A New GM (1, 1) Model Based on Piecewise Rational Quadratic Monotonicity-Preserving Interpolation Spline

Wenqing Chen and Yuanpeng Zhu

Abstract—In the classical GM (1, 1) model, in order to reduce the randomness of the original nonnegative sequence, the original nonnegative sequence is cumulatively generated so as to obtain a monotone increasing 1-AGO sequence. The method of background value construction will directly affect the accuracy and applicability of the model. Therefore, the reconstruction of the model background value has great significance to improve its matching and prediction precision. In order to improve the GM (1, 1) model, we provide a more logical formula for calculating background value, which is based on a $C^1$ monotonicity-preserving piecewise rational quadratic interpolation spline, and thereby establishing a new GM (1, 1) model. Numerical examples show that the new GM (1, 1) model is more effective and accurate compared with the classical GM (1, 1) model.

Index Terms—Background value, GM(1, 1) model, Grey theory, Monotonicity-Preserving interpolation spline

I. INTRODUCTION

Let an original non-negative and uniformly-spaced sequence be
\[ X^{(0)} = \{ x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n) \} \]
(1)

The method of cumulative generation of the original sequence is the main idea of the classical grey prediction GM (1,1) model proposed by Deng in [1], [2], so as to reduce the randomization of the original data and obtain a significantly monotonicity increasing 1-AGO sequence \( X^{(1)} \). Then a first order grad prediction differential equation on the sequence \( X^{(1)} \) is established. And the differential equation is numerically solved by least square method to estimate the parameters. Finally, the original data is simulated and predicted by using the inverse cumulative generation operation.

The 1-AGO sequence \( X^{(1)} \) is given as follows
\[ X^{(1)} = \{ x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n) \} \]
(2)

where
\[ x^{(1)}(k) = \sum_{i=0}^{k} x^{(0)}(k) = x^{(1)}(k-1) + x^{(0)}(k), \quad k = 1, 2, \ldots, n \]
(3)

From Eq. (3), the 1-AGO sequence \( X^{(1)} \) has the feature of monotonicity-increasing. Suppose that \( x^{(1)}(t) \) satisfies the following first order grad forecasting differential equation
\[ \frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b, \]
(4)

where the grey developmental coefficient \( a \) and the grey control parameter \( b \) are the parameters in the model to be estimated.

The solution of the above differential equation with the initial condition \( X^{(1)}(1) = X^{(1)}(1) \) is as follows
\[ x^{(1)}(t) = \left( x^{(1)}(1) - \frac{b}{a} \right) e^{-at} + \frac{b}{a}. \]
(5)

Therefore, in order to obtain the prediction model of the original data sequence, we need to identify the effect of the grey development coefficient \( a \) and the grey control parameter \( b \) in Eq. (4). To this end, we do the integral accumulation on both sides of Eq. (4) for \( [k, k+1], \quad k = 1, 2, \ldots, n-1 \), then we can get
\[ \int_{k}^{k+1} \frac{dx^{(1)}(t)}{dt} dt + a \int_{k}^{k+1} x^{(1)}(t) dt = b, \]

that is
\[ x^{(1)}(k+1) - x^{(1)}(k) + a \int_{k}^{k+1} x^{(1)}(t) dt = b, \]
or
\[ x^{(0)}(k+1) + a \int_{k}^{k+1} x^{(1)}(t) dt = b. \]
(6)

Let background value be \( z^{(1)}(k+1) \) defined as the integral
\[ z^{(1)}(k+1) = \int_{k}^{k+1} x^{(1)}(t) dt. \]

In order to calculate the background value \( z^{(1)}(k+1) \), we need to integrate \( x^{(1)}(t) \), which requires the values of \( a \) and \( b \) to be given in advance. However, from the Eq. (6), the values of \( a \) and \( b \) are determined by the values of the original sequence and structure form of the background value. Consequently, to estimate the values of \( a \) and \( b \), we must use some methods to estimate the background value \( z^{(1)}(k+1) \), which is a key factor affecting the simulation error \( \pi \) and the quality of the predicting model.

In the classical GM (1,1) model, we use the piecewise linear polynomial interpolation \( L(t) := (k + 1 - t)x^{(1)}(k) + (t - k)x^{(1)}(k+1) \) to approximate \( x^{(1)}(t) \), see [1], [2]; then we get the estimated background value \( z^{(1)}(k+1) \) as follows
\[ z^{(1)}(k+1) = \int_{k}^{k+1} x^{(1)}(t) dt \approx \int_{k}^{k+1} L(t) dt = \frac{1}{2} \left[ x^{(1)}(k) + x^{(1)}(k+1) \right]. \]
(7)

For each interval \([k, k+1], \quad k = 1, 2, \ldots, n-1\), by substituting the estimated background value \( z^{(1)}(k+1) \) into
Eq. (6) and further using the least square method, the values of the parameters $a$ and $b$ can be estimated by the following formula

$$\begin{bmatrix} a \\ b \end{bmatrix} = \left(G^T G\right)^{-1} G^T X,$$

where

$$X = \begin{bmatrix} x^0(2) \\ x^0(3) \\ \vdots \\ x^0(n) \end{bmatrix}, \quad G = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{bmatrix}.$$ 

Finally, the estimated solution to the differential Eq. (4) with the initial condition $X(1)(1) = X(1)(1)$ is obtained, as shown below

$$\tilde{z}^{(1)}(t) = [x^{(1)}(1) - \frac{b}{a}] e^{-a(t-1)} + \frac{b}{a}, \quad k = 1, 2, \ldots \quad (8)$$

We thus get the following grey prediction equation

$$\tilde{x}^{(1)}(k + 1) = \tilde{x}^{(1)}(k + 1) - \tilde{x}^{(1)}(k), \quad k = 1, 2, \ldots \quad (9)$$

![Fig. 1. Prediction error source diagram of conventional GM(1,1) model.](image)
are also given to prove the value of the new developed schemes. And section IV presents the conclusion.

II. \( C^1 \) MONOTONICITY-PRESERVING PIECEWISE CUBIC INTERPOLATION SPLINE

According to Eq. (3), the 1-AGO sequence \( X^{(1)} \) has the feature of monotonicity-increasing, that is \( x^{(1)}(k) \leq x^{(1)}(k+1), \forall k \). The fitting exponential curve \( x^{(1)}(t) \) to the 1-AGO sequence \( X^{(1)} \) is also monotonicity-increasing and has infinite smoothness. Therefore, we develop a \( C^1 \) monotonicity-preserving rational quadratic interpolation spline to interpolate the 1-AGO sequence, so as to reconstruct the curve \( x^{(1)}(t) \).

For the discrete data \( (k, x^{(1)}(k)), k = 1, 2, \ldots, n \), we denote \( d^{(1)}(k) \) as the derivative value at node \( t = k \). For \( t \in [k, k+1], \) in [22], a monotonicity-preserving rational quadratic interpolation spline is constructed as follows

\[
R(t) = x^{(1)}(k) + \frac{x^{(0)}(k+1)[x^{(0)}(k+1)s^2 + d^{(1)}(k)s(1-s)]}{x^{(0)}(k+1) + d^{(1)}(k)+2x^{(0)}(k+1)s(1-s)},
\]

where \( s = t - k \in [0, 1] \). We can see that the interpolation spline given in Eq. (10) has \( O(h^2) \) convergence.

From formula (10), by direct calculation, we have

\[
\begin{aligned}
R(k) &= x^{(1)}(k), \quad R(k+1) = x^{(1)}(k+1), \\
R'(k) &= d^{(1)}(k), \quad R'(k+1) = d^{(1)}(k+1),
\end{aligned}
\]

which indicates that \( R(k^-) = R(k^+), \) \( R'(k^-) = R'(k^+) \). This means that the rational quadratic interpolation spline defined by Eq. (10) is \( C^1 \) continuous for arbitrary nonzero local parameter.

In practical application, the derivative values of the rational quadratic interpolation spline at the nodes should be estimated first. In this paper, we calculate the derivative value by the following method

\[
\begin{aligned}
d^{(1)}(1)m &= x^{(1)}(2) - x^{(1)}(1), \\
d^{(1)}(k) &= \frac{1}{2} [x^{(1)}(k+1) - x^{(1)}(k-1)], \\
d^{(1)}(n) &= x^{(1)}(n) - x^{(1)}(n-1),
\end{aligned}
\]

where \( k = 2, 3, \ldots, n - 1 \).

Obviously, for the monotonicity increasing 1-AGO sequence \( X^{(1)} \), the derivative value determined by Eq. (11) is non-negative, which means \( d^{(1)}(k) \geq 0, \forall k \). Without loss of generality, for \( t \in [k, k+1], \) direct computation gives that

\[
R'(t) = \frac{[x^{(0)}(k+1)]^2}{r(t)} \left\{ \frac{1}{2} \left[ x^{(0)}(k+1) + x^{(0)}(k) \right] s^2 + 2x^{(0)}(k+1)s(1-s) + \frac{1}{2} \left[ x^{(0)}(k) + x^{(0)}(k-1) \right] (1-s)^2 \right\},
\]

where

\[
r(t) = x^{(0)}(k+1) + \frac{1}{2} x^{(0)}(k+1) + x^{(0)}(k) + \frac{1}{2} x^{(0)}(k-1) \} s (1-s).
\]

From these, we can see that \( R'(t) \geq 0 \), which implies that the interpolation spline \( R(t) \) is monotonicity-preserving.

III. ESTABLISH NEW GM(1,1) MODEL

For the original non-negative sequence \( X^{(0)} = \{x^{(0)}(0), x^{(0)}(2), \ldots, x^{(0)}(n)\} \), we first calculate its 1-AGO sequence \( X^{(1)} = \{x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n)\} \). Then for the 1-AGO sequence \( X^{(1)} \), we use the \( C^1 \) monotonicity-preserving piecewise rational quadratic interpolation spline \( R(t) \) to reconstruct the exponential curve \( x^{(1)}(t) \). For the interval \( [k, k+1], \) we estimate the background value \( z^{(1)}(k+1) = \int_k^{k+1} x^{(1)}(t) dt \) by the following method

\[
z^{(1)}(k+1) = \int_k^{k+1} x^{(1)}(t) dt \\
= \frac{A + Bt^2 + C(1-t)}{A + Bt + C(1-t)} dt
\]

where

\[
A = x^{(1)}(k), B = [x^{(0)}(k+1)]^2, C = x^{(0)}(k+1)x^{(0)}(k+1), D = x^{(0)}(k+1), E = d^{(1)}(k+1) + d^{(1)}(k) - 2x^{(0)}(k+1).
\]

If \( E \neq 0 \), then

\[
(1 + \frac{D}{E}) = \frac{1}{2} \left[ 4x^{(0)}(k+1) + x^{(1)}(k+1) + x^{(1)}(k+1) - x^{(1)}(k) - x^{(1)}(k+1) \right] \neq 0,
\]

so there is

\[
\int_k^{k+1} R(t) dt = \frac{A + Bt^2 + C(1-t)}{A + Bt + C(1-t)} dt
\]

where

\[
(1 + \frac{D}{E}) < 0, \quad A + Bt + C(1-t)
\]

when \( (1 + \frac{D}{E}) > 0, \quad A + Bt + C(1-t), \) when \( E = 0 \).

Then, the estimated background value was substituted into the grey differential equation equation (6). And we further use the least square method to solve Eq. (6). The formula is as follows

\[
(\begin{array}{c}
\frac{a}{b}
\end{array}) = (G^T G)^{-1} G^T X,
\]

where

\[
X = \begin{bmatrix}
x^{(0)}(2) \\
x^{(0)}(3) \\
\vdots \\
x^{(0)}(n)
\end{bmatrix}, \quad G = \begin{bmatrix}
-z^{(1)}(2) & 1 \\
z^{(1)}(3) & 1 \\
\vdots \\
z^{(1)}(n) & 1
\end{bmatrix}
\]

Finally, we obtain the following estimated solution to the differential equation (4) with the initial condition \( X^{(1)}(1) = X^{(1)}(1) \) as follows

\[
z^{(1)}(t) = \frac{a}{b} \left[ x^{(1)}(t) - \frac{b}{a} e^{-a(t-1)} + \frac{b}{a} \right].
\]

We thus get the following grey prediction equation

\[
x^{(0)}(k+1) = x^{(1)}(k+1) - x^{(1)}(k) \]

\[
= (1 - e^a) \left[ x^{(0)}(1) + \frac{a}{b} e^{-ak}, \quad k = 1, 2, \ldots \right.
\]

We shall give several simulation results of practice examples to show that the new GM(1,1) model based on \( C^1 \) monotonicity-preserving piecewise rational quadratic interpolation spline improves prediction accuracy compared to the classical GM (1,1) model. In the following examples, the relative error is computed by

\[
\varepsilon = \frac{x^{(0)}(k) - z^{(0)}(k)}{x^{(0)}(k)}
\]

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**Example 1:** In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 7$ given in [23]. In addition, we compare the results predicted by our new GM(1,1) model with the GM(1,1) model and the method proposed in [23]. Table I and Fig. 2 give the numerical results. The results show that our model has the best prediction effect compared with the other two prediction models, and it performs very well in predicting data with the exponential growth trend.

![Fig. 2. Graphic results for example 1.](image)

**Example 2:** In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 12$ given in [8]. Similarly, we compare the results predicted by our new GM(1,1) model with the GM(1,1) model and the method proposed in [7]. Table II and Fig. 3 give the numerical results. The results turn out that the new GM(1,1) model still performs the best among the three prediction models. In addition, its prediction accuracy is significantly higher than the classical GM(1,1) model.

![Fig. 3. Graphic results for example 2.](image)

**Example 3:** In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 9$ given in [24]. Table III and Fig. 4 give the numerical results.

![Example 4: In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 10$ given in [25]. Table IV and Fig. 5 give the numerical results.](image)

**Example 4:** In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 10$ given in [25]. Table IV and Fig. 5 give the numerical results.

**Example 5:** In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 14$ given in [26]. Table V and Fig. 6 give the numerical results.

![Example 5: In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 14$ given in [26]. Table V and Fig. 6 give the numerical results.](image)

**Example 6:** In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 7$ given in [27]. Table VI and Fig. 7 give the numerical results.

The Figs. 2-7 above show the 1-AGO data of Table 1-6 and the curves of piecewise linear interpolant, monotonic-preserving quadratic interpolation spline $R(t)$. It can be seen from Examples 1–6 that the average relative error $\tau$ of the new GM (1,1) model is lower than that of the classical GM (1,1) model, which means that the new GM (1,1) model can improve the quality of the forecasting model.
TABLE I
NUMERICAL RESULTS FOR EXAMPLE 1.

<table>
<thead>
<tr>
<th>$x^{(0)}$</th>
<th>Classical GM(1,1)</th>
<th>New GM(1,1)</th>
<th>The Model in [23]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Prediction data</td>
<td>Relative error $\varepsilon$ (%)</td>
<td>Prediction data</td>
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<tr>
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<td>2.9836</td>
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<tr>
<td>4.4511</td>
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<td>6.6402</td>
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<td>$\pi$ (%)</td>
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TABLE II
NUMERICAL RESULTS FOR EXAMPLE 2.

<table>
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<tr>
<th>$x^{(0)}$</th>
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<th>New GM(1,1)</th>
<th>The Model in [7]</th>
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<tbody>
<tr>
<td></td>
<td>Prediction data</td>
<td>Relative error $\varepsilon$ (%)</td>
<td>Prediction data</td>
</tr>
<tr>
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<td>110852</td>
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<td>110852</td>
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<td>135175</td>
<td>117980</td>
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<td>19.93</td>
<td>17.01</td>
<td>17.37</td>
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IV. CONCLUSION

By using a $C^1$ monotonicity-preserving piecewise rational quadratic interpolation spline to reconstruct the background value, we have established a new GM (1,1) model. Numerical examples show that the new GM (1,1) model has smaller prediction error than the classical one, especially in reliability and validity of the prediction, and this model performs better when the original data are presented with convexity in time series. Future work will concentrate on exploring more applications of the new GM (1,1) model, such as scientific decision-making in electricity production and manufactures.

TABLE V
NUMERICAL RESULTS FOR EXAMPLE 5.

<table>
<thead>
<tr>
<th>$x^{(0)}$</th>
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<th>New GM(1,1)</th>
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<td></td>
<td>Prediction data</td>
<td>Relative error $\varepsilon$ (%)</td>
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<tr>
<td>64832.05</td>
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<td>$\pi$ (%)</td>
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REFERENCES


Fig. 6. Graphic results for example 5.

<table>
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<tr>
<th>( x^{(0)} )</th>
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<th>New GM(1,1)</th>
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<td>Relative error ( \varepsilon ) (%)</td>
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\( \varepsilon \) (%)

7.9564

5.0447

Fig. 7. Graphic results for example 6.