathrm{H}_{s_1 x} \cap \mathrm{F}_{x s_2} \\ &\supseteq \mathrm{H}_x \cap \mathrm{F}_x \\ &= (\mathrm{H}_x \text{ ó } \mathrm{F})_x, \end{aligned}$$

which implies that $\mathrm{H} \text{ ó } \mathrm{F} \prec \mathrm{H} \odot \mathrm{F}$.

Conversely, assume that 2) holds. Let x be any element of S. Since $L[x] = x \cup Sx$ and $R[x] = x \cup xS$ are a left ideal and a right ideal on S, respectively, we have $\mathrm{C}_{L[x]}$ and $\mathrm{C}_{R[x]}$ are a hesitant fuzzy left ideal and a hesitant fuzzy right ideal on S, respectively. Then we have,

$$\begin{aligned} \left(\mathrm{C}_{L[x]R[x]}\right)_x &= \left(\mathrm{C}_{L[x]} \odot \mathrm{C}_{R[x]}\right)_x \\ &\supseteq \left(\mathrm{C}_{L[x]} \text{ ó } \mathrm{C}_{R[x]}\right)_x \\ &= \left(\mathrm{C}_{L[x] \cap R[x]}\right)_x \\ &= [0,1], \end{aligned}$$

which implies that $x \in L[x]R[x] = (x \cup Sx)(x \cup xS) = x^2 \cup x^2 S \cup Sx^2 \cup Sx^2 S$. Consequently, S is an intra-regular semigroup.

We now introduces the notion of hesitant fuzzy idempotent sets on semigroups.

Definition 4.7: A hesitant fuzzy set H on a semigroup S is called a **hesitant fuzzy idempotent** on S if $\mathrm{H} = \mathrm{H} \odot \mathrm{H}$.

Next, we characterize a regular and intra-regular semigroup in terms of hesitant fuzzy sets.

Theorem 4.8: Let S be a semigroup. Then the following conditions are equivalent.

1) S is regular and intra-regular.

2) Every hesitant fuzzy quasi-ideal on S is a hesitant fuzzy idempotent set.

3) $\mathrm{H} \text{ ó } \mathrm{F} \prec \mathrm{H} \odot \mathrm{F}$ for every hesitant fuzzy generalized bi-ideals H and F on S.

Proof: 1)$\Rightarrow$3). Assume that S is regular and intra-regular. Let H and F be any hesitant fuzzy generalized bi-ideals on S, and x any element of S. By assumption, there exist elements s_1, s_2 and s_3 such that $x = x s_1 x$ and $x = s_2 x^2 s_3$. Then we have,

$$\begin{aligned} x &= x s_1 x \\ &= x s_1 x s_1 x \\ &= x s_1 (s_2 x^2 s_3) s_1 x \end{aligned}$$

$$= (xs_1s_2x)(xs_3s_1x),$$

which means that

$$\begin{aligned}(H \odot F)_x &= \bigcup_{x=yz} H_y \cap F_z \\ &\supseteq H_{xs_1s_2x} \cap F_{xs_3s_1x} \\ &\supseteq H_x \cap F_x \\ &= (H \sqcap F)_x.\end{aligned}$$

Consequently, $H \sqcap F \prec H \odot F$.

It is obvious that, $3) \Rightarrow 2)$.

$2) \Rightarrow 1)$. Assume that 2) holds. Let Q be any quasi-ideal of S and x any element of S. Then we have, C_Q is a hesitant fuzzy quasi-ideal on S. By assumption, $(C_{QQ})_x = (C_Q \odot C_Q)_x = (C_Q)_x = [0,1]$, which implies that $x \in QQ$. Therefore, $Q = QQ$ and hence S is regular and intra-regular.

From Theorem 4.8 we can easily obtain the following theorem.

Theorem 4.9: Let S be a semigroup. Then the following conditions are equivalent.

1) Every hesitant fuzzy quasi-ideal on S is a hesitant fuzzy idempotent set.
2) Every hesitant fuzzy bi-ideal on S is a hesitant fuzzy idempotent set.
3) $H \sqcap F \prec H \odot F$ for every hesitant fuzzy quasi-ideals H and F on S.
4) $H \sqcap F \prec H \odot F$ for every hesitant fuzzy quasi-ideal H and every hesitant fuzzy bi-ideal F on S.
5) $H \sqcap F \prec H \odot F$ for every hesitant fuzzy quasi-ideal H and every hesitant fuzzy generalized bi-ideal F on S.
6) $H \sqcap F \prec H \odot F$ for every hesitant fuzzy bi-ideal H and every hesitant fuzzy quasi-ideal F on S.
7) $H \sqcap F \prec H \odot F$ for every hesitant fuzzy bi-ideals H and F on S.
8) $H \sqcap F \prec H \odot F$ for every hesitant fuzzy bi-ideal H and every hesitant fuzzy generalized bi-ideal F on S.
9) $H \sqcap F \prec H \odot F$ for every hesitant fuzzy generalized bi-ideal H and every hesitant fuzzy quasi-ideal F on S.
10) $H \sqcap F \prec H \odot F$ for every hesitant fuzzy generalized bi-ideal H and every hesitant fuzzy bi-ideal F on S.
11) $H \sqcap F \prec H \odot F$ for every hesitant fuzzy generalized bi-ideals H and F on S.

We now state another characterization theorem for regular and intra-regular semigroups.

Theorem 4.10: Let S be a semigroup. Then the following conditions are equivalent.

1) S is regular and intra-regular.
2) $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$ for every hesitant fuzzy right ideal H and every hesitant fuzzy left ideal F on S.
3) $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$ for every hesitant fuzzy generalized bi-ideals H and F on S.

Proof: $1) \Rightarrow 3)$. Assume that S is regular and intra-regular. Let H and F be any hesitant fuzzy generalized bi-ideals on S. It is easy to see that, $H \sqcap F \prec H \odot F$. Then we have, $H \sqcap F = F \sqcap H \prec F \odot H$, we can conclude that $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$.

It is obvious that, $3) \Rightarrow 2)$.

$2) \Rightarrow 1)$. Assume that 2) holds. Let H and F be any hesitant fuzzy right ideal and any hesitant fuzzy left ideal on S, respectively. By assumption, $H \sqcap F \prec (H \odot F) \sqcap (F \odot H) \prec F \odot H$ and $H \sqcap F \prec (H \odot F) \sqcap (F \odot H) \prec H \odot F$. It is easy to see that, $H \sqcap F = F \odot H$. Consequently, S is regular and intra-regular.

By using Theorem 4.10, we have the following theorem.

Theorem 4.11: Let S be a semigroup. Then the following conditions are equivalent.

1) $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$ for every hesitant fuzzy right ideal H and every hesitant fuzzy left ideal F on S.
2) $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$ for every hesitant fuzzy right ideal H and every hesitant fuzzy quasi-ideal F on S.
3) $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$ for every hesitant fuzzy right ideal H and every hesitant fuzzy bi-ideal F on S.
4) $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$ for every hesitant fuzzy right ideal H and every hesitant fuzzy generalized bi-ideal F on S.
5) $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$ for every hesitant fuzzy left ideal H and every hesitant fuzzy quasi-ideal F on S.
6) $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$ for every hesitant fuzzy left ideal H and every hesitant fuzzy bi-ideal F on S.
7) $H \sqcap F \prec (H \odot F) \sqcap (F \odot H)$ for every hesitant fuzzy left ideal H and every hesitant fuzzy generalized bi-ideal F on S.

8) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec (\mathrm{H} \odot \mathrm{F})\,\text{ó}\,(\mathrm{F} \odot \mathrm{H})$ for every hesitant fuzzy quasi-ideals H and F on S.

9) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec (\mathrm{H} \odot \mathrm{F})\,\text{ó}\,(\mathrm{F} \odot \mathrm{H})$ for every hesitant fuzzy quasi-ideal H and every hesitant fuzzy bi-ideal F on S.

10) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec (\mathrm{H} \odot \mathrm{F})\,\text{ó}\,(\mathrm{F} \odot \mathrm{H})$ for every hesitant fuzzy quasi-ideal H and every hesitant fuzzy generalized bi-ideal F on S.

11) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec (\mathrm{H} \odot \mathrm{F})\,\text{ó}\,(\mathrm{F} \odot \mathrm{H})$ for every hesitant fuzzy bi-ideals H and F on S.

12) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec (\mathrm{H} \odot \mathrm{F})\,\text{ó}\,(\mathrm{F} \odot \mathrm{H})$ for every hesitant fuzzy bi-ideal H and every hesitant fuzzy generalized bi-ideal F on S.

13) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec (\mathrm{H} \odot \mathrm{F})\,\text{ó}\,(\mathrm{F} \odot \mathrm{H})$ for every hesitant fuzzy generalized bi-ideals H and on S.

We now state another characterization theorem for is regular and intra-regular semigroups.

Theorem 4.12: Let S be a semigroup. Then the following conditions are equivalent.

1) S is regular and intra-regular.

2) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy quasi-ideal H and every hesitant fuzzy right ideal F on S.

3) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy generalized bi-ideals H and F on S.

Proof: 1)$\Rightarrow$3). Assume that S is regular and intra-regular. Let H and F be any hesitant fuzzy generalized bi-ideals on S and x any element of S. Since S is regular and intra-regular, there exist elements s_1, s_2 and s_3 of S such that $x = xs_1x$ and $x = s_2x^2s_3$. Then we have,

$$\begin{aligned} x &= xs_1x \\ &= xs_1xs_1xs_1x \\ &= xs_1(s_2x^2s_3)s_1(s_2x^2s_3)s_1x \\ &= (xs_1s_2x)(xs_3s_1s_2x)(xs_3s_1x). \end{aligned}$$

By assumption,

$$\begin{aligned} (\mathrm{H} \odot \mathrm{F} \odot \mathrm{H})_x &= \bigcup_{x=yz} \mathrm{H}_y \cap (\mathrm{F} \odot \mathrm{H})_z \\ &= \bigcup_{x=yz} \left(\mathrm{H}_y \cap \left(\bigcup_{z=ab} \mathrm{F}_a \cap \mathrm{H}_b \right) \right) \\ &\supseteq \mathrm{H}_{xs_1s_2x} \cap \mathrm{F}_{xs_3s_1s_2x} \cap \mathrm{H}_{xs_3s_1x} \\ &\supseteq \mathrm{H}_x \cap \mathrm{F}_x \cap \mathrm{H}_x \\ &= (\mathrm{H}\,\text{ó}\,\mathrm{F})_x, \end{aligned}$$

which means $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$.

It is obvious that, 3)$\Rightarrow$2).

3)$\Rightarrow$1). Assume that 3) holds. Let H and F be any hesitant fuzzy quasi-ideal and any hesitant fuzzy right ideal on S, respectively. Since Y is a hesitant fuzzy right ideal on S, we have $\mathrm{H} = \mathrm{H}\,\text{ó}\,\mathrm{Y} = \mathrm{H} \odot \mathrm{Y} \odot \mathrm{H}$, which implies that S is regular. By assumption,

$$\begin{aligned} \mathrm{H}\,\text{ó}\,\mathrm{F} &\prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H} \\ &\prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{Y} \\ &\prec \mathrm{H} \odot \mathrm{F}. \end{aligned}$$

Consequently, S is regular and intra-regular.

From Theorem 4.12 we can easily obtain the following theorem.

Theorem 4.13: Let S be a semigroup. Then the following conditions are equivalent.

1) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy quasi-ideal H and every hesitant fuzzy left ideal F on S.

2) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy quasi-ideal H and every hesitant fuzzy right ideal F on S.

3) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy quasi-ideals H and F on S.

4) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy quasi-ideal H and every hesitant fuzzy bi-ideal F on S.

5) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy quasi-ideal H and every hesitant fuzzy generalized bi-ideal F on S.

6) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy bi-ideal H and every hesitant fuzzy left ideal F on S.

7) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy bi-ideal H and every hesitant fuzzy right ideal F on S.

8) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy bi-ideal H and every hesitant fuzzy quasi-ideal F on S.

9) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy bi-ideals H and F on S.

10) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy bi-ideal H and every hesitant fuzzy generalized bi-ideal F on S.

11) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy generalized bi-ideal H and every hesitant fuzzy left ideal F on S.

12) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy generalized bi-ideal H and every hesitant fuzzy right ideal F on S.

13) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy generalized bi-ideal H and every hesitant fuzzy quasi-ideal F on S.

14) $\mathrm{H}\,\text{ó}\,\mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy generalized bi-ideal H and every hesitant fuzzy bi-ideal F on S.

15) $\mathrm{H} \text{ ó } \mathrm{F} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{H}$ for every hesitant fuzzy generalized bi-ideals H and F on S.

Now we characterize a regular and intra-regular semigroup in terms of hesitant fuzzy ideals.

Theorem 4.14: Let S be a semigroup. Then the following conditions are equivalent.

1) S is regular and intra-regular.

2) $\mathrm{H} \text{ ó } \mathrm{F} \text{ ó } \mathrm{G} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{G}$ for every hesitant fuzzy quasi-ideal H, every hesitant fuzzy right ideal F and every hesitant fuzzy left ideal G on S.

3) $\mathrm{H} \text{ ó } \mathrm{F} \text{ ó } \mathrm{G} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{G}$ for every hesitant fuzzy generalized bi-ideal H, every hesitant fuzzy right ideal F and every hesitant fuzzy left ideal G on S.

Proof: $1) \Rightarrow 3)$. Assume that S is regular and intra-regular. Let H, F and G be any hesitant fuzzy generalized bi-ideal, any hesitant fuzzy right ideal and any hesitant fuzzy left ideal on S, respectively. Next, let x any element of S. Since S is regular and intra-regular, there exist elements s_1, s_2 and s_3 of S such that $x = xs_1x$ and $x = s_2x^2s_3$. Thus we have, $x = (xs_1s_2x)(xs_3s_1s_2x)(xs_3s_1x)$. By assumption,

$$
\begin{aligned}
(\mathrm{H} \odot \mathrm{F} \odot \mathrm{G})_x &= \bigcup_{x=yz} \mathrm{H}_y \cap (\mathrm{F} \odot \mathrm{G})_z \\
&= \bigcup_{x=yz} \left(\mathrm{H}_y \cap \left(\bigcup_{z=ab} \mathrm{F}_a \cap \mathrm{G}_b \right) \right) \\
&\supseteq \mathrm{H}_{xs_1s_2x} \cap \mathrm{F}_{xs_3s_1s_2x} \cap \mathrm{G}_{xs_3s_1x} \\
&\supseteq \mathrm{H}_x \cap \mathrm{F}_x \cap \mathrm{G}_x \\
&= (\mathrm{H} \text{ ó } \mathrm{F} \text{ ó } \mathrm{G})_x,
\end{aligned}
$$

which implies that $\mathrm{H} \text{ ó } \mathrm{F} \text{ ó } \mathrm{G} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{G}$.

It is obvious that, $3) \Rightarrow 2)$.

$2) \Rightarrow 1)$. Assume that 2) holds. Let H, F and G be any hesitant fuzzy quasi-ideal, any hesitant fuzzy right ideal and any hesitant fuzzy left ideal on S, respectively. By assumption,

$$
\begin{aligned}
\mathrm{H} \text{ ó } \mathrm{F} &= \mathrm{H} \text{ ó } \mathrm{F} \text{ ó } \mathrm{Y} \\
&\prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{Y} \\
&\prec \mathrm{H} \odot \mathrm{F}.
\end{aligned}
$$

On the other hand, since Y is a hesitant fuzzy right ideal on S, we have

$$
\begin{aligned}
\mathrm{F} \text{ ó } \mathrm{H} &= \mathrm{F} \text{ ó } \mathrm{Y} \text{ ó } \mathrm{H} \\
&= \mathrm{F} \odot \mathrm{Y} \odot \mathrm{H} \\
&\prec \mathrm{F} \odot \mathrm{H},
\end{aligned}
$$

which means that $\mathrm{F} \text{ ó } \mathrm{H} = \mathrm{F} \odot \mathrm{H}$. Consequently, S is regular and intra-regular.

The following theorem can be easily seen.

Theorem 4.15: Let S be a semigroup. Then the following conditions are equivalent.

1) $\mathrm{H} \text{ ó } \mathrm{F} \text{ ó } \mathrm{G} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{G}$ for every hesitant fuzzy quasi-ideal H, every hesitant fuzzy right ideal F and every hesitant fuzzy left ideal G on S.

2) $\mathrm{H} \text{ ó } \mathrm{F} \text{ ó } \mathrm{G} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{G}$ for every hesitant fuzzy bi-ideal H, every hesitant fuzzy right ideal F and every hesitant fuzzy left ideal G on S.

3) $\mathrm{H} \text{ ó } \mathrm{F} \text{ ó } \mathrm{G} \prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{G}$ for every hesitant fuzzy generalized bi-ideal H, every hesitant fuzzy right ideal F and every hesitant fuzzy left ideal G on S.

V. Hesitant Fuzzy Sets on Semisimple Semigroups

In this section, we characterize left (right) quasi-regular, quasi-regular intra-regular, left quasi-regular and semisimple semigroups by the properties of their hesitant fuzzy ideals. Also we prove that every hesitant fuzzy two-sided ideal on a semigroup S is a hesitant fuzzy interior ideal if and only if S is a semisimple semigroup.

Recall that a semigroup S is called **left (right) quasi-regular** if every left (right) ideal of S is hesitant fuzzy idempotent, and is called **quasi-regular** if every left ideal and every right ideal of S is hesitant fuzzy idempotent.

It is easy to see that, a semigroup S is left (right) quasi-regular if and only if $x \in SxSx (x \in xSxS)$, that is, there exist elements s_1 and s_2 in S such that $x = s_1xs_2x$ $(x = xs_1xs_2)$.

Now we characterize a left (right) quasi-regular semigroup in terms of hesitant fuzzy ideals.

Theorem 5.1: Let S be a semigroup. Then the following conditions are equivalent.

1) S is left (right) quasi-regular.

2) Every hesitant fuzzy left (right) ideal is hesitant fuzzy idempotent.

Proof: First assume that S is a left quasi-regular semigroup. Let H be any hesitant fuzzy left ideal on S and x any element of S. By assumption, there exist elements s_1 and s_2 of S such that $x = s_1xs_2x$. Then we have,

$$
\begin{aligned}
(\mathrm{H} \odot \mathrm{H})_x &= \bigcup_{x=yz} \mathrm{H}_y \cap \mathrm{H}_z \\
&\supseteq \mathrm{H}_{s_1x} \cap \mathrm{H}_{s_2x} \\
&\supseteq \mathrm{H}_x \cap \mathrm{H}_x \\
&= \mathrm{H}_x,
\end{aligned}
$$

which means that $\mathrm{H} \prec \mathrm{H} \odot \mathrm{H}$. On the other hand, since H is a hesitant fuzzy left ideal on S, we have $\mathrm{H} \odot \mathrm{H} \prec \mathrm{H}$. Therefore, $\mathrm{H} \odot \mathrm{H} = \mathrm{H}$ and hence H is a hesitant fuzzy idempotent set on S.

Conversely, assume that every hesitant fuzzy left ideal on S is hesitant fuzzy idempotent. Let x be any element of S.

Since $L[x]=x\cup Sx$ is a left ideal of S, we have $\mathrm{C}_{L[x]}$ is a hesitant fuzzy left ideal on S. By assumption,

$$\begin{aligned}\left(\mathrm{C}_{L[x]L[x]}\right)_x &= \left(\mathrm{C}_{L[x]}\odot \mathrm{C}_{L[x]}\right)_x\\ &= \left(\mathrm{C}_{L[x]}\right)_x\\ &= [0,1],\end{aligned}$$

which means that $x\in L[x]L[x]=(x\cup Sx)(x\cup Sx)=$ $x^2\cup xSx\cup Sx^2\cup SxSx.$ Consequently, S is left quasi-regular.

Now we characterize a quasi-regular semigroup in terms of hesitant fuzzy quasi-ideals.

Theorem 5.2: Let S be a semigroup. Then the following conditions are equivalent.

1) S is quasi-regular.

2) $\mathrm{H}=(\mathrm{H}\odot \mathrm{Y})^2$ ó $(\mathrm{Y}\odot \mathrm{H})^2$ for every hesitant fuzzy quasi-ideal H on S.

Proof: First assume that S is a left quasi-regular semigroup. It follows from Theorem 5.1 that the hesitant fuzzy right ideal $\mathrm{H}\odot \mathrm{Y}$ is a hesitant fuzzy idempotent set on S. Similarly, we can prove that, $\mathrm{Y}\odot \mathrm{H}$ is a hesitant fuzzy idempotent set on S. Then we have,

$$\begin{aligned}(\mathrm{H}\odot \mathrm{Y})^2 \text{ ó } (\mathrm{Y}\odot \mathrm{H})^2 &= (\mathrm{H}\odot \mathrm{Y}) \text{ ó } (\mathrm{Y}\odot \mathrm{H})\\ &\prec \mathrm{H}.\end{aligned}$$

On the other hand, let x be any element of S. By assumption, there exist elements s_1, s_2 on S such that $x=xs_1xs_2$, which implies that

$$\begin{aligned}(\mathrm{H}\odot \mathrm{Y})^2_x &= \bigcup_{x=yz}(\mathrm{H}\odot \mathrm{Y})_y\cap(\mathrm{H}\odot \mathrm{Y})_z\\ &\supseteq (\mathrm{H}\odot \mathrm{Y})_y\cap(\mathrm{H}\odot \mathrm{Y})_z\\ &= \left(\bigcup_{y=ab}\mathrm{H}_a\cap \mathrm{Y}_b\right)\cap\left(\bigcup_{z=cd}\mathrm{H}_c\cap \mathrm{Y}_d\right)\\ &\supseteq \mathrm{H}_x\cap \mathrm{Y}_{s_1}\cap \mathrm{H}_x\cap \mathrm{Y}_{s_2}\\ &= \mathrm{H}_x.\end{aligned}$$

This implies that, $\mathrm{H}\prec(\mathrm{H}\odot \mathrm{Y})^2$. Similarly, we can prove that, $\mathrm{H}\prec(\mathrm{Y}\odot \mathrm{H})^2$. Therefore,

$$\mathrm{H}\prec(\mathrm{H}\odot \mathrm{Y})^2 \text{ ó } (\mathrm{Y}\odot \mathrm{H})^2$$

and hence $\mathrm{H}=(\mathrm{H}\odot \mathrm{Y})^2$ ó $(\mathrm{Y}\odot \mathrm{H})^2$.

Conversely, assume that 2) holds. Let H be any hesitant fuzzy right ideal on S. Since H is a hesitant fuzzy quasi-ideal on S, we have

$$\begin{aligned}\mathrm{H} &= (\mathrm{H}\odot \mathrm{Y})^2 \text{ ó } (\mathrm{Y}\odot \mathrm{H})^2\\ &\prec (\mathrm{H}\odot \mathrm{Y})^2\\ &\prec \mathrm{H}^2\\ &\prec \mathrm{H}\odot \mathrm{Y}\\ &= \mathrm{H},\end{aligned}$$

which means that $\mathrm{H}=\mathrm{H}^2$. Hence, it follows from Theorem 5.1 that S is right quasi-regular. Similarly, we can prove that, S is left quasi-regular. Consequently, S is quasi-regular.

Now we characterize a intra-regular and left quasi-regular semigroup in terms of hesitant fuzzy ideals.

Theorem 5.3: Let S be a semigroup. Then the following conditions are equivalent.

1) S is both intra-regular and left quasi-regular.

2) H ó F ó $\mathrm{G}\prec \mathrm{H}\odot \mathrm{F}\odot \mathrm{G}$ for every hesitant fuzzy left ideal H, for every hesitant fuzzy right ideal F and for every hesitant fuzzy quasi-ideal G on S.

3) H ó F ó $\mathrm{G}\prec \mathrm{H}\odot \mathrm{F}\odot \mathrm{G}$ for every hesitant fuzzy left ideal H, for every hesitant fuzzy right ideal F and for every hesitant fuzzy generalized bi-ideal G on S..

Proof: 1)$\Rightarrow$3). Assume that S is both intra-regular and left quasi-regular. Let H, F and G be any hesitant fuzzy left ideal, any hesitant fuzzy right ideal and any hesitant fuzzy generalized bi-ideal on S, respectively. Next, let x any element of S. By assumption, there exist elements s_1, s_2, s_3 and s_4 of S such that $x=s_1x^2s_2$ and $x=s_3xs_4x$. Then we have,

$$x=s_3xs_4x=s_3(s_1x^2s_2)s_4x=(s_3s_1x)((xs_2s_4)x)$$

which means that

$$\begin{aligned}(\mathrm{H}\odot \mathrm{F}\odot \mathrm{G})_x &= \bigcup_{x=yz}\mathrm{H}_y\cap(\mathrm{F}\odot \mathrm{G})_z\\ &\supseteq \mathrm{H}_{s_3s_1x}\cap(\mathrm{F}\odot \mathrm{G})_z\\ &\supseteq \mathrm{H}_x\cap\left(\bigcup_{z=ab}\mathrm{F}_a\cap \mathrm{G}_b\right)\\ &\supseteq \mathrm{H}_x\cap \mathrm{F}_{xs_2s_4}\cap \mathrm{G}_x\\ &\supseteq \mathrm{H}_x\cap \mathrm{F}_x\cap \mathrm{G}_x\\ &= (\mathrm{H} \text{ ó } \mathrm{F} \text{ ó } \mathrm{G})_x.\end{aligned}$$

Consequently, H ó F ó $\mathrm{G}\prec \mathrm{H}\odot \mathrm{F}\odot \mathrm{G}$.

It is obvious that, 3)$\Rightarrow$2).

2)$\Rightarrow$1). Assume that 2) holds. Let H, F and G be any hesitant fuzzy left ideal, any hesitant fuzzy right ideal and any hesitant fuzzy quasi-ideal on S, respectively. Then, since Y is a hesitant fuzzy right ideal on S, we have

$$\begin{aligned}\mathrm{H} &= \mathrm{H} \text{ ó } \mathrm{Y} \text{ ó } \mathrm{H}\\ &\prec \mathrm{H}\odot \mathrm{Y}\odot \mathrm{H}\\ &\prec \mathrm{H}\odot \mathrm{H}\\ &\prec \mathrm{H}\odot \mathrm{Y}\\ &\prec \mathrm{H},\end{aligned}$$

which implies that $\mathrm{H} = \mathrm{H} \odot \mathrm{H}$. By assumption, H ó F $=$ H ó Y ó F $= \mathrm{H} \odot \mathrm{Y} \odot \mathrm{F} \prec \mathrm{H} \odot \mathrm{F}$. Consequently, S is both intra-regular and left quasi-regular.

From Theorem 5.3 we can easily obtain the following theorem.

Theorem 5.4: Let S be a semigroup. Then the following conditions are equivalent.

1) H ó F ó G $\prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{G}$ for every hesitant fuzzy left ideal H, for every hesitant fuzzy right ideal F and for every hesitant fuzzy quasi-ideal G on S.

2) H ó F ó G $\prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{G}$ for every hesitant fuzzy left ideal H, for every hesitant fuzzy right ideal F and for every hesitant fuzzy bi-ideal G on S.

3) H ó F ó G $\prec \mathrm{H} \odot \mathrm{F} \odot \mathrm{G}$ for every hesitant fuzzy left ideal H, for every hesitant fuzzy right ideal F and for every hesitant fuzzy generalized bi-ideal G on S.

Now, we proved that every hesitant fuzzy two-sided ideal on a semigroup S is a hesitant fuzzy interior ideal if and only if S is a semisimple semigroup.

Theorem 5.5: Let S be a semisimple semigroup. Then the following conditions are equivalent.

1) H is a hesitant fuzzy interior ideal on S.

2) H is a hesitant fuzzy two-sided ideal on S.

Proof: First assume that H is a hesitant fuzzy interior ideal on S. Let x and y be any elements of S. Since S is a semisimple semigroup, there exist elements s_1, s_2 and s_3 of S such that $x = s_1 x s_2 x s_3$. By assumption, $\mathrm{H}_{xy} = \mathrm{H}_{(s_1 x s_2 x s_3)y} = \mathrm{H}_{(s_1 x s_2)x(s_3 y)} \supseteq \mathrm{H}_x$, which means that H is a hesitant fuzzy right ideal on S. Similarly, we can prove that, H is a hesitant fuzzy left ideal on S. Consequently, H is a hesitant fuzzy two-sided ideal on S.

It is obvious that, $2) \Rightarrow 1)$.

Now we characterize a semisimple semigroup in terms of hesitant fuzzy ideals.

Theorem 5.6: Let S be a semigroup. Then the following conditions are equivalent.

1) S is semisimple.

2) Every hesitant fuzzy two-sided ideal on S is a hesitant fuzzy idempotent set.

3) H ó F $= \mathrm{H} \odot \mathrm{F}$ for every hesitant fuzzy interior ideals H and F on S.

Proof: $1) \Rightarrow 3)$. Assume that S is a semisimple semigroup. Let H and F be any hesitant fuzzy interior ideals on S. Then by Theorem 5.5, H is a hesitant fuzzy two-sided ideal on S. Since Y is a hesitant fuzzy interior ideal on S, we have $\mathrm{H} \odot \mathrm{F} \prec \mathrm{H} \odot \mathrm{Y} \prec \mathrm{H}$ and $\mathrm{H} \odot \mathrm{F} \prec \mathrm{Y} \odot \mathrm{F} \prec \mathrm{F}$, which means that $\mathrm{H} \odot \mathrm{F} \prec$ H ó F. Next, let x be any element of S. By assumption, there exist elements s_1, s_2, s_3 and s_4 in S such that $x = (s_1 x s_2)(s_3 x s_4)$. Thus we have,

$$\begin{aligned}(\mathrm{H} \odot \mathrm{H})_x &= \bigcup_{x=yz} \mathrm{H}_y \cap \mathrm{F}_z \\ &\supseteq \mathrm{H}_{s_1 x s_2} \cap \mathrm{F}_{s_3 x s_4} \\ &\supseteq \mathrm{H}_x \cap \mathrm{F}_x \\ &= (\text{H ó F})_x,\end{aligned}$$

it is easy to see that, H ó F $\prec \mathrm{H} \odot \mathrm{F}$. Consequently, H ó F $= \mathrm{H} \odot \mathrm{F}$.

It is obvious that, $3) \Rightarrow 2)$.

$2) \Rightarrow 1)$. Assume that 2) holds. Let x be any element in S. Since $J[x] = x \cup xS \cup Sx \cup SxS$ is a two-sided ideal of S, we have $\mathrm{C}_{J[x]}$ is a hesitant fuzzy two-sided ideal on S. Then we have, $\left(\mathrm{C}_{J[x]J[x]}\right)_x = \left(\mathrm{C}_{J[x]} \odot \mathrm{C}_{J[x]}\right)_x = \left(\mathrm{C}_{L[x]}\right)_x = [0,1]$, which means that $x \in J[x]J[x] =$

$(x \cup xS \cup Sx \cup SxS)(x \cup xS \cup Sx \cup SxS) = x^2$
$\cup x^2 S \cup xSxS \cup xSx \cup xSxS \cup xSSxS \cup SxxS \cup$
$SxSx \cup SxSxS \cup SxSxSxSxS \cup SxSSx \cup SxSSxS$.

Consequently, S is a semisimple semigroup.

The following theorem can be easily seen.

Theorem 5.7: Let S be a semigroup. Then the following conditions are equivalent.

1) Every hesitant fuzzy two-sided ideal on S is a hesitant fuzzy idempotent set.

2) Every hesitant fuzzy interior ideal on S is a hesitant fuzzy idempotent set.

3) H ó F $= \mathrm{H} \odot \mathrm{F}$ for every hesitant fuzzy two-sided ideals H and F on S..

4) H ó F $= \mathrm{H} \odot \mathrm{F}$ for every hesitant fuzzy two-sided ideal H and every hesitant fuzzy interior ideal F on S.

5) H ó F $= \mathrm{H} \odot \mathrm{F}$ for every hesitant fuzzy interior ideal H and every hesitant fuzzy two-sided ideal F on S.

6) H ó F $= \mathrm{H} \odot \mathrm{F}$ for every hesitant fuzzy interior ideals H and F on S.

7) $\mathrm{H} \odot \mathrm{F} = \mathrm{F} \odot \mathrm{H}$ and $\mathrm{H} \odot \mathrm{H} = \mathrm{H}$ for every hesitant fuzzy two-sided ideals H and F on S.

8) $\mathrm{H} \odot \mathrm{F} = \mathrm{F} \odot \mathrm{H}$ and $\mathrm{H} \odot \mathrm{H} = \mathrm{H}$ for every hesitant fuzzy interior ideals H and F on S.

Let x be an element of a semigroup S. Then we write $R_x = \{s \in S : xsx = x\}$. Next, let H be a hesitant fuzzy semigroup of a semigroup S. For every $x \in S$, there exists an element $s \in R_x$ such that $\mathrm{H}_s \supseteq \mathrm{H}_x$ if $\mathrm{H}_x \neq \varnothing$, then H is called a **hesitant fuzzy regular subsemigroup** on S.

Theorem 5.8: Let S be a semigroup. Then the following conditions are equivalent.

1) A is a regular subsemigroup of S.
2) C_A is a hesitant fuzzy regular subsemigroup on S.

Proof: First assume that A is a regular subsemigroup of S. Obviously C_A is a hesitant fuzzy semigroup of S. Let x be an element of S. If $\left(\mathrm{C}_A\right)_x \neq \varnothing$, i.e. $\left(\mathrm{C}_A\right)_x = [0,1]$, then $x \in A$, and furthermore from the regularity of A we know that there exists an element s of R_x such that $s \in A$, i.e., $\left(\mathrm{C}_A\right)_s = [0,1]$. Therefore, $\mathrm{H}_s \supseteq \mathrm{H}_x$ and hence C_A is a hesitant fuzzy regular subsemigroup on S.

Conversely, assume that C_A is a hesitant fuzzy regular subsemigroup on S. It is easy to see that A is a subsemigroup of S. Let x be an element of S. In addition, if $\mathrm{H}_s \supseteq \mathrm{H}_x = [0,1]$, then $\mathrm{H}_s = [0,1]$, which implies that $s \in A$. Hence A is a regular subsemigroup of S.

VI. Conclusion

In the structural theory of hesitant fuzzy algebraic systems, hesitant fuzzy sets with special properties always play an important role. In this work, we focus on a particular topic related to hesitant fuzzy algebra, which develops fuzzy versions of semigroups. Specifically, we study the theory of left (right) regular, completely regular, intra-regular, regular, left (right) quasi-regular, quasi-regular intra-regular, left quasi-regular and semisimple semigroups. We introduce the notions of hesitant fuzzy semiprime sets and hesitant fuzzy idempotent sets on semigroups and some properties of them are obtained. We also characterized left (right) regular, completely regular, intra-regular, regular, left (right) quasi-regular, quasi-regular intra-regular, left quasi-regular and semisimple semigroups by the properties of their hesitant fuzzy ideals. At the end we prove that every hesitant fuzzy two-sided ideal on a semigroup S is a hesitant fuzzy interior ideal if and only if S is a semisimple semigroup.

References

[1] M.P. Drazin, “A partial order in completely regular semigroups,” *Journal of Algebra*, vol. 98, no. 2, pp. 362-374, 1986.
[2] Y. Hao, and Y. Luo, “Directed graphs and combinatorial properties of completely regular semigroups,’ Semigroup Forum , vol. 81, pp. 524 - 530, 2010.
[3] Y. He, “Weakly J -covered completely regular semigroups,” Semigroup Forum, vol. 66, pp. 97 - 109, 2003.
[4] P. Julath and M. Siripitukdet, “Some characterizations of fuzzy bi-ideals and fuzzy quasi-ideals of semigroups,” *Engineering Letters*, vol. 25, no. 2, pp. 160-166, 2017.
[5] J. Kavikumar, A.B. Khamis and R.B. Roslan, “Bipolar-valued fuzzy finite switchboard state machines,” *Proceedings of the World Congress on Engineering and Computer Science*, vol. 1, pp. 571-576, 2012.
[6] R.A.R. Monzo, and B.M. Vernikov, “Identities determining varieties of semigroups with completely regular power,” *Semigroup Forum*, vol. 82, pp. 384-388, 2011.
[7] M. Petrich, “Certain relations on the congruence lattice of a completely regular semigroup,” Semigroup Forum, vol. 59, pp. 264 - 281, 1999.
[8] M. Petrich, “Certain partial orders on semigroups,” *Czechoslovak Mathematical Journal*, vol. 51, no. 126, pp. 415-432, 2001.
[9] P.G. Trotter, “Completely regular semigroups,” *Semigroup Forum*, vol. 64, pp. 333-336, 2002.
[10] B.M. Vernikov, “Completely regular semigroup varieties whose free objects have weakly permutable fully Invariant congruences,” *Semigroup Forum*, vol. 68, pp. 154-158, 2004.
[11] L.G. Xin, W.Z. Pan, and L. Yun, “The lattice generated by some subvarieties of completely regular semigroups,” *Science in China Series A: Mathematics*, vol. 51, no. 12, pp. 2343-2352, 2008.
[12] H. Yang, “Two operators on the lattice of completely regular semigroup varieties,” *Semigroup Forum*, vol. 64, pp. 397-424, 2002.
[13] H. Yu, Z. Wang, T. Wu, and M. Ye, “Classification of some τ -congruence-free completely regular semigroups,” *Semigroup Forum*, vol. 84, pp. 308-322, 2012.
[14] J. Zhang, and R. Shen, “Some relations on completely regular semigroups,” *Semigroup Forum*, vol. 79, pp. 119-127, 2009.
[15] K.S. Ajan, and F.J. Pastijn, “Finitely presented groups and completely regular semigroups,” *Journal of Pure and Applied Algebra*, vol. 108, pp. 219-230, 1996.
[16] V. Gould, and P. Smith, “Orders and straight left orders in completely regular semigroups,” *Monatsh. Math.*, vol. 131, pp. 193-214, 2000.
[17] G. Mashevitzky, “Unification types of completely regular semigroups,” *Journal of Automated Reasoning*, vol. 29, pp. 171-182, 2002.
[18] I. Dolinka, “On free spectra of finite completely regular semigroups and monoids,” *Journal of Pure and Applied Algebra*, vol. 213, pp. 1979-1990, 2009.
[19] A. Khan, M. Shabir, and Y.B. Jun, “Intuitionistic fuzzy quasi-ideals of ordered semigroups,” *Russian Mathematics*, vol. 54, no. 12, pp. 59-71, 2010.
[20] A. Khan, Y.B. Jun, N.H. Sarmin, and F.M. Khan, “Ordered semigroups characterized by $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideals,” *Neural Comput & Applic*, vol. 21, no. 1, pp. S121-S132, 2012.
[21] S.K. Maity, and R. Ghosh, “On quasi completely regular semirings,” *Semigroup Forum*, vol. 89, pp. 422-430, 2014.
[22] N. Kehayopulu, and M. Tsingelis, “Characterization of some types of ordered semigroups in terms of fuzzy sets,” *Lobachevskii Journal of Mathematics*, vol. 29, no.1, pp. 14-20, 2008.
[23] M. Khan, B. Davvaz, and T. Aziz, “Some characterizations of intra-regular Abel Grassmann's groupoids by their generalized fuzzy quasi-ideals,” *Afr. Mat.*, vol. 26, pp. 899-912, 2015.
[24] A. Khan, R. Khan, and Y.B. Jun, “Uni-soft structure applied to ordered semigroups,” *Soft Comput*, vol. 21, pp. 1021-1030, 2017.
[25] A. Khan, N.H. Sarmin, B. Davvaz, and F.M. Khan, “New types of fuzzy bi-ideals in ordered semigroups,” *Neural Comput & Applic*, vol. 21, pp. S295-S305, 2012.
[26] N.Yaqoob, R. Chinram, A. Ghareeb, and M. Aslam, “Left almost semigroups characterized by their interval valued fuzzy ideals,” *Afr. Mat.*, vol. 24, pp. 231-245, 2013.
[27] F. Yousafzai, N. Yaqoob, and A. Zeb, “On generalized fuzzy ideals of ordered AG-groupoids,” *Int. J. Mach. Learn. & Cyber.*, vol. 7, pp. 995-1004, 2016.
[28] N. Kehayopulu, and M. Tsingelis, “Intra-regular ordered semigroups in terms of fuzzy sets,” *Lobachevskii Journal of Mathematics*, vol. 30, no.1, pp. 23-29, 2009.
[29] A. Khan, Y.B. Jun, and M.Z. Abbas, “Characterizations of ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy interior ideals,” *Neural Comput & Applic*, vol. 21, pp. 433-440, 2012.
[30] B. Li, and Y. Feng, “Intuitionistic (λ, μ) -fuzzy sets in Γ -semigroups,” Journal of Inequalities and Applications, vol. 107, pp. 1 - 9, 2013.
[31] S. Kar, P. Sarkar, and V.L. Fotea, “On some interval-valued fuzzy hyperideals of semihypergroupss,” *Afr. Mat.*, vol. 26, pp. 1171-1186, 2015.
[32] F. Yousafzai, W. Khan, A. Khan and K. Hila, “I–V fuzzy set theory applied on po-LA-semigroups,” *Afr. Mat.*, vol. 27, pp. 23-35, 2016.
[33] A. Ali, M. Khan, and F.G. Shi, “On fuzzy soft intra-regular Abel-Grassmann's groupoids,” *Afr. Mat.*, vol. 28, pp. 171-187, 2017.
[34] M. Khan, S. Anis, M. Faisal, and K.P. Shum, “Intra-regular Abel Grassmann's Groupoids determined by $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sets,” *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, vol. 88, no. 1, pp. 81-87, 2018.

[35] Y.B. Jun, K.J. Lee, and S.Z. Song, "Hesitant fuzzy bi-ideals in semigroups," Commun. Korean Math. Soc., vol. 30, no. 3, pp. 143 - 154, 2015.
[36] A. Ali, M. Khan, and F.G. Shi, "Hesitant fuzzy ideals in Abel-Grassmann's groupoids," *Italian Journal of Pure and Applied Mathematics*, vol. 35, pp. 537-556, 2015.
[37] Y.B. Jun, K.J. Lee, and C.H. Park, "Hesitant fuzzy semigroups with two frontiers," *Commun. Korean Math. Soc.*, vol. 31, no. 1, pp. 17-25, 2016.
[38] Y.B. Jun, K.J. Lee, and C.H. Park, "Hesitant fuzzy ideals in semigroups with two frontiers," *Journal of Basic and Applied Engineering Research*, vol. 4, no. 6, pp. 385-388, 2017.
[39] M.Y. Abbasi, A.F. Talee, S.A. Khan, and K. Hila, "A hesitant fuzzy set approach to ideal theory in Γ-semigroups," *Advances in Fuzzy Systems*, vol. 2018, pp. article ID 5738024, 6 pages, 2018.
[40] K. Zhu, J. Wang, and Y. Yang, "A new approach to rough lattices and rough fuzzy lattices based on fuzzy ideals," *IAENG International Journal of Applied Mathematics*, vol. 49, no.4, pp. 408-414, 2019.