A New GM (1, 1) Model Based on Piecewise Rational Quadratic Monotonicity-Preserving Interpolation Spline

Wenqing Chen and Yuanpeng Zhu*

Abstract—**In the classical GM (1,1) model, in order to reduce the randomness of the original nonnegative sequence, the original nonnegative sequence is cumulatively generated so as to obtain a monotone increasing 1-AGO sequence. The method of background value construction will directly affect the accuracy and applicability of the model. Therefore, the reconstruction of the model background value has great significance to improve its matching and prediction precision. In order to improve the GM (1,1) model, we provide a more logical formula for calculating background value, which is based on a C^1 monotonicity-preserving piecewise rational quadratic interpolation spline, and thereby establishing a new GM (1,1) model. Numerical examples show that the new GM (1,1) model is more effective and accurate compared with the classical GM (1,1) model.**

Index Terms—**Background value, GM(1, 1) model, Grey theory, Monotonicity-Preserving interpolation spline**

I. Introduction

Let an original non-negative and uniformly-spaced sequence be

$$X^{(0)} = \left\{ x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n) \right\}. \tag{1}$$

The method of cumulative generation of the original sequence is the main idea of the classical grey prediction GM (1,1) model proposed by Deng in [1], [2], so as to reduce the randomization of the original data and obtain a significantly monotonically increasing 1-AGO sequence $X^{(1)}$. Then a first order grad prediction differential equation on the sequence $X^{(1)}$ is established. And the differential equation is numerically solved by least square method to estimate the parameters. Finally, the original data is simulated and predicted by using the inverse cumulative generation operation.

The 1-AGO sequence $X^{(1)}$ is given as follows

$$X^{(1)} = \left\{ x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n) \right\}, \tag{2}$$

where

$$\begin{aligned} x^{(1)}(k) &= \sum_{i=0}^{k} x^{(0)}(k) \\ &= x^{(1)}(k-1) + x^{(0)}(k), k = 1, 2, \cdots, n. \end{aligned} \tag{3}$$

From Eq. (3), the 1-AGO sequence $X^{(1)}$ has the feature of monotonicity-increasing. Suppose that $x^{(1)}(t)$ satisfies the following first order grad forecasting differential equation

$$\frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b, \tag{4}$$

where the grey developmental coefficient a and the grey control parameter b are the parameters in the model to be estimated.

The solution of the above differential equation with the initial condition $\widetilde{X}^{(1)}(1) = X^{(1)}(1)$ is as follows

$$x^{(1)}(t) = \left[x^{(1)}(1) - \frac{b}{a} \right] e^{-a(t-1)} + \frac{b}{a}. \tag{5}$$

Therefore, in order to obtain the prediction model of the original data sequence, we need to identify the effect of the grey development coefficient a and the grey control parameter b in Eq. (4). To this end, we do the integral accumulation on both sides of Eq. (4) for $[k, k+1], k = 1, 2, \cdots, n-1$, then we can get

$$\int_k^{k+1} \frac{dx^{(1)}(t)}{dt} dt + a \int_k^{k+1} x^{(1)}(t) dt = b,$$

that is

$$x^{(1)}(k+1) - x^{(1)}(k) + a \int_k^{k+1} x^{(1)}(t) dt = b,$$

or

$$x^{(0)}(k+1) + a \int_k^{k+1} x^{(1)}(t) dt = b. \tag{6}$$

Let background value be $z^{(1)}(k+1) := \int_k^{k+1} x^{(1)}(t) dt$.

In order to calculate the background value $z^{(1)}(k+1)$, we need to integrate $x^{(1)}(t)$, which requires the values of a and b to be given in advance. However, from the Eq. (6), the values of a and b are determined by the values of the original sequence and structure form of the background value. Consequently, to estimate the values of a and b, we must use some methods to estimate the background value $z^{(1)}(k+1)$, which is a key factor affecting the simulation error $\overline{\varepsilon}$ and the quality of the predicting model.

In the classical GM (1,1) model, we use the piecewise linear polynomial interpolation $L(t) := (k+1-t)\, x^{(1)}(k) + (t-k)\, x^{(1)}(k+1)$ to approximate $x^{(1)}(t)$, see [1], [2] , then we get the estimated background value $z^{(1)}(k+1)$ as follows

$$\begin{aligned} z^{(1)}(k+1) &= \int_k^{k+1} x^{(1)}(t) dt \\ &\approx \int_k^{k+1} L(t) dt \\ &= \tfrac{1}{2} \left[x^{(1)}(k) + x^{(1)}(k+1) \right]. \end{aligned} \tag{7}$$

For each interval $[k, k+1]$, $k = 1, 2, \cdots, n-1$, by substituting the estimated background value $z^{(1)}(k+1)$ into

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Eq. (6) and further using the least square method, the values of the parameters a and b can be estimated by the following formula

$$\begin{pmatrix} a \\ b \end{pmatrix} = \left(G^T G\right)^{-1} G^T X,$$

where

$$X = \begin{bmatrix} x^0(2) \\ x^0(3) \\ \vdots \\ x^0(n) \end{bmatrix}, \; G = \begin{pmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{pmatrix}.$$

Finally, the estimated solution to the differential Eq. (4) with the initial condition $\widetilde{X}^{(1)}(1) = X^{(1)}(1)$ is obtained, as shown below

$$\widetilde{x}^{(1)}(t) = \left[x^{(1)}(1) - \frac{b}{a}\right] e^{-a(t-1)} + \frac{b}{a}, \quad k = 1, 2, \ldots. \quad (8)$$

We thus get the following grey prediction equation

$$\widetilde{x}^{(0)}(k+1) = \widetilde{x}^{(1)}(k+1) - \widetilde{x}^{(1)}(k), \quad k = 1, 2, \ldots. \quad (9)$$

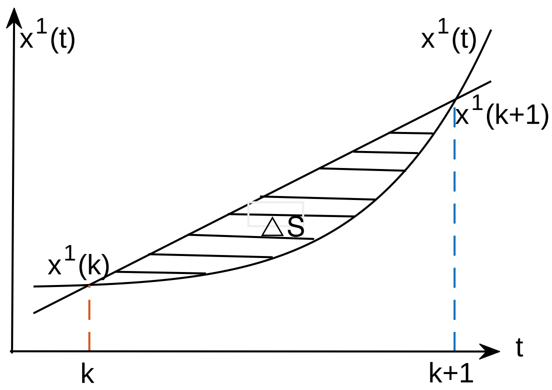

Fig. 1. Prediction error source diagram of conventional GM(1,1) model.

From (7), it can be seen that the classical GM (1,1) model uses the average of adjacent values to construct the background value $z^{(1)}(k+1)$. Its geometric meaning is to substitute the straight ladder region for the trapezoidal area based on the edge of exponential curve $x^{(1)}(t)$, as shown in Fig. 1. Nevertheless, this method has a disadvantage that when the 1-AGO data sequence varies greatly, the prediction result will have a large error (ΔS) that increases exponentially. Therefore, the classical GM (1,1) model has certain limitations in practical applications. As pointed out in [3], [4], [5], the construction of the background value $z^{(1)}(k+1)$ directly determines the prediction accuracy of the GM (1, 1) model. In [6], Li and Dai reconstructed $x^{(1)}(t)$ by a high-order Newton interpolation polynomial. And the background value $z^{(1)}(k+1)$ was estimated based on the Newton-Cores integral. However, as shown in [6], when there are a mess of data, the high-order Newton interpolation polynomial may have the Runge phenomenon. So the truncation error may be very large, and additionally, the numerical stability cannot be guaranteed when calculating the approximate value of Newton-Cores integral. In [7], Tang and Xiang estimated the background value $z^{(1)}(k+1)$ by reconstructing $x^{(1)}(t)$ on the interval $[k, k+1]$ using the piecewise quadratic interpolation polynomial. It provides low computational complexity and good numerical stability. In [8], Wang et al. used piecewise cubic interpolation spline to reconstruct $x^{(1)}(t)$ and obtained the estimated background value $z^{(1)}(k+1)$. Avoiding Runge phenomenon of high-order polynomial and having a better approximation order are the advantages of this method. However, all the above methods ignore the important monotone increasing characteristic of the curve $x^{(1)}(t)$ that to be reconstructed. If the reconstructed curve loses the monotonicity-increasing characteristic of $x^{(1)}(t)$, it will also cause a large error in the background value $z^{(1)}(k+1)$. Therefore, the exact approximation function of $x^{(1)}(t)$ is the key to enhancing the estimation of the background value and improving the effectiveness and accuracy. Thus it is of great significance to construct monotonicity-preserving interpolation splines, and many methods have been proposed. Recently, Qin and his colleagues have committed to the establishments of new splines, such as the rational polynomial interpolation spline, the rational trigonometric interpolation spline and the piecewise bivariate rational interpolation spline; see [9], [10], [11], and the references quoted therein.

The classical GM(1,1) model is easy to use and efficient in time series prediction and it represents the main idea of the methodology of the grey models. In recent years, a lot of grey prediction models have been developed based on the analogous methods of the GM(1,1). For instance, FGM(1,1), the NGM model, INDGM, TDPGM(1,1), Grey polynomial model, seasonal GM(1,1), etc; see [12], [13], [14], [15], [16], [17], and the references quoted therein. However, as the traditional GM (1,1) model has certain limitations in some cases, many researchers have proposed numerous new methods to improve the GM (1,1) model; see for example [18], [19], [20], [21].

In this paper, we shall use a C^1 monotonicity-preserving piecewise rational quadratic interpolation spline developed in [22] to reconstruct the curve $x^{(1)}(t)$ and present a new scheme to estimate the background value $z^{(1)}(k+1)$. It improves on the schemes in some ways:

(1) The Lagrange polynomial interpolation scheme developed in [6] may have Runge phenomenon. Whereas, the given monotonicity-preserving piecewise rational quadratic interpolation spline method can avoid this situation, thus helping to reduce the error and improve the numerical stability.

(2) The existing methods developed in [3], [4], [5], [6], [7], [8] ignore the important monotone increasing characteristic of the curve $x^{(1)}(t)$, which is considered in this paper.

(3) Our paper refers to monotonicity-preserving piecewise rational quadratic interpolation spline in [22] to reconstruct $x^{(1)}(t)$, and the spline has $O(h^3)$ convergence.

(4) In the numerical examples, we compare the new model with the traditional model and other models in the paper. The prediction results show that our new model can improve the prediction accuracy in practice.

The rest of this paper is organized as follows. In section II, the construction of C^1 monotonicity-preserving piecewise rational quadratic interpolation spline with monotonicity is given. In section III, a new GM (1,1) model based on C^1 monotonicity-preserving rational quadratic interpolation spline is constructed in detail. Several numerical examples are also given to prove the value of the new developed schemes. And section IV presents the conclusion.

II. C^1 MONOTONICITY-PRESERVING PIECEWISE CUBIC INTERPOLATION SPLINE

According to Eq.(3), the 1-AGO sequence $X^{(1)}$ has the feature of monotonicity-increasing, that is $x^{(1)}(k) \le x^{(1)}(k+1)$, $\forall k$. The fitting exponential curve $x^{(1)}(t)$ to the 1-AGO sequence $X^{(1)}$ is also monotonicity-increasing and has infinite smoothness. Therefore, we develop a C^1 monotonic-preserving rational quadratic interpolation spline to interpolate the 1-AGO sequence, so as to reconstruct the curve $x^{(1)}(t)$.

For the discrete data $\left(k, x^{(1)}(k)\right)$, $k = 1, 2, \ldots, n$, we denote $d^{(1)}(k)$ as the derivative value at node $t = k$. For $t \in [k, k+1]$, in [22], a monotonicity-preserving rational quadratic interpolation spline is constructed as follows

$$\begin{aligned} R(t) &= x^{(1)}(k) \\ &+ \frac{x^{(0)}(k+1)\left[x^{(0)}(k+1)s^2 + d^{(1)}(k)s(1-s)\right]}{x^{(0)}(k+1) + \left[d^{(1)}(k+1) + d^{(1)}(k) - 2x^{(0)}(k+1)\right]s(1-s)}, \end{aligned} \tag{10}$$

where $s = t - k \in [0, 1]$. We can see that the interpolation spline given in Eq. (10) has $O(h^3)$ convergence.

From formula (10) , by direct calculation, we have

$$\begin{cases} R(k) = x^{(1)}(k),\ R(k+1) = x^{(1)}(k+1), \\ R'(k) = d^{(1)}(k),\ R'(k+1) = d^{(1)}(k+1), \end{cases}$$

which indicates that $R(k^-) = R(k^+), R'(k^-) = R'(k^+)$. This means that the rational quadratic interpolation spline defined by Eq. (10) is C^1 continuous for arbitrary nonzero local parameter.

In practical application, the derivative values of the rational quadratic interpolation spline at the nodes should be estimated first. In this paper, we calculate the derivative value by the following method

$$\begin{cases} d^{(1)}(1) = x^{(1)}(2) - x^{(1)}(1), \\ d^{(1)}(k) = \frac{1}{2}\left[x^{(1)}(k+1) - x^{(1)}(k-1)\right], \\ d^{(1)}(n) = x^{(1)}(n) - x^{(1)}(n-1). \end{cases} \tag{11}$$

where $k = 2, 3, \ldots, n-1$.

Obviously, for the monotonicity increasing 1-AGO sequence $X^{(1)}$, the derivative value determined by Eq. (11) is non-negative, which means $d^{(1)}(k) \ge 0, \forall\ k$. Without loss of generality, for $t \in [k, k+1]$, direct computation gives that

$$\begin{aligned} R'(t) = \frac{\left[x^{(0)}(k+1)\right]^2}{r^2(t)} &\left\{\tfrac{1}{2}\left[x^{(0)}(k+1) + x^{(0)}(k)\right]s^2 \right. \\ &+ 2x^{(0)}(k+1)s(1-s) \\ &\left. + \tfrac{1}{2}\left[x^{(0)}(k) + x^{(0)}(k-1)\right](1-s)^2\right\}, \end{aligned}$$

where

$$\begin{aligned} r(t) &= x^{(0)}(k+1) \\ &+ \left\{-\tfrac{3}{2}x^{(0)}(k+1) + x^{(0)}(k) + \tfrac{1}{2}x^{(0)}(k-1)\right\}s(1-s). \end{aligned}$$

From these, we can see that $R'(t) \ge 0$, which implies that the interpolation spline $R(t)$ is monotonicity-preserving.

III. ESTABLISH NEW GM(1,1) MODEL

For the original non-negative sequence $X^{(0)} = \left\{x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)\right\}$, we first calculate its 1-AGO sequence $X^{(1)} = \left\{x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n)\right\}$. Then for the 1-AGO sequence $X^{(1)}$, we use the C^1 monotonicity-preserving piecewise rational quadratic interpolation spline $R(t)$ to reconstruct the exponential curve $x^{(1)}(t)$. For the interval $[k, k+1]$, we estimate the background value $z^{(1)}(k+1) = \int_k^{k+1} x^{(1)}(t)dt$ by the following method

$$\begin{aligned} z^{(1)}(k+1) &= \int_k^{k+1} x^{(1)}(t)dt \\ &\approx \int_k^{k+1} R(t)dt. \end{aligned}$$

Let $A = x^{(1)}(k), B = \left[x^{(0)}(k+1)\right]^2, C = x^{(0)}(k+1)d^{(1)}(k), D = x^{(0)}(k+1), E = d^{(1)}(k+1) + d^{(1)}(k) - 2x^{(0)}(k+1)$. If $E \ne 0$, then

$$\begin{aligned} \left(\tfrac{1}{4} + \tfrac{D}{E}\right) = \tfrac{1}{8E}&\left[4x^{(0)}(k+1) + x^{(1)}(k+2)\right. \\ &\left. + x^{(1)}(k+1) - x^{(1)}(k) - x^{(1)}(k-1)\right] \ne 0, \end{aligned}$$

so there is

$$\begin{aligned} \int_k^{k+1} R(t)dt &= \int_k^{k+1}\left[A + \frac{Bt^2 + Ct(1-t)}{D + Et(1-t)}\right]dt \\ &= \begin{cases} A + \frac{C-B}{E} + \frac{2CD - 2BD - BE}{E^2\sqrt{-\frac{1}{4} - \frac{D}{E}}}\left[\arctan \frac{\frac{1}{2}}{\sqrt{-\frac{1}{4} - \frac{D}{E}}}\right], \\ when\ \left(\frac{1}{4} + \frac{D}{E}\right) < 0, \\ A + \frac{C-B}{E} + \left(\frac{2CD - 2BD - BE}{2E^2\sqrt{\frac{1}{4} + \frac{D}{E}}}\right)\ln\left|\frac{E\left(\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{D}{E}}\right)^2}{D}\right|, \\ when\ \left(\frac{1}{4} + \frac{D}{E}\right) > 0, \\ A + \frac{2B+C}{6D},\ when\ \ E = 0. \end{cases} \end{aligned}$$

Then, the estimated background value was substituted into the grey differential equation equation (6). And we further use the least square method to solve Eq. (6). The formula is as follows

$$\begin{pmatrix} a \\ b \end{pmatrix} = \left(G^T G\right)^{-1} G^T X,$$

where

$$X = \begin{bmatrix} x^0(2) \\ x^0(3) \\ \vdots \\ x^0(n) \end{bmatrix}, G = \begin{pmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{pmatrix}.$$

Finally, we obtain the following estimated solution to the differential equation (4) with the initial condition $\widetilde{X}^{(1)}(1) = X^{(1)}(1)$ as follows

$$\widetilde{x}^{(1)}(t) = \left[x^{(1)}(1) - \frac{b}{a}\right]e^{-a(t-1)} + \frac{b}{a}.$$

We thus get the following grey prediction equation

$$\begin{aligned} x^{(0)}(k+1) &= x^{(1)}(k+1) - x^{(1)}(k) \\ &= (1 - e^a)\left[x^{(1)}(1) - \tfrac{b}{a}\right]e^{-ak},\ \ k = 1, 2, \ldots. \end{aligned}$$

We shall give several simulation results of practice examples to show that the new GM(1,1) model based on C^1 monotonicity-preserving piecewise rational quadratic interpolation spline improves prediction accuracy compared to the classical GM (1,1) model. In the following examples, the relative error is computed by

$$\varepsilon = \frac{\left|\overline{x}^{(0)}(k) - x^{(0)}(k)\right|}{x^{(0)}(k)}.$$

Example 1: In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 7$ given in [23]. In addition, we compare the results predicted by our new GM(1,1) model with the GM(1,1) model and the method proposed in [23]. Table I and Fig. 2 give the numerical results. The results show that our model has the best prediction effect compared with the other two prediction models, and it performs very well in predicting data with the exponential growth trend.

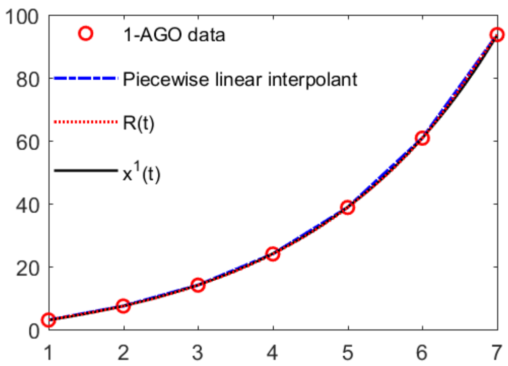

Fig. 2. Graphic results for example 1.

Example 2: In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 12$ given in [8]. Similarly, we compare the results predicted by our new GM(1,1) model with the GM(1,1) model and the method proposed in [7]. Table II and Fig. 3 give the numerical results. The results turn out that the new GM(1,1) model still performs the best among the three prediction models. In addition, its prediction accuracy is significantly higher than the classical GM(1,1) model.

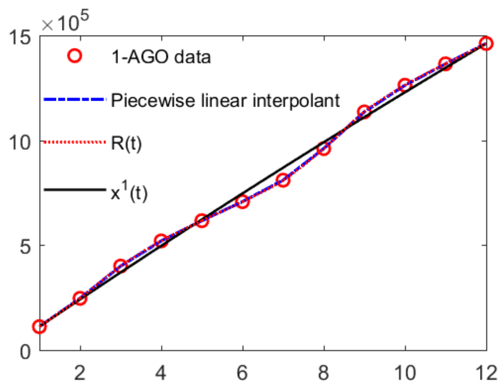

Fig. 3. Graphic results for example 2.

Example 3: In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 9$ given in [24]. Table III and Fig. 4 give the numerical results.

Example 4: In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 10$ given in [25]. Table IV and Fig. 5 give the numerical results.

Example 5: In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 14$ given in [26]. Table V and Fig. 6 give the numerical results.

TABLE III
NUMERICAL RESULTS FOR EXAMPLE 3.

$x^{(0)}$	Classical GM(1,1)		New GM(1,1)	
	Prediction data	Relative error ε (%)	Prediction data	Relative error ε (%)
0.0200	0.020000	0	0.020000	0
0.0191	0.019290	0.9929	0.019301	1.0506
0.0176	0.017450	0.8486	0.017460	0.7949
0.0159	0.015788	0.7109	0.015795	0.6600
0.0144	0.014280	0.8202	0.014289	0.7723
0.0129	0.012920	0.1575	0.012926	0.2029
0.0117	0.011689	0.0979	0.011693	0.0556
0.0105	0.010574	0.7067	0.010578	0.7464
0.0095	0.009566	0.6958	0.009570	0.7325
$\bar{\varepsilon}$ (%)		0.5589		0.5572

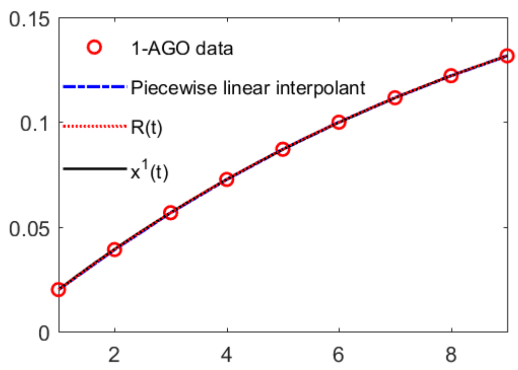

Fig. 4. Graphic results for example 3.

TABLE IV
NUMERICAL RESULTS FOR EXAMPLE 4.

$x^{(0)}$	Classical GM(1,1)		New GM(1,1)	
	Prediction data	Relative error ε (%)	Prediction data	Relative error ε (%)
897	897.00	0	897.00	0
897	939.52	4.7407	939.54	4.7423
890	898.25	0.9275	898.30	0.9324
876	858.80	1.9637	858.87	1.9558
848	821.07	3.1751	821.17	3.1642
814	785.01	3.5616	785.12	3.5475
779	750.53	3.6551	750.66	3.6379
738	717.56	2.7698	717.71	2.7492
669	686.04	2.5470	686.21	2.5721
600	655.90	9.3175	656.09	9.3477
$\bar{\varepsilon}$ (%)		3.2658		3.2649

Example 6: In this example, we consider the non-negative data $x^{(0)}(k)$, $k = 1, 2, \cdots, 7$ given in [27]. Table VI and Fig. 7 give the numerical results.

The Figs. 2-7 above show the 1-AGO data of Table 1-6 and the curves of piecewise linear interpolant, monotonic-preserving quadratic interpolation spline $R(t)$. It can be seen from Examples 1–6 that the average relative error $\bar{\varepsilon}$ of the new GM (1,1) model is lower than that of the classical GM (1,1) model, which means that the new GM (1,1) model can improve the quality of the forecasting model.

TABLE I
NUMERICAL RESULTS FOR EXAMPLE 1.

$x^{(0)}$	Classical GM(1,1)		New GM(1,1)		The Model in [23]	
	Prediction data	Relative error ε (%)	Prediction data	Relative error ε (%)	Prediction data	Relative error ε (%)
2.9836	2.9836	0	2.9836	0	2.9836	0
4.4511	4.3804	1.5884	4.4925	0.9292	4.4561	0.1123
6.6402	6.5006	2.1027	6.6826	0.6383	6.6132	0.4066
9.9061	9.6469	2.6161	9.9404	0.3465	9.8146	0.9237
14.7781	14.3162	3.1226	14.7865	0.0569	14.5657	1.4373
22.0464	21.2454	3.6331	21.9951	0.2327	21.6168	1.9486
32.8893	31.5285	4.1374	32.7180	0.5209	32.0812	2.4570
$\bar{\varepsilon}$ (%)		2.4576		0.3892		1.0408

TABLE II
NUMERICAL RESULTS FOR EXAMPLE 2.

$x^{(0)}$	Classical GM(1,1)		New GM(1,1)		The Model in [7]	
	Prediction data	Relative error ε (%)	Prediction data	Relative error ε (%)	Prediction data	Relative error ε (%)
110852	110852	0	110852	0	110852	0
135175	117980	12.72	130093	3.76	127821	5.41
153647	119117	22.47	128585	16.31	126664	17.66
120296	128264	6.62	127094	5.65	125830	4.68
96362	121422	26.27	125621	30.36	124380	29.23
90798	122592	35.01	124165	36.75	123253	35.70
102591	123773	20.65	122725	19.63	122137	19.11
150534	124965	16.99	121303	19.42	121031	19.63
175123	126168	27.95	119896	31.54	119934	31.52
127148	113383	10.83	118507	6.80	114848	9.76
102085	128610	25.98	117133	14.74	117772	15.47
97103	129849	33.72	115775	19.23	116705	20.21
$\bar{\varepsilon}$ (%)		19.93		17.01		17.37

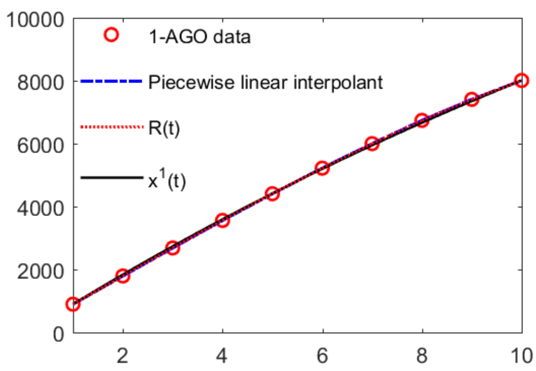

Fig. 5. Graphic results for example 4.

IV. CONCLUSION

By using a C^1 monotonicity-preserving piecewise rational quadratic interpolation spline to reconstruct the background value, we have established a new GM (1,1) model. Numerical examples show that the new GM (1,1) model has smaller prediction error than the classical one, especially in reliability and validity of the prediction, and this model performs better when the original data are presented with convexity in time series. Future work will concentrate on exploring more applications of the new GM (1,1) model, such as scientific decision-making in electricity production and manufactures.

TABLE V
NUMERICAL RESULTS FOR EXAMPLE 5.

$x^{(0)}$	Classical GM(1,1)		New GM(1,1)	
	Prediction data	Relative error ε (%)	Prediction data	Relative error ε (%)
64832.05	64832.05	0	64832.05	0
71847.09	57476.77	20.001	57600.77	19.829
78646.30	67165.21	14.598	67327.40	14.392
86293.10	78486.76	9.046	78696.50	8.803
93887.95	91716.70	2.312	91985.41	2.026
105557.09	107176.71	1.534	107518.32	1.858
125761.85	125242.71	0.413	125674.17	0.070
143143.63	146353.96	2.243	146895.86	2.621
168850.20	171023.78	1.287	171701.11	1.688
198739.27	199852.01	0.560	200695.05	0.984
245352.80	233539.60	4.815	234584.98	4.389
278541.09	272905.57	2.023	274197.66	1.559
334839.41	318907.39	4.758	320499.45	4.283
386086.72	372663.28	3.477	374619.89	2.970
$\bar{\varepsilon}$ (%)		4.791		4.677

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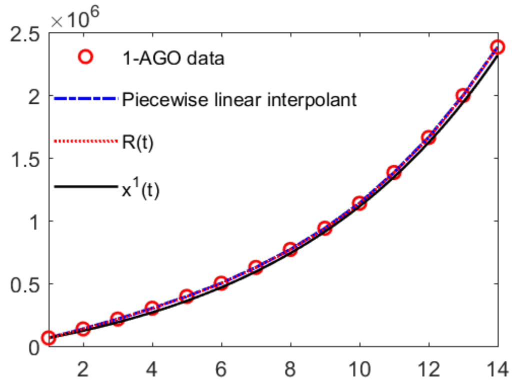

Fig. 6. Graphic results for example 5.

TABLE VI
NUMERICAL RESULTS FOR EXAMPLE 6.

$x^{(0)}$	Classical GM(1,1)		New GM(1,1)	
	Prediction data	Relative error ε (%)	Prediction data	Relative error ε (%)
21.1	21.1	0	21.1	0
26.6	21.4246	19.4566	22.0435	17.1299
36.1	32.7131	9.3820	33.7675	6.4613
52.3	49.9496	4.4941	51.7270	1.0957
80.1	76.2679	4.7842	79.2384	1.0757
126.8	116.4532	8.1599	121.3820	4.2729
196.3	177.8122	9.4182	185.9401	5.2776
$\bar{\varepsilon}$ (%)		7.9564		5.0447

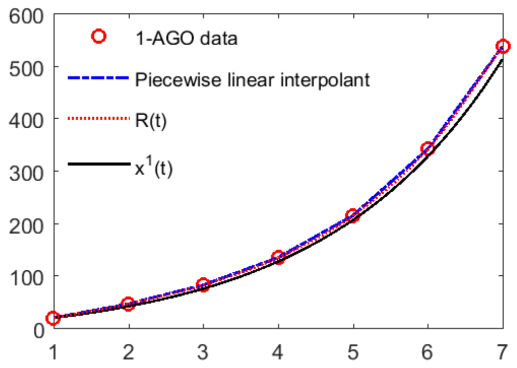

Fig. 7. Graphic results for example 6.

2018A030310381) and the Hundred-Step Ladder "Climbing" Program of South China University of Technology, China.

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