# CNF-Structure: Stabilizers, Orbits, Fibre-Transversals

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*Abstract*— **The structure of CNF formulas as well as classes of formulas is considered here from a group theoretic perspective, provided by the action under the complementation group. For this study again, the fibre view approach to CNF on basis of the concepts of base hypergraphs and its fibre-transversals is exploited extensively. Several CNF classes are investigated which are defined via orbits of complementation subgroups. Besides their stabilizer properties we also study to some extent the satisfiability aspects of those classes. In that context it turns out that several results regarding stabilizer properties or satisfiability aspects valid for fibre-transversals cannot be transfered when replacing them with arbitrary CNF formulas. Further we present an algorithm for computing the isotropy groups of fibre formulas, and investigate the lifting process to the total case. The members of several concrete subclasses of CNF are treated thereby such as linear or symmetric formulas.**

***Keywords: CNF, orbit, satisfiability, stabilizer, base hypergraph***

## 1 Introduction

A fundamental open question in mathematics is the NP versus P problem which is attacked within the theory of NP-completeness [9]. The genuine and one of the most important NP-complete [6] problems is the propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas. More precisely, SAT is the natural NP-complete problem and thus lies at the heart of computational complexity theory. Moreover, SAT plays an essential role in the theory of designing exact algorithms. Numerous applications can be treated within the CNF-SAT approach on basis of the high expressiveness of the CNF language, so enabling polynomial-time reductions of computational problems occuring in the main applicational areas [10, 11]. Important applications of SAT are, e.g., formal verification [25], bounded model checking [5] or artificial intelligence. In industrial applications most often the modelling CNF formulas are of a specific structure. And therefore it would be desirable to have fast algorithms for such instances. Also from the structural point of view one is interested in classes for which SAT can be solved in polynomial time. There are known several structured classes having a polynomially bounded time complexity, such as quadratic formulas, (extended and q-)Horn formulas, matching formulas, nested and co-nested formulas etc. [1, 3, 4, 8, 12, 13, 14, 15, 23, 26]. On basis of the complementation operation on CNF formulas, in this work we investigate the orbit structure and also the isotropy groups of formulas and formula classes which also had been started in a previous paper [19]. Besides the fundamental aspect of exploring the CNF structure, another motivation behind this research is the fact that formulas with large isotropy groups have small orbits. On the other hand the generator sets of isotropy groups are of polynomial size. This might enabling one to compute class invariants more efficiently, specifically those that are connected to the satisfiability of formulas like the monotonicity index. The hope here is to identify new subclasses of CNF which behave easy for SAT-decision as well as to gain new structural insight into CNF-SAT in general. Further it turns out that a useful tool in revealing the structure of CNF-SAT is provided by linear formulas (LCNF). Note that the complexity of various satisfiability problems on linear formula classes is well studied, confer e.g. [20, 22]. Again the fibre view approach to CNF on basis of the notions of the base hypergraph and the fibre-transversals [16] provides the conceptional basis for this study. After the presentation of some preliminaries regarding the CNF or group contexts, the fibre view perspective is briefly recapitulated for convenience, as the whole paper relies on the corresponding terminology. Specifically several results directly are related to structures defined over fibre-transversals. Section 3 presents basic results on the complementation operation on the CNF formulas such as stabilizers and orbits. It also provides a fixed-parameter tractability assertion regarding certain formula classes with bounded sizes of its orbits. Section 4 studies formulas and formula classes defined via orbits of subgroups of the complementation group. Several results regarding the stabilizers of such structured objects are presented. Section 5 is devoted to design an algorithm for computing the stabilizer of a fibre formula. Section 6 is intended to present stabilizer results for formulas of a given structure such as the linear case or symmetric formulas. Section 7 briefly returns to orbit classes for discussing to some extent their satisfiability properties. Here again the specific role of fi-

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bre transversals turns out. Finally several open problems and future work perspectives are outlined.

## 2 Notation and Preliminaries

A Boolean or propositional variable $x$ taking values from $\{0,1\}$ can appear as a positive literal which is $x$ or as a negative literal which is the negated variable $\overline{x}$. To *flip* or *complement* a literal always means to negate the underlying variable. Setting a literal to 1 means to set the corresponding variable accordingly. A clause $c$ is a finite non-empty disjunction of different literals and it is represented as a set $c = \{l_1, \ldots, l_k\}$. If all literals in $c$ are complemented one gets $c^\gamma$. A clause containing no negative literal is called *positive*. A clause containing only negated variables is called *negative*. A *unit* clause contains exactly one literal. A conjunctive normal form formula $C$, for short formula, is a finite conjunction of different clauses and is considered as a set of these clauses $C = \{c_1, \ldots, c_m\}$. Let $|C|$ be refered to as the *size* of $C$. $C^\gamma$ is the formula obtained from $C$ by transfering $c \to c^\gamma$ for all $c \in C$. A formula can also be empty which is denoted as $\varnothing$. Let CNF be the collection of all formulas. For a formula $C$ (clause $c$), by $V(C)$ ($V(c)$) denote the set of variables occurring in $C$ ($c$). Let $\mathrm{CNF}_+$ ($\mathrm{CNF}_-$) denote that part of CNF containing only positive (negative) clauses. A formula $C \in \mathrm{CNF}$ is called *linear* if it contains no complementary unit clauses and additionally every pair $c_i, c_j \in C$, $i \neq j$, satisfies $|V(c_i) \cap V(c_j)| \leq 1$. By LCNF the class of linear formulas is denoted. Given $C \in \mathrm{CNF}$, let $A(C) := \{c \in C : c^\gamma \notin C\}$ and $S(C) := \{c \in C : c^\gamma \in C\}$ defining the classes $\mathcal{A} := \{C \in \mathrm{CNF} : C = A(C)\}$ of *antisymmetric* and $\mathcal{S} := \{C \in \mathrm{CNF} : C = C^\gamma\}$ of *symmetric* formulas [21]. Note that $\mathcal{A} \cap \mathcal{S} = \{\varnothing\}$, and that for every non-empty $C \in \mathrm{CNF}$ one has $C = A(C) \cup S(C)$ as disjoint union. However, clearly $\mathcal{S} \cup \mathcal{A}$ is a proper subset of CNF. Let $\mathcal{S}_\pm \subseteq \mathcal{S}$ contain all formulas $C = C \cup C^\gamma$, where $\varnothing \neq C \in \mathrm{CNF}_+$. For a finite set $M$, let $2^M$ denote its powerset. As usual for a positive integer $n$, let $[n] := \{1, \ldots, n\}$, and for convenience we set $[0] := \varnothing$. For any set system, i.e., a finite family of sets $\mathcal{M} := \{M_i : i \in [r]\}$ which all are subsets over any base set, as usual we set $\bigcup \mathcal{M} := \bigcup_{i \in [r]} M_i$. Throughout log means the logarithm function with respect to base 2; and groups always are assumed to be finite. Given a group $G$, recall that the order of any subgroup of $G$ is a divisor of its cardinality $|G|$ according to a central theorem of Lagrange. Let $\mathrm{Gn}(G)$ denote a set of generators of $G$. Let $< g > \leq G$ denote the cyclic subgroup generated by $g \in G$. Further recall that every abelian group can be written as a direct product of cyclic subgroups. Given $C \in \mathrm{CNF}$, SAT asks whether there is a truth assignment $t : V(C) \to \{0,1\}$ such that there is no $c \in C$ all literals of which are set to 0. If such an assignment exists it is called a *model* of $C$. Let $\mathrm{SAT} \subseteq \mathrm{CNF}$ denote the collection of all formulas for which there is a model. Clauses containing a complemented pair of literals are always satisfied. Hence, it is assumed throughout that clauses only contain literals over different variables, i.e., $|V(c)| = |c|$. As usual iff means if and only if.

### 2.1 Base Hypergraphs of CNF's

For convenience, let us briefly recall the fibre structure of CNF which is closely related to the notion of the base hypergraph, both concepts were introduced in [16]. So, the hyperedge set $B(C)$ of the base hypergraph $\mathcal{H}(C) = (V(C), B(C))$ assigned to a formula $C \in \mathrm{CNF}$ is defined as $B(C) := \{V(c) : c \in C\} \in \mathrm{CNF}_+$. The collection of all clauses $c$ such that $V(c) = b$, for a fixed $b \in B(C)$, is the fibre $C_b$ of $C$ over $b$ yielding the fibre-decomposition $C = \bigcup_{b \in B(C)} C_b$ of $C$. Therefore $b$ is also refered to as the *base point* of the fibre. Conversely, a hypergraph $\mathcal{H} = (V, B)$ can be regarded as base hypergraph if its vertex set $V$ is a non-empty finite set of Boolean variables such that for every $x \in V$ there is a $b \in B$ containing $x$. Every $b \in B$ is assumed to be non-empty. Further throughout, if not stated otherwise, it is assumed that $C \neq \varnothing$, $B \neq \varnothing$. By $W_b := \{c : V(c) = b\}$ denote the collection of all possible clauses over a fixed $b \in B$. By definition, a hypergraph $\mathcal{H} = (V, B)$ is *linear* if $|b \cap b'| \leq 1$, for all distinct $b, b' \in B$, and $\mathcal{H}$ is *exact linear* if $\leq$ above is replaced with $=$. Recall that a hypergraph $\mathcal{H} = (V, B)$ is called *loopless* if $|b| \geq 2$, for all $b \in B$ [2]. Observe that $\mathcal{H}(C)$ is loopless iff $C$ is free of unit clauses, and $\mathcal{H}(C)$ is (exact) linear if $C$ is (exact) linear. The *intersection graph* $\mathcal{I}(\mathcal{H})$ of $\mathcal{H} = (V, B)$ gets a vertex for each $b \in B$ and there is exactly one edge joining a pair of vertices $b \neq b'$ iff $b \cap b' \neq \varnothing$. A hypergraph $\mathcal{H}$ is called *connected* iff $\mathcal{I}(\mathcal{H})$ is connected in the usual sense. A hypergraph $\mathcal{H}$ is called *Sperner* if no hyperedge is contained in any other hyperedge of $\mathcal{H}$ [2]. Clearly, any loopless and linear hypergraph is Sperner; the converse however does not hold true in general.. The set of all clauses over $\mathcal{H}$ is $K_\mathcal{H} := \bigcup_{b \in B} W_b$. A *$\mathcal{H}$-based* formula is a subset $C \subseteq K_\mathcal{H}$ such that $C_b := C \cap W_b \neq \varnothing$, for every $b \in B$. For a $\mathcal{H}$-based $C \subseteq K_\mathcal{H}$, let $\bar{C} := K_\mathcal{H} \setminus C$ be its complement formula. If $C$ satisfies $\bar{C}_b := W_b \setminus C_b \neq \varnothing$, for all $b \in B$, then $\bar{C}$ also is $\mathcal{H}$-based. A *fibre-transversal* of $K_\mathcal{H}$ is a $\mathcal{H}$-based formula $F \subset K_\mathcal{H}$ such that $|F \cap W_b| = 1$, for every $b \in B$, this clause is denoted as $F(b)$. By $\mathcal{F}(K_\mathcal{H})$ denote the set of all fibre-transversals of $K_\mathcal{H}$. Observe that, given a linear base hypergraph $\mathcal{H}$ then every fibre-transversal $F \in \mathcal{F}(K_\mathcal{H})$ is linear. Similarly, a linear formula always is a fibre-transversal over its base hypergraph, as it is assumed to be free of complementary unit clauses. A *compatible* fibre-transversal is defined by the property that $\bigcup_{b \in B} F(b) \in W_V$. $\mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})$ is the set of all compatible fibre-transversals of $K_\mathcal{H}$. As an example for a compatible fibre-transversal, consider the base hypergraph with variable set $V := \{x_1, x_2, x_3\}$ and $B := \{b_1 := x_1x_2, b_2 := x_1x_3, b_3 := x_2x_3\}$. Then, e.g., the clauses $c_1 := x_1\bar{x}_2 \in W_{b_1}$,$c_2 := x_1\bar{x}_3 \in W_{b_2}$

and $c_3 := \bar{x}_2\bar{x}_3 \in W_{b_3}$, denoted as literal strings, form a compatible fibre-transversal of the corresponding $K_{\mathcal{H}}$, because $c_1 \cup c_2 \cup c_3 = x_1\bar{x}_2\bar{x}_3 \in W_V$. A *diagonal* fibre-transversal $F$ is defined through the property that for each $F' \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ one has $F \cap F' \neq \varnothing$. Finally, let $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ be the collection of all diagonal fibre-transversals of $K_{\mathcal{H}}$. According to [17] a base hypergraph $\mathcal{H}$ is called diagonal iff $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \varnothing$. As for the total clause set $K_{\mathcal{H}}$ we can define fibre-transversals for a $\mathcal{H}$-based formula $C \subset K_{\mathcal{H}}$ as follows. A fibre-transversal $F$ of $C$ contains exactly one clause of each fibre $C_b$ of $C$. The collection of all fibre-transversals of $C$ is denoted as $\mathcal{F}(C)$. We also have compatible and diagonal fibre-transversals of $C$ via $\mathcal{F}_{\text{comp}}(C) := \mathcal{F}(C) \cap \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$, and $\mathcal{F}_{\text{diag}}(C) := \mathcal{F}(C) \cap \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. For a base-hypergraph $\mathcal{H}$ and a class $\mathcal{C} \subseteq \text{CNF}$ let $\mathcal{C}(\mathcal{H}) := \{C \in \mathcal{C} : \mathcal{H}(C) = \mathcal{H}\}$, denote the $\mathcal{H}$-based fraction of $\mathcal{C}$. For convenience let us cite a central fact in that context shown in [16]. It characterizes the satisfiability of a formula $C$ in terms of compatible fibre-transversals in its based complement formula $\bar{C}$.

**Theorem 1** *[16] For $\mathcal{H} = (V, B)$, let $C \subset K_{\mathcal{H}}$ be a $\mathcal{H}$-based formula such that $\bar{C}$ is $\mathcal{H}$-based, too. Then $C$ is satisfiable iff $\bar{C}$ admits a compatible fibre-transversal $F$. Moreover, the union $c := \bigcup F^{\gamma}$ of all clauses in $F^{\gamma}$ is a clause with $V(c) = V$ and corresponds to a model of $C$.*

## 3 Basic Concepts and Results

For a fixed finite and non-empty set of propositional variables $V$, let $B_t = 2^V$ and $\mathcal{H}_t = (V, B_t)$. Denote by $\text{CNF} := 2^{K_{\mathcal{H}_t}}$ the set of all CNF formulas with $V(C) \subseteq V$, $B(C) =: B \subseteq B_t$. Let $c^X$ be the clause obtained from $c \in K_{\mathcal{H}_t}$ by complementing all variables in $X \cap V(c)$, where $X$ is an arbitrary subset of $V$, for short we set $c^{\gamma} := c^{V(c)}$, and further $c^{\varnothing} := c$. This *complementation operation* $\varphi(c, X) := c^X$ acting on $K_{\mathcal{H}_t}$ induces an action on CNF by observing that $\{c\} \in \text{CNF}$: For $C = \{c_1, \ldots, c_m\} \in \text{CNF}$ and $X \in 2^V$ let $\varphi : \text{CNF} \times 2^V \to \text{CNF}$, such that $\varphi(C, X) := \{c_1^X, \ldots, c_m^X\} =: C^X \in \text{CNF}$. Again set $C^{\gamma} := C^{V(C)}$ in case that all variables in $C$ are complemented, and $C^{\varnothing} := C$. Thus formally we obtain the *$G_V$-action* on CNF of the abelian group $G_V := (2^V, \oplus)$ with neutral element $\varnothing$. Indeed, let $X, Y \in G_V$, then $(C^X)^Y = C^{X \oplus Y}$. Further we set $\varnothing^X := \varnothing \in \text{CNF}$, for every $X \in G_V$. In case $V(C) \subsetneq V$, the relevant subgroup of $G_V$ is $G_{V(C)} = (2^{V(C)}, \oplus)$. We shall use the abbreviation $E := \{\varnothing\} \leq G_V$ for the trivial group. By $\mathcal{O}(C) := \{C^X : X \in G_{V(C)}\} = \{C^X : X \in G_V\}$ denote the *($G_V$-)orbit* of $C$ in CNF yielding the classes of an equivalence relation on CNF. This quotient space $\text{CNF}/G_V$ therefore usually is called the *orbit space*. Recall that a group acts transitively on its orbits. Let $G_{V(C)}(C) := \{X \in G_{V(C)} : C^X = C\}$ denote the *isotropy group* also called *stabilizer* of $C \in \text{CNF}$. For a fibre-subformula $C_b \subseteq C$, there are two kinds of isotropy groups, namely, the first $G_{V(C)}(C_b) := \{X \in G_{V(C)} : C_b^X = C_b\}$ which is refered to as the *total level*. And the second $G_b(C_b) := \{X \in G_b : C_b^X = C_b\}$, which is refered to as the *fibre level*, where $G_b := (2^b, \oplus)$, $V(C_b) = b$; thus $G_b(C_b) \leq G_{V(C)}(C_b)$. Similarly, for any $c \in K_{\mathcal{H}_t}$ with $b := V(c)$ one has its stabilizer $G_b(c) := \{X \in G_b : c^X = c\}$ on the fibre level, and also that one on the total level, namely $G_V(c) := \{X \in G_V : c^X = c\}$.

**Lemma 1** *Given $\mathcal{H} = (V, B)$, and non-trivial subgroups $G \leq G_V$ and $H \leq G_b$ then $|\text{Gn}(G)| \leq |V|$ and $|\text{Gn}(H)| \leq |b|$, $b \in B$.*

Proof. Clearly $\text{Gn}(G_V) = \{\{x\} : x \in V\}$ thus $|\text{Gn}(G)| \leq |\text{Gn}(G_V)| = |V|$. Also $\text{Gn}(G_b) = \{\{x\} : x \in b\}$ hence $|\text{Gn}(H)| \leq |b|$, $b \in B$. □

**Lemma 2** *Let $C \in \text{CNF}$ with $\mathcal{H}(C) =: (V, B)$.*

*(i) $X \in G_V(C)$ is equivalent with $c \in C \Leftrightarrow c^X \in C$.*

*(ii) $X \in G_V(C)$ iff $X \in G_V(C_b)$, for all $b \in B$.*

Proof. $X \in G_V(C)$ means $C^X = C$ and (i) follows directly. Addressing (ii) observe that for distinct $b, b' \in B$ there is no $X \in G_V$ such that $C_b^X = C_{b'}$. Thus $X \in G_V(C)$ iff $C^X = \bigcup_{b \in B} C_b^X = C = \bigcup_{b \in B} C_b$ iff $C_b^X = C_b$, $b \in B$, iff $X \in G_V(C_b)$, $b \in B$. □

Assertion (ii) above shall be restated as follows, for convenience:

**Corollary 1** $G_V(C) = \bigcap_{b \in B} G_V(C_b)$, *where $\mathcal{H}(C) = (V, B)$.*

More generally we set for the stabilizer of a formula class:

**Definition 1** $G_V(\mathcal{C}) := \{X \in G_V : C \in \mathcal{C} \Rightarrow C^X \in \mathcal{C}\}$ *is called the* isotropy group (or stabilizer) *of the class $\mathcal{C} \subseteq \text{CNF}$.*

Indeed that is a group, as for $X, Y \in G_V(\mathcal{C})$ and $C \in \mathcal{C}$ assume $C^X =: C' \in \mathcal{C}$ then $C^{X \oplus Y^{-1}} = C^{X \oplus Y} = C'^Y \in \mathcal{C}$ hence $X \oplus Y^{-1} \in G_V(\mathcal{C})$. Further note that according to the theorem of Lagrange every subgroup $G \leq G_V$ here is of order $2^{e(G)}$ with the integer $e(G) := \log |G| \geq 0$. A mapping $g : \text{CNF} \to \text{CNF}$ is *$G_V$-equivariant*, by definition, if $g(C^X) = [g(C)]^X$, for every $X \in G_V$ and every $C \in \text{CNF}$. As shown in [18] one has:

**Lemma 3** $G_{V(C)}(C') = G_{V(C)}(C)$ *for all $C' \in \mathcal{O}(C)$.*

As usual a *fixed point* of an operation [24] is the unique member of an 1-point *invariant* (also called *stable*) subspace, so by definition its isotropy group equals the whole group. According to Theorem 4 proven in [17] one has:

**Lemma 4** *$\varnothing \neq C \in$ CNF is a fixed point of the $G_V$-action iff $C_b = W_b$, for all $b \in B(C)$.*

**Lemma 5** *Let $C \in$ CNF, such that $\mathcal{H}(C) = \mathcal{H}(\bar{C}) =: \mathcal{H}$ then $G_V(C) = G_V(\bar{C})$.*

PROOF. Let $X \in G_V(C)$ and assume $c \in C \cap \bar{C}^X$ then $c^X \in \bar{C}$ hence $c^X \notin C$ yielding a contradiction. So, $C \cap \bar{C}^X = \varnothing$ which directly implies $\bar{C} = \bar{C}^X$, because also $\mathcal{H}(\bar{C}^X) = \mathcal{H}$. Thus it is $X \in G_V(\bar{C})$. The reverse inclusion follows by exchanging the roles of $C, \bar{C}$. □

**Lemma 6** *For $C, C' \in$ CNF with $V(C) = V(C') =: V$, assume $G_V(C) = G_V(C')$ then $|\mathcal{O}(C)| = |\mathcal{O}(C')|$.*

PROOF. Let $G := G_V(C)$, then $G \leq G_V$ is a normal subgroup so that the left coset space $G_V/G$ is the group of cosets $YG := \{Y \oplus X : X \in G\}$, for every $Y \in G_V$. Clearly, $|YG| = |G|$ and $C^{Y'} = C^Y$, for all $Y' \in YG$, implying $|\mathcal{O}(C)| = |G_V/G| = |\mathcal{O}(C')|$ by assumption. □

Note that the last equation in the previous proof directly corresponds to the usual orbit rule of group theory relating the orbit size with the stabilizer index. Recall that for $C \in$ CNF the value $\mu(C) := \min\{\min\{|C'_+|, |C'_-|\} : C' \in \mathcal{O}(C)\}$ is the monotonicity index [17] of $C$. Hence $\mu$ is a *class invariant* having the same value for all orbit members. Moreover, as shown in [17] one has $C \in$ SAT iff $\mu(C) = 0$. Recall that the fixed-parameter tractability (FPT) w.r.t. parameter $k$ means a worst case upper bound for the computational time complexity of the form $O(p(n,k)g(k))$, for instances of size $n$, where $p$ is a polynomial and $g$ is an arbitrary function of the parameter $k$ only, cf. e.g. [7].

**Theorem 2** *For a constant positive integer $k$, let $\mathcal{C} := \mathcal{C}(k) \subseteq$ CNF be such that $|\mathrm{Gn}(G_{V(C)}(C))| \geq |V(C)| - k$, and such that $\mathrm{Gn}(G_{V(C)}(C))$ can be computed in polynomial time, for every $C \in \mathcal{C}$, then SAT is FPT w.r.t. $k$, for instances from $\mathcal{C}$.*

PROOF. For $C \in \mathcal{C}$, let $G := G_{V(C)}$ and let $H := G_{V(C)}(C)$ be the isotropy group of $C$. The factor group $G/H$ may be identified with a set of representatives of its cosets. We claim that $\mathrm{Gn}(G/H) = \mathrm{Gn}(G) \setminus \mathrm{Gn}(H)$. Indeed, first assume that there is any $X \in \mathrm{Gn}(H)$ which is also in $\mathrm{Gn}(G/H)$ then $X \in H$ and also $X \in G/H$ implying $X = \varnothing \in G/H$ but $\varnothing \notin \mathrm{Gn}(H)$ providing a contradiction. Next let $Y \in \mathrm{Gn}(G) \setminus \mathrm{Gn}(H)$ then clearly $\varnothing \neq H \in G/H$ therefore $Y \in \mathrm{Gn}(G/H)$. So, by assumption one has $|\mathrm{Gn}(G/H)| = |\mathrm{Gn}(G)| - |\mathrm{Gn}(H)| \leq k$, and according to the proof of Lemma 6, $\mathcal{O}(C) = \{C^{\bigoplus_{Y \in Z} Y} : Z \subseteq \mathrm{Gn}(G/H)\}$. Thus $|\mathcal{O}(C)| \leq 2^k$ meaning that $\mu(C)$ can be computed in FPT time $O(p(|C|, |V(C)|)2^k)$ where $p$ is an appropriate polynomial. □

## 4 Orbits, Orbit Classes, and Stabilizers

This section considers rather structured formulas respectively formula classes. These are defined over a subgroup of the complementation operation as orbit formulas or different types of orbit classes. We are interested in their stabilizers as well as its satisfiability aspects. To that end, let us fix an arbitrary base hypergraph $\mathcal{H} = (V, B)$ used throughout the section.

**Definition 2** *For $b \in B$, subgroups $G \leq G_V$, $H \leq G_b$,*

*(1)* $\mathrm{R}_b(G) := \{X \cap b : X \in G\}$ *is the* $(b)$-restriction *of* $G$,

*(2)* $\mathrm{L}_b(H) := \{X \in G_V : X \cap b \in H\}$ *is the* $(G_V)$-lift *of* $H$.

As an example consider $V = \{u, x, y, z\}$, $b = \{x, z\}$, $G := \{\varnothing, \{x, y\}\} \leq G_V$. Then one obtains $\mathrm{R}_b(G) = \{\varnothing, \{x\}\} =: H \leq G_b$, and $\mathrm{L}_b(H) = 2^{\{u,x,y\}} \leq G_V$. For every $b \in B$ one obviously has $\mathrm{R}_b(E) = E$ and $\mathrm{L}_b(E) = 2^{V \setminus b}$. More generally, any lift or restriction turns out to be a group, and further one has:

**Lemma 7** *For $b \in B$, and any non-trivial subgroups $G \leq G_V$, $H \leq G_b$, one has:*

*(i)* $\mathrm{R}_b(G) \leq G_b$, $e(\mathrm{R}_b(G)) \leq \min\{e(G), b\}$, $H \leq \mathrm{L}_b(H) \leq G_V$.

*(ii)* $G \leq \mathrm{L}_b(\mathrm{R}_b(G))$, $H = \mathrm{R}_b(\mathrm{L}_b(H))$.

*(iii)* $\mathrm{Gn}(\mathrm{L}_b(H)) = \{\{x\} : x \in V \setminus b\} \cup \mathrm{Gn}(H)$.

PROOF. Recall that $|G| = 2^e$, where $e := e(G) \geq 0$, as a subgroup of $G_V$. As $b \in G_V$, one has $\{X \cap b : X \in G\} \subseteq G_V \cap G_b$ hence $|\mathrm{R}_b(G)| \leq G_b$, and also $|\mathrm{R}_b(G)| \leq 2^e$. Moreover, $\mathrm{R}_b(G)$ is a subgroup of $G_b$. Indeed, for any $X, Y^{-1} = Y \in G$, let $X_b := X \cap b$, $Y_b := Y \cap b \in \mathrm{R}_b(G)$ then one has

$$\begin{aligned} X_b \oplus Y_b &= (X_b \cup Y_b) \setminus (X_b \cap Y_b) \\ &= (X \cup Y) \cap b \setminus (X \cap Y) \cap b \\ &= (X \oplus Y) \cap b \end{aligned}$$

which can be verified easily. Thus $X_b \oplus Y_b \in \mathrm{R}_b(G)$ being a subgroup. Hence there is $e_b := e(\mathrm{R}_b(G)) \geq 0$ such that $e_b \leq \min\{b, e\}$ and $|\mathrm{R}_b(G)| = 2^{e_b}$. Next, choose arbitrary $X, Y \in G_V$ such that as previously defined $X_b, Y_b \in H$. Then reversing the sequence of equations above one directly obtains $(X \oplus Y) \cap b = X_b \oplus Y_b \in H$ implying $X \oplus Y \in \mathrm{L}_b(H)$ thus being a subgroup of $G_V$. Further, $H \subseteq G_b \subseteq G_V$ therefore $H \subseteq \mathrm{L}_b(H)$ thus $H \leq \mathrm{L}_b(H)$, hence (i) is verified. Both assertions in (ii) are obvious. For (iii), observe that $H \leq \mathrm{L}_b(H)$, as $H \leq G_b \leq G_V$ therefore $\mathrm{Gn}(H) \subseteq \mathrm{Gn}(\mathrm{L}_b(H))$, as by assumption $H \neq E$. The elements of $H$ are the only members of $\mathrm{L}_b(H)$ which are subsets of $b$. Additionally, for fixed $X \in H$ one has $X \cup Y = X \oplus Y \in \mathrm{L}_b(H)$ for every $Y \in 2^{V \setminus b}$. As $\mathrm{Gn}(2^{V \setminus b}) = \{\{x\} : x \in V \setminus b\}$, (iii) is verified, because by assumption $\mathrm{Gn}(H) \neq \{\varnothing\}$. □

Note that in general the first part $G \leq \mathrm{L}_b(\mathrm{R}_b(G))$ of (ii) above cannot be sharpened to $G = \mathrm{L}_b(\mathrm{R}_b(G))$, which can be seen via $\mathrm{L}_b(\mathrm{R}_b(E)) = 2^{V \setminus b}$ and $E < 2^{V \setminus b}$. From part (iii) and its proof above one obtains:

**Corollary 2** *$\mathrm{L}_b(H) = \bigcup_{Y \in 2^{V \setminus b}} YH$ as the union of cosets in $G_V/H$. for every not necessarily non-trivial $H \leq G_b$.*

However, the first relation in part (ii) of Lemma 7 can be sharpened by replacing its right hand side with the smallest group contained in all restriction lifts of a fixed group, so one obtains:

**Proposition 1** *Let $G \leq G_V$ then*

*(a) $G \leq \bigcap_{b \in B} \mathrm{L}_b(\mathrm{R}_b(G))$.*

*(b) If there is $U \subseteq V$ with $G = 2^U$ then*

$$G = \bigcap_{b \in B} \mathrm{L}_b(\mathrm{R}_b(G))$$

PROOF. According to Lemma 7 it is $G \leq \mathrm{L}_b(\mathrm{R}_b(G))$ for all $b \in B$ therefore $G \leq \bigcap_{b \in B} \mathrm{L}_b(\mathrm{R}_b(G))$. For (b), let $Y \in \bigcap_{b \in B} \mathrm{L}_b(\mathrm{R}_b(G))$ then $Y \cap b \in \mathrm{R}_b(G)$, hence there is $X(b) \in G$ with $Y \cap b = X(b) \cap b$, for all $b \in B$. Clearly as $\bigcup B = V$ one has $Y = Y \cap V = Y \cap \bigcup B = \bigcup_{b \in B}(Y \cap b) = \bigcup_{b \in B}(X(b) \cap b)$. On the other hand, $X(b) \cap b \subseteq \bigcup \mathrm{Gn}(G)$, for all $b \in B$. Since by assumption $\mathrm{Gn}(G) = \{\{x\} : x \in U\}$, thus $\bigcup \mathrm{Gn}(G) = U$, it follows that $\bigcup_{b \in B}(X(b) \cap b) = Y \in G$, and with (a) one obtains (b). □

As an example that in general statement (a) does not hold true as an equation take the following example of a Sperner base hypergraph, even more restricted as consisting of disjoint base points, only: Let $V = \{u, x, y, z\}$, $B = \{b_1, b_2\}$ where $b_1 = \{u, x\}, b_2 = \{y, z\}$. Let $G := \{\varnothing, \{u, y\}, \{u, z\}, \{y, z\}\} \leq G_V$. Then $H_1 := \mathrm{R}_{b_1}(G) = 2^{\{u\}}$ and $H_2 := \mathrm{R}_{b_2}(G) = 2^{\{y,z\}}$. Further one has $\mathrm{L}_{b_1}(H_1) = 2^{\{u,y,z\}}$ and $\mathrm{L}_{b_2}(H_2) = 2^V$ hence $\mathrm{L}_{b_1}(H_1) \cap \mathrm{L}_{b_2}(H_2) = 2^{\{u,y,z\}} > G$. However, note that the converse of statement (b) above does not hold true in general by considering a Sperner counterexample: Let again $V = \{u, x, y, z\}$, $B = \{b_1, b_2\}$, but where now $b_1 = \{u, x, y\}, b_2 = \{u, x, z\}$. Let $G := \{\varnothing, \{u, x\}\} \leq G_V$. Then $H_1 := \mathrm{R}_{b_1}(G) = G = \mathrm{R}_{b_2}(G) =: H_2$. Further one has $\mathrm{L}_{b_1}(H_1) = \{\varnothing, \{z\}, \{u, x\}, \{u, x, z\}\}$ and $\mathrm{L}_{b_2}(H_2) = \{\varnothing, \{y\}, \{u, x\}, \{u, x, y\}\}$ hence $\mathrm{L}_{b_1}(H_1) \cap \mathrm{L}_{b_2}(H_2) = G$. An obvious but useful fact directly following from the definitions is stated next.

**Fact 1** *For $b \in B$, and a fibre formula $C_b \subseteq W_b$ the stabilizers of the fibre and the total levels are related as $\mathrm{L}_b(G_b(C_b)) = G_V(C_b)$.*

For a subgroup $G \leq G_V$, and $c \in K_{\mathcal{H}}$ let $\mathcal{O}_G(c) := \mathcal{O}_G(\{c\}) = \{c^X : X \in G\}$ denote the $G$-orbit of $c$. If $G = G_V$ we write $\mathcal{O}(c)$ instead of $\mathcal{O}_G(c)$. The ($G$-)orbit of a clause can be regarded as a formula. The ($G$-)orbit of a formula $C$ naturally yields a subclass of CNF, namely $\mathcal{O}_G(C) := \{C^X : X \in G\}$. Of specific interest however is a closely related class which is obtained from a formula via collecting the $G$-orbits of its clauses in the following sense.

**Definition 3** *Let $c \in K_{\mathcal{H}}$.*

*(1) The fibre formula $\mathcal{O}_G(c) \subset W_{V(c)}$ is called the $G$(-orbit)-formula (of $c$).*

*(2) For a fixed $\mathcal{H}$-based formula $C$, let $\mathcal{C}_G(C) := \{\mathcal{O}_G(c) : c \in C\}$ denote the $G$(-orbit)-family (of $C$), whereas $\mathcal{O}_G(C)$ is refered to as the $G$(-orbit)-class (of $C$).*

Some useful fact are collected next regarding $G$-formulas and their stabilizers on the fibre and the total levels.

**Lemma 8** *Let $G \leq G_V$, $b \in B$, $H \leq G_b$, and $c \in W_b$ then one has:*

*(i) $\mathcal{O}_G(c) \subseteq W_b$ and $W_b = \mathcal{O}(c)$. Specifically, $W_b$ is bijective to $G_b$, $b \in B$.*

*(ii) $G_b(c) = E$, $G_V(c) = 2^{V \setminus b}$.*

*(iii) $|\mathcal{O}_H(c)| = |H|$, $G_b(\mathcal{O}_H(c)) = H$.*

*(iv) $|\mathcal{O}_G(c)| = |\mathrm{R}_b(G)|$, $G_V(\mathcal{O}_G(c)) = \mathrm{L}_b(\mathrm{R}_b(G))$.*

PROOF. As $W_b = \{c \in K_{\mathcal{H}} : V(c) = b\}$ one has for given $c \in W_b$ and any $X \in G_V$ that $V(c^X) = V(c^{X \cap b}) = V(c)$ hence $\mathcal{O}(c) \subseteq W_b$, and also $\mathcal{O}_G(c) = \mathcal{O}_{\mathrm{R}_b(G)}(c) \subseteq W_b$. As $|\mathcal{O}(c)| = |G_V \cap 2^b| = |2^b| = |W_b|$, we have $\mathcal{O}(c) = W_b$ and $|W_b| = |G_b|$. The first equation of (ii) is obvious, as $c \neq \varnothing$. In view of Fact 1 one has $G_V(c) = \mathrm{L}_b(E)$ yielding the second equation. The first equation of (iii) is implied by (ii) as $E \cap H = E$. For the second one, obviously $H \leq G_b(\mathcal{O}_H(c))$ is true as $H$ operates transitive on its orbit. Reversely, for a non-trivial member $Y \in G_b(\mathcal{O}_H(c))$ of the stabilizer of the $G$-formula of $c$, one specifically has $Y \in G_b$, $b \cap Y \neq \varnothing$, and $c^Y \in \mathcal{O}_H(c)$. Further observe that if there was $Z \in G_b$ with $c^Y = c^Z$ then $c = c^{Y \oplus Z}$, hence $Y \oplus Z = \varnothing$ meaning $Y = Z$, therefore $Y \in H$, thus (iii). As stated in the proof of (i) one has $\mathcal{O}_G(c) = \mathcal{O}_{\mathrm{R}_b(G)}(c)$. So, identifying $H := \mathrm{R}_b(G) \leq G_b$ the first equation of (c) yields the first one of (iv). The second equation here finally is implied by the second one of (iii) together with Fact 1. □

On basis of the fibre-decomposition $C = \bigcup_{b\in B} C_b$ of a formula, it is possible to characterize the stabilizer of a $G$-familie with respect to its fibre-wise subfamilies as follows:

**Proposition 2** *Let $G \leq G_V$ and $C$ be $\mathcal{H}$-based then:*

*(a) $\mathcal{C}_G(C) = \bigcup_{b\in B} \mathcal{C}_b$, with $\varnothing \neq \mathcal{C}_b := \{\mathcal{O}_G(c) : c \in C_b\}$.*

*(b) $G_V(\mathcal{C}_G(C)) = \bigcap_{b\in B} G_V(\mathcal{C}_b)$.*

Proof. Since $C$ is $\mathcal{H}$-based, $\mathcal{C}_b \neq \varnothing$ is true for every $b \in B$. Let $G \leq G_V$ then $\mathcal{C}_G(C) = \{\mathcal{O}_G(c) : c \in C\} = \bigcup_{b\in B}\{\mathcal{O}_G(c) : c \in C_b\}$ because the $G$-orbit of every $c \in C$ with $V(c) = b$ is a fibre formula $\mathcal{O}_G(c) \subseteq W_b$. Now (a) directly implies the subgroup relation $\bigcap_{b\in B} G_V(\mathcal{C}_b) \leq G_V(\mathcal{C}_G(C))$. Reversely, let $X \in G_V(\mathcal{C}_G(C))$ and $D \in \mathcal{C}_b$, for arbitrary $b \in B$. Then there is $d \in C_b$ with $D = \mathcal{O}_G(d)$. Therefore $D^X = \mathcal{O}_G(d)^X = \mathcal{O}_G(d^X) \in \mathcal{C}_G(C)$. Thus $d^X \in C_b$ as $V(d) = b = V(d^X)$ implying $D^X \in \mathcal{C}_b$ yielding $X \in G_V(\mathcal{C}_b)$, for every $b \in B$, proving (b). □

**Definition 4** *Given $c, c' \in W_b$ then due to Lemma 8 there exists, by transitivity, a unique* transition member *$Y(c, c') := V(c \oplus c') \in G_b$ with $c' = c^{Y(c,c')}$ , where $c \oplus c'$ is regarded as a set of literals.*

**Proposition 3** *Let $C$ be $\mathcal{H}$-based and $G \leq G_V$.*

*(a) $|\mathcal{O}_G(C)| = |G/[G \cap G_V(C)]|$ and $|\mathcal{C}_G(C)| = |C|$.*

*(b) $G \leq G_V(\mathcal{O}_G(C))$ and $G \leq G_V(\mathcal{C}_G(C))$.*

Proof. The first equation is a direct consequence of the orbit rule of group theory as $G \cap G_V(C) \leq G$. The second equation of (a) is obvious, as $C$ is assumed to be non-empty. For (b), recall Def. 1, and let $Y \in G$. For arbitrary $D \in \mathcal{O}_G(C)$, there is $X \in G$ with $D = C^X$, thus $D^Y = C^{Y\oplus X} \in \mathcal{O}_G(C)$, so $Y \in G_V(\mathcal{O}_G(C))$. Next let $D \in \mathcal{C}_G(C)$, so there is $c \in C$, $V(c) = b$ such that $D = \mathcal{O}_G(c)$, hence $D^Y = \mathcal{O}_G(c) \in \mathcal{C}_G(C)$ because $G \leq \mathrm{L}_b(\mathrm{R}_b(G)) = G_V(\mathcal{O}_G(c))$ due to Lemmata 7 and 8, so one concludes $Y \in G_V(\mathcal{C}_G(C)$. □

Note that in general $G \neq G_V(\mathcal{O}_G(C))$ which can be seen already in the case of one fibre only, i.e., $V = b$: Let $b := \{u, x, y, z\}$ and consider the clause $c := \{u, x, \bar{y}, \bar{z}\} \in W_b$. Let $\mathrm{Gn}(H) := \{\{u, x\}, \{y, z\}\}$, and consider the $H$-formula $\mathcal{O}_H(c) = \{c, d, e, f\} =: C \subseteq W_b$ then $d := c^{\{u,x\}}$, $e := c^{\{y,z\}}$, $f := c^{\{u,x,y,z\}}$. Setting $G := \{\varnothing, \{y\}\}$ yields $\mathcal{O}_G(C) = \{C, C^{\{y\}}\}$. But one verifies that $c^{\{y\}} = e^{\{z\}}$, $f^{\{y\}} = d^{\{z\}}$ implying $C^{\{z\}} = C^{\{y\}}$, and also $C^{\{y,z\}} = C$. Thus $\{z\} \in G_V(\mathcal{O}_G(C)) > G$. A similar argument shows that generally $G_V(\mathcal{C}_G(C)) \geq G$. However for the case $G = G_V$ one has $G_V(\mathcal{O}(C)) = G_V$, and $G_V(\mathcal{C}_{G_V}(C)) = G_V$, by corresponding transitivity arguments.

**Lemma 9** *For a $\mathcal{H}$-based formula $C$, and $G \leq G_V$ one has:*

*(i) $\bigcup \mathcal{O}_G(C) = \bigcup \mathcal{C}_G(C)$.*

*(ii) $G \leq G_V(\bigcup \mathcal{O}_G(C))$, $G \leq G_V(\bigcup \mathcal{C}_G(C))$.*

Proof. The membership of a $c \in \bigcup \mathcal{O}_G(C)$ is equivalent with the existence of an $X \in G$ such that $c \in C^X$. Which is the same as that there is $d \in C$ with $c = d^X$ equivalent with $c \in \mathcal{O}_G(d)$ which is the same as $c \in \bigcup \mathcal{C}_G(C)$ yielding (i). Next, for fixed $Y \in G$, one has by transitivity $\mathcal{O}_G(C) = \mathcal{O}_G(C^Y)$ thus $[\bigcup \mathcal{O}_G(C)]^Y = \bigcup_{X\in G} C^{X\oplus Y} = \bigcup \mathcal{O}_G(C^Y) = \bigcup \mathcal{O}_G(C)$. So, $G \leq G_V(\bigcup \mathcal{O}_G(C))$, and the part (ii) of the Lemma follows from its first assertion. □

Specializing on a quite restricted type of $\mathcal{H}$-based formulas, the fibre-transversals the following further connections between orbit classes and families are valid:

**Proposition 4** *Let $F \in \mathcal{F}(K_{\mathcal{H}})$ and $G \leq G_V$.*

*(a) All members from $\mathcal{C}_G(F)$ are mutually disjoint fibre formulas, not necessarily of equal size. The members of $\mathcal{O}_G(F)$ all are distinct fibre-transversals, of equal size but in general not mutually disjoint.*

*(b) Every member of $\mathcal{O}_G(F)$ is a transversal of $\mathcal{C}_G(F)$, and vice versa. Meaning that each member of $\mathcal{O}_G(F)$ is constituted of exactly one clause from every member of $\mathcal{C}_G(F)$, and vice versa.*

Proof. Assume that there are distinct clauses $F(b) =: c$, $F(b') =: c'$ of $F$ with $d \in \mathcal{O}_G(c) \cap \mathcal{O}_G(c')$. Then $b = V(d) = b'$ because $\mathcal{O}_G(c) \subseteq W_b$, $\mathcal{O}_G(c') \subseteq W_{b'}$ according to Lemma 8 (i). So, a contradiction to $b \neq b'$ is obtained as $F$ is a fibre transversal. Hence $\mathcal{C}_G(F)$ consists of mutually disjoint members. According to Lemma 8 (iv) the size of a $G$-formula $\mathcal{O}_G(c)$ depends on $V(c) \in B$, hence may vary over $B$. Next according to Theorem 5 (i), in [17], one has $F^X \in \mathcal{F}(K_{\mathcal{H}})$, for every $X \in G_V$, and due to (ii) of the same result one has $F^X \neq F$, for any non-trivial $X \in G_V$, hence $F^X \neq F^Y$, for distinct $X, Y \in G_V$. Both statements imply that $\mathcal{O}_G(F)$ consists of distinct fibre-transversals. The total number of clauses in $\mathcal{O}_G(F)$ in the sense of the sum of the sizes of its members is $|F| \cdot |G|$ according to Prop. 3. From (a) together with Lemma 9 and Lemma 8 (iv) it follows that $|F| \cdot |G| \geq |F| \cdot (\min_{b\in B} |\mathrm{R}_b(G)|) \geq \sum_{c\in F} |\mathcal{O}_G(c)| = |\bigcup \mathcal{C}_G(F)| = |\bigcup \mathcal{O}_G(F)|$. So, if there is $b \in B$ with $|\mathrm{R}_b(G)| < |G|$ then $|F| \cdot |G| > |\bigcup \mathcal{O}_G(F)|$, and in this case the members of $\mathcal{O}_G(F)$ cannot be mutually disjoint, finishing the proof of (a). Finally, let $C \in \mathcal{O}_G(F)$,

$C' \in \mathcal{C}_G(F)$ be chosen arbitrarily. Then there are unique $X \in G$ and $c \in F$ such that $C = F^X$ and $C' = \mathcal{O}_G(c)$. Therefore one obtains $C \cap C' = F^X \cap \mathcal{O}_G(c) = \{c^X\}$. Thus $|C \cap C'| = 1$ for every pair $(C, C') \in \mathcal{O}_G(F) \times \mathcal{C}_G(F)$, hence (b). □

For the stabilizers of $G$-orbit classes resp. $G$-orbit families of fibre-transversals one obtains:

**Theorem 3** *Let $G \le G_V$ be an abitrary subgroup, and $F \in \mathcal{F}(K_{\mathcal{H}})$.*

*(a) $G$ is the stabilizer of $\mathcal{O}_G(F)$, $\bigcup \mathcal{O}_G(F)$, $\bigcup \mathcal{C}_G(F)$.*

*(b) $G_V(\mathcal{C}_G(F)) = \bigcap_{b\in B} \mathrm{L}_b(\mathrm{R}_b(G))$.*

*(c) If there is $U \subseteq V$ with $G = 2^U$ then*

$$G_V(\mathcal{C}_G(F)) = G$$

.

Proof. Due to Prop. 3 one has $G \le G_V(\mathcal{O}_G(F))$. Moreover, $G \le G_V(\bigcup \mathcal{O}_G(F)) = G_V(\bigcup \mathcal{C}_G(F))$ according to Lemma 9. For considering the reverse inclusions of (a), first fix an arbitrary non-trivial member $Y \in G_V(\mathcal{O}_G(F))$. Then one specifically has $F^Y \in \mathcal{O}_G(F)$. Suppose that $Y \notin G$ then there is $X \in G$ such that $F^X = F^Y$ thus $F = F^{X\oplus Y}$, hence $X \oplus Y = \varnothing$. The last implication is valid here because $X \oplus Y$ cannot yield a permutation of the clauses in $F$ as they belong to mutually distinct fibres. So, it follows that $Y = X$ yielding a contradiction, therefore $Y \in G$, meaning $G_V(\mathcal{O}_G(F)) = G$. Now, let $C := \bigcup \mathcal{O}_G(F)$, and $Y \in G_V(C)$ then $Y \in G_V$. Let $D \in \mathcal{O}_G(F)$ be arbitrary then there is $X \in G$ with $D = F^X$ therefore $D^Y = F^{X\oplus Y}$ yielding a fibre-transversal. On the other hand $c^Y \in C$, for every $c \in D$, thus $D^Y \subset C$. It is claimed that now one obtains $D^Y \in \mathcal{O}_G(F)$ implying $Y \in G_V(\mathcal{O}_G(F)) = G$, as shown above, yielding $G = G_V(\bigcup \mathcal{O}_G(F))$. To verify the claim, suppose that there are $d_i \in D$, with $b_i := V(d_i)$, such that $d_i^Y \in F^{Z_i}$, $Z_i \in G$, $i = 1, 2$. As $D = F^X$, there are $c_i \in F$ such that $d_i = c_i^X$. Thus one has $c_i^{X\oplus Y} \in F^{Z_i}$, meaning $c_i^{X\oplus Y} = c^{Z_i}$, as only $c_i$ is a clause here over $b_i$, implying $X \oplus Y = Z_i$, $i = 1, 2$. So, $Z_1 = Z_2$ verifying the claim. Finally, $G = G_V(\bigcup \mathcal{C}_G(F))$ is implied by Lemma 9, so (a) is verified. Regarding (b) observe that $\mathcal{C}_b = \{\mathcal{O}_G(F(b))\}$ is exactly the collection of all members in $\mathcal{C}_G(F)$ over the base point $b$, for every $b \in B$, and $\mathcal{C}_G(F) = \bigcup_{b\in B} \mathcal{C}_b$. Clearly $G_V(\mathcal{C}_b) = G_V(\{\mathcal{O}_G(F(b))\}) = G_V(\mathcal{O}_G(F(b))) = \mathrm{L}_b(\mathrm{R}_b(G))$ where the last equality is implied by Lemma 8 (iv) because $\mathcal{O}_G(F(b))$ is a $G$-formula. Therefore in view of Prop. 2 it is $G_V(\mathcal{C}_G(C)) = \bigcap_{b\in B} G_V(\mathcal{C}_b) = \mathrm{L}_b(\mathrm{R}_b(G))$ yielding assertion (b). Finally assertion (c) is implied by (b) and Prop. 1. □

As any linear formula is assumed to be free of complementary unit clauses, it specifically is a fibre-transversal yielding the next result on basis of Theorem 3:

**Corollary 3** *If $G \le G_V$ and $C \in$ LCNF with $\mathcal{H}(C) = \mathcal{H}$, then $G = G_V(\mathcal{O}_G(C)) = G_V(\bigcup \mathcal{O}_G(C)) = G_V(\bigcup \mathcal{C}_G(C))$. Further $G_V(\mathcal{C}_G(C)) = \bigcap_{b\in B} \mathrm{L}_b(\mathrm{R}_b(G))$, and if there is $U \subseteq V$ with $G = 2^U$ then $G_V(\mathcal{C}_G(C)) = G$.*

## 5 Formula Stabilizers: The Fibre Case

The specific case of fibre formulas is studied next. We aim at providing an algorithm for computing the stabilizer of an arbitrary fibre formula generalizing Lemma 8 valid for $(G)$-formulas only. So, throughout this section, let $C$ be a non-empty fibre formula meaning $C \subseteq W_b$ where $b := V(C)$, and let $E \ne H \le G_b$ be a proper subgroup of the complementation group.

**Lemma 10** *(i) For two distinct $H$-orbits, $\mathcal{O} := \mathcal{O}_H(c), \mathcal{O}' := \mathcal{O}_H(c')$, $c, c' \in W_b$, there is exactly one $X \in G_b \setminus H$ which is composed of generators in $\mathrm{Gn}(G_b) \setminus \mathrm{Gn}(H)$ only, such that $\mathcal{O}^X = \mathcal{O}'$. This unique $X$ is called the* primitive (orbit) transition element*.*

*(ii) Let $C = \bigcup_{i\in[s]} \mathcal{O}_i$ be the union of $s \ge 1$ disjoint $H$-orbits: $\mathcal{O}_i := \mathcal{O}_H(c_i)$, where $c_i \in C \subseteq W_b$. Then every $X \in G_b(C) \setminus H$ provides a non-trivial 2-regular permutation $\pi_X$ of $[s]$ with $\mathcal{O}_i^X = \mathcal{O}_{\pi_X(i)}$, $i \in [s]$.*

Proof. Let $c \in \mathcal{O}$, $c' \in \mathcal{O}'$, where the orbits are assumed to be distinct. Then $Y(c, c') \in G_b \setminus H$ recalling Def. 4. By transitivity for every $c_i \in \mathcal{O}$ there is a unique $X_i \in H$ such that $c^{X_i} = c_i$ yielding the unique member $c_i^{Y(c,c')} = c'^{X_i} \in \mathcal{O}'$. Thus $Y(c, c')$ provides a bijection from $\mathcal{O}$ to $\mathcal{O}'$ meaning $\mathcal{O}^{Y(c,c')} = \mathcal{O}'$. As $Y(c, c') \subseteq b$ and $\mathrm{Gn}(H) \subseteq 2^b$ we have $X := Y(c, c') \setminus \bigcup \mathrm{Gn}(H) \subseteq b$. Clearly, also $X \in G_b \setminus H$ and so $\mathcal{O}^X = \mathcal{O}'$ as above. Moreover $X$ is unique: Let $\tilde{c} \in \mathcal{O}$, $\tilde{c}' \in \mathcal{O}'$ be another pair of clauses then there are unique $\tilde{Y}, \tilde{Y}' \in H$ with $\tilde{c}^{\tilde{Y}} = c$, $\tilde{c}'^{\tilde{Y}'} = c'$. It follows that $Y(\tilde{c}, \tilde{c}') = \tilde{Y} \oplus Y(c, c') \oplus \tilde{Y}'$ implying $Y(\tilde{c}, \tilde{c}') \setminus \bigcup \mathrm{Gn}(H) = Y(c, c') \setminus \bigcup \mathrm{Gn}(H) = X$. For proving (2), let $G := G_b(C)$ be the isotropy group of $C$ on the fibre level, and $X \in G \setminus H$. According to (1), the $H$-orbits $\mathcal{O}_i^X$, $i \in [s]$, are pairwise different, and their union must yield $C$ as $X \in G$. Thus $X$ induces a bijection $\pi_X$ on $[s]$ such that $\mathcal{O}_i^X = \mathcal{O}_{\pi_X(i)} \subseteq C$, $i \in [s]$. As $< X >$ is cyclic of order 2, this permutation decomposes into disjoint transpositions, i.e., 2-cycles, namely $\pi_X = (i_1, \pi_X(i_1)) \cdots (i_r, \pi_X(i_r))$, for $r = s/2$, $i_j = \min([s] \setminus \{i_k, \pi(i_k) : k \in [j-1]\})$, for all $j \in [r]$, implying (2). □

Observe that under the assumptions in Lemma 10 one has $H \le G_b(C)$, so the previous proof directly implies:

**Corollary 4** *Let $C$ be the union of $s \ge 1$ disjoint $H$-orbits of clauses in $W_b$, for an integer $s$. If there is $X \in G_b(C) \setminus H$, then $s$ is even.*

**Theorem 4** *The isotropy group $G_b(C)$ of a fibre formula $C \subseteq W_b$ can be computed in time $O(|b|^2 \cdot |C|^2 \cdot \log^2 |C|)$ as a direct product of cyclic subgroups.*

PROOF. Again let $G := G_b(C)$. If $C = \varnothing$ we have $G = G_b$. Otherwise compute $G$ by iteratively enlarging the number of factors in the current direct product of cyclic groups $H$, as long as there is a new generator $X \in \mathrm{Gn}(G)$ yielding the next factor $< X >$. Initially setting $H := E$, $C$ can be regarded as the union of $s := |C| \geq 1$ pairwise disjoint $H$-orbits $\mathcal{O}_i := \{c_i\}$, $i \in [s]$. If $s = 1 \mod 2$ the procedure stops with $G := H$ according to Corollary 4. Otherwise, one has to check in the current iteration whether there is $X \in G \setminus H$. To that end, let $c_i$ be an arbitrary member of the orbit $\mathcal{O}_i$, $i \in [s]$. Considering these clauses as the vertices of a complete graph $K_s$, we label every edge $c_i - c_j$ by its unique primitive orbit transition member $X_{i,j} := Y(c_i, c_j) \setminus \bigcup \mathrm{Gn}(H)$, $i, j \in [s], i < j$, where $\mathrm{Gn}(H)$ is the generator set in the current iteration. Then due to Lemma 10 our problem is equivalent to identify a perfect matching in $K_s$ such that all its members carry an equal label $X$ which therefore belongs to $G$, as it provides a bijection of the current set of orbits. One might implement a clever version of a minimum weight perfect matching algorithm, which however for dense graphs is rather slow. On the other hand we do not need a matching, only a suitable label which can be determined faster as follows: By the lexicographic order, based on a pre-ordering of the variables in $b$, sort all primitive orbit transition elements $X_{i,j}$ in a sequence $T$. Now equal labels are grouped together. Finally linearly pass through $T$ in search for a first consecutive subsequence $t = (X_{i_1,j_1}, \ldots, X_{i_r,j_r})$ of $T$ having the following properties: (i) $|t| = s/2 =: r$, (ii) all its elements are equal to an $X$, and (iii) $\sum_{k=1}^{r}(i_k + j_k) = s(s+1)/2$. Observe that these conditions ensure that the corresponding edges $c_{i_k} - c_{j_k}$, $k \in [r]$, form a perfect matching in $K_s$ of equal label $X$. The lexicographic sorting of $O(s^2)$ labels can be executed in time $O(|b|^2 \cdot s^2 \cdot \log s)$ dominating the time amount for computing the primitive transition elements relying on Lemma 1, as well as the time amount for the subsequence search. If there is a subsequence as required yielding label $X$, then set $H \leftarrow H \times < X >$ and join each pair $\mathcal{O}_i$, $\mathcal{O}_{\pi_X(i)}$ of the current orbits to the new $H$-orbit $\mathcal{O}_i \cup \mathcal{O}_{\pi_X(i)}$ according to Lemma 10 (b). Then $s \leftarrow s/2$ is the new number of orbits. Otherwise, the procedure stops with $G := H$. The joining operation clearly is dominated by the sorting bound as stated above. As every newly added cyclic group factor corresponds to exactly one generator of the isotropy group we have at most $|b|$ such iterations due to Lemma 1. On the other hand the number of iterations is bounded by $\log |C| \leq |b|$ because of the repeated joining process. So the overall upper bound for the time complexity amounts to $O(|b|^2 \cdot |C|^2 \cdot \log^2 |C|)$. $\square$

Regarding classes of arbitrary fibre formulas over the same fibre one has.

**Proposition 5** *Let $\mathcal{C}_b \subseteq 2^{W_b} \subset \mathrm{CNF}$ be a non-empty class of fibre formulas over the same base point $b$, i.e., $B(C) = \{b\}$, for all $C \in \mathcal{C}_b$. Then*

$$\bigcap_{C \in \mathcal{C}_b} G_b(C) \leq G_b(\mathcal{C}_b) \leq G_b\left(\bigcup \mathcal{C}_b\right)$$

*where in general equality does not hold true in either relation.*

PROOF. Let $X \in \bigcap_{C \in \mathcal{C}_b} G_b(C)$ and $D \in \mathcal{C}_b$ then $X \in G_b(D)$ and $D^X \in \mathcal{C}_b$ thus $\bigcap_{C \in \mathcal{C}_b} G_b(C) \leq G_b(\mathcal{C}_b)$. Let $c, c' \in W_b$ and $H < G_b$ be a proper subgroup such that $C := \mathcal{O}_H(c) \neq \mathcal{O}_H(c') =: C'$ and set $\mathcal{C}_b := \{C, C'\}$. According to Corollary 4 one has $G_b(C) = G_b(C') = H$ thus $\bigcap_{C \in \mathcal{C}_b} G_b(C) = H$. On basis of Lemma 10 (a) there is the primitive orbit transition member $X \in G_b \setminus H$ such that $C^X = \mathcal{O}_H(c)^X = \mathcal{O}_H(c') = C'$ meaning $X \in G_b(\mathcal{C}_b) > H$, establishing the first proper subgroup relation above. Next assume that $X \in G_b(\mathcal{C}_b)$ then $[\bigcup \mathcal{C}_b]^X = \bigcup_{C \in \mathcal{C}_b} C^X = \bigcup \mathcal{C}_b$ hence $G_b(\mathcal{C}_b) \leq G_b(\bigcup \mathcal{C}_b)$. Let $c \in W_b$ and set $\mathcal{C}_b := \{\{c\}, W_b \setminus \{c\}\}$ meaning $G_b(\bigcup \mathcal{C}_b) = G_b(W_b) = G_b$. Further $G_b(\mathcal{C}_b) = E$ because either member has a trivial stabilizer which is implied by the contraposition of Corollary 4 for $H = E$. Moreover there is no $X \in G_b$ which enables a permutation of both formulas in $\mathcal{C}_b$ providing an extreme counterexample to $G_b(\mathcal{C}_b) = G_b(\bigcup \mathcal{C}_b)$. $\square$

Observe that under the settings as above in general one cannot conclude that $G_b(\mathcal{C}_b) = E$ if there is a member $C \in \mathcal{C}_b$ having this property. Take e.g. $\mathcal{C}_b = \{\{c\} : c \in W_b\}$, then either member admits a trivial stabilizer but $G_b(\mathcal{C}_b) = G_b$. Instead one derives the following:

**Corollary 5** *Let $b \in B$ and let $\mathcal{C}_b$ be a non-empty class of mutually disjoint fibre formulas over the same base point $b$. Let $\mathcal{C}_b(k)$ be the collection of all members in $\mathcal{C}_b$ having equal size $k$. If there is an odd integer $s$, and an odd size $k$ such that $|\mathcal{C}_b(k)| = s$ then $G_b(\mathcal{C}_b) = E$.*

PROOF. Let $S := \{|C| : C \in \mathcal{C}_b\}$ be the set of all size values occuring in $\mathcal{C}_b$. For odd $k \in S$ and odd $s = |\mathcal{C}_b(k)|$ defining $C(k) := \bigcup \mathcal{C}_b(k)$ yields a fibre formula of odd length $s' := s \cdot k$. Further $C(k) = \bigcup_{c \in C(k)} \mathcal{O}_E(c)$ is the union of $s'$ disjoint $E$-orbits as the members of $\mathcal{C}_b(k)$ are mutually disjoint. According to the contraposition of Corollary 4 and in view of Prop. 5 one obtains $E = G_b(C(k)) \geq G_b(\mathcal{C}_b(k))$ thus $G_b(\mathcal{C}_b(k)) = E$. Now suppose there is $X \in G_b(\mathcal{C}_b)$, and let $C \in \mathcal{C}_b$ with $j = |C| \in S$, so $C \in \mathcal{C}_b(j)$. Since $|C^Y| = |C|$, for every $Y \in G_b$, it follows that also $C^X \in \mathcal{C}_b(j)$ meaning $\mathcal{C}_b(j)^X = \mathcal{C}_b(j)$, or $X \in G_b(\mathcal{C}_b(j))$, for every $j \in S$. Thus one obtains $X \in \bigcap_{j\in S} G_b(\mathcal{C}_b(j)) = E$, as the intersection joins also $G_b(\mathcal{C}_b(k)) = E$ yielding $G_b(\mathcal{C}_b) \leq E$ finishing the proof. □

**Theorem 5** *Let $b \in W_b$, $s$ be a positive integer and $\mathcal{C}_i \subseteq 2^{W_b}$, such that all its members have equal size $k_i$, $i \in [s]$. For $\mathcal{C} := \bigcup_{i\in[s]} \mathcal{C}_i$ one has $G_b(\mathcal{C}) = \bigcap_{i\in[s]} G_b(\mathcal{C}_i)$.*

PROOF. The subgroup relation $\bigcap_{i\in[s]} G_b(\mathcal{C}_i) \leq G_b(\mathcal{C})$ is obvious and the reverse relation follows as in the previous proof. □

## 6 Total Stabilizers and Class Members

This section focuses on lifting the fibre stabilizers to the total level. Specifically we investigate the stabilizers of formulas in some concrete CNF classes. To that end, again fix a base hypergraph $\mathcal{H} =: (V, B)$ and recall that $G_b(C_b) \leq G_b$ denotes the isotropy group of $C_b$ over $b$, $b \in B$.

**Lemma 11** *For a subgroup $G \leq G_V$, and $b \in B$, let $C = C_b$ be the union of $s > 0$ $G$-orbits of clauses in $W_b$. Then on basis of Def. 4, for fixed $c' \in C$ and $M_E := \bigcup_{c\in C_b} Y(c', c)$, one has*

$$G_V(C) = \begin{cases} \mathrm{L}_b(2^{M_E}), & \text{if } \log s = |M_E| - e(\mathrm{R}_b(G)) \geq 0 \\ \mathrm{L}_b(\mathrm{R}_b(G)), & \text{if } s \text{ is odd} \end{cases}$$

PROOF. Let $e := e(\mathrm{R}_b(G)) \geq 0$ hence $|\mathcal{O}_G(c)| = |\mathcal{O}_{\mathrm{R}_b(G)}(c)| = 2^e$, for every $c \in C_b$ thus $|C| = s \cdot 2^e$. If $s$ is odd, by contraposing Corollary 4 one has $G_b(C) = \mathrm{R}_b(G)$ directly implying $G_V(C) = \mathrm{L}_b(\mathrm{R}_b(G))$. Next, assume $\log s = |M_E| - e \geq 0$ where $M_E := \bigcup_{c\in C_b} Y(c', c) \in G_b$ and define $G' := \{Y(c', c) : c \in C_b\} \subset G_b$ for any fixed $c' \in C_b$. Then obviously $C_b = \{c'^X : X \in G'\}$. Hence $C_b$ equals exactly one $G'$-orbit, respectively, one $\mathrm{L}_b(G')$-orbit iff $G' \leq G_b \Leftrightarrow \mathrm{L}_b(G') \leq G_V$, which is claimed to be true. Therefore $\mathrm{L}_b(\mathrm{R}_b(G')) = \mathrm{L}_b(G')$ is the isotropy group of $C$ according to the result previously proven. To establish the claim, observe that all $Y(c', c)$ are pairwise distinct therefore $|G'| = |C_b| = s2^e$ implying $\log |G'| = \log s + e = |M_E|$. Hence $|G'| = |2^{M_E}|$ and, as every member in $G'$ is a subset of $M_E$, it follows that $G' = 2^{M_E} \leq G_b$. Here one has $G' = \mathrm{R}_b(G)$ if $\log s = 0$ which means an odd $s$, finishing the proof. □

**Corollary 6** *Let $U \subseteq V$ with $G = 2^U \leq G_V$. For $\mathcal{H}$-based $C$ such that $C_b$ is the union of $s_b > 0$ $G$-orbits of clauses in $W_b$, and such that $s_b$ is odd, for every $b \in B$ it holds that $G_V(C) = G$.*

PROOF. According to Lemma 11 $G_V(C_b) = \mathrm{L}_b(\mathrm{R}_b(G))$ because $s_b$ is odd, for all $b \in B$. Therefore by Corollary 1 it follows that $G_V(C) = \bigcap_{b\in B} \mathrm{L}_b(\mathrm{R}_b(G))$. Hence, using Prop. 1 one derives the assertion because of the assumption that $G = 2^U$ for $U \subseteq V$. □

Given $X \in G_b(C_b)$ and setting $\tau(X) := \{X \cup U : U \in 2^{V\setminus b}\}$, $\Pi_b := \bigcup_{X\in G_b(C_b)} \tau(X)$ one obtains:

**Theorem 6** *$G_V(C) = \bigcap_{b\in B} \mathrm{L}_b(G_b(C_b))$, for a $\mathcal{H}$-based formula $C$. Moreover $\mathrm{L}_b(G_b(C_b)) = \Pi_b \leq G_V$, $b \in B$.*

PROOF. Clearly $G_V(C_b) = \{X \in G_V : C_b^X = C_b\} = \{X \in G_V : X \cap b \in G_b(C_b)\} = \mathrm{L}_b(G_b(C_b)) =: \mathrm{L}_b \leq G_V$, for every $b \in B$, according to Lemma 7 (i). Therefore $G_V(C) = \bigcap_{b\in B} \mathrm{L}_b$ immediately follows on the total level relying on Corollary 1. Further one has $\bigcap_{b\in B} \mathrm{L}_b \leq G_V$ and the first statement is settled. Addressing the last assertion let $Y \in \Pi_b \subseteq 2^V$ then there is a unique $X \in G_b(C_b)$ such that $Y \in \tau(X)$. Hence there is $U \in 2^{V\setminus b}$ : $Y = X \cup U$ implying $Y \cap b = X \cap b = X \in G_b(C_b)$ as $U \cap b = \varnothing$. So $Y \in \mathrm{L}_b$. Reversely, let $Y \in \mathrm{L}_b$ then there is $X \in G_b(C_b) : Y \cap b = X$ implying $X \subseteq Y$ and $U := Y \setminus X \in 2^{V\setminus b}$. Thus $Y = X \cup U \in \tau(b)$ establishing $\mathrm{L}_b = \Pi_b$, $b \in B$, also yielding $\Pi_b \leq 2^V$. □

Further one obtains the following sufficient condition for the trivial isotropy group $E$.

**Corollary 7** *Let $C \in \mathrm{CNF}$ be $\mathcal{H}$-based such that $|C_b|$ is odd, for all $b \in B$ then $G_V(C) = E$.*

PROOF. Relying on Lemma 11 the assumption implies $G_V(C_b) = \mathrm{L}_b(E) = 2^{V\setminus b}$, for every $b \in B$, using the proof of Theorem 6. Thus due to the same theorem $G_V(C) = \bigcap_{b\in B} 2^{V\setminus b} = 2^{V\setminus \bigcup B} = 2^{\varnothing} = E$. □

**Theorem 7** *For every $\mathcal{H}$-based $C \in \mathrm{LCNF} \cup \mathrm{CNF}_+ \cup \mathrm{CNF}_-$ one has $G_V(C) = E$.*

PROOF. As by assumption members of LCNF are considered to be free of unit clauses it follows for any linear or monotone formula $C$ that $|C_b| = 1$ for every $b \in B$. Thus the assertion is implied by Corollary 7. □

In terms of the intersection graph one has for $\mathcal{S}_\pm$:

**Lemma 12** *For $\mathcal{H}$-based $C \in \mathcal{S}_\pm$, let $\{\mathcal{I}_1, \ldots, \mathcal{I}_k\}$ be the set of connected components of the intersection graph $\mathcal{I}(\mathcal{H})$ of $\mathcal{H}$. Then one has $\mathrm{Gn}(G_V(C)) = \{X_i := \bigcup_{b\in V(\mathcal{I}_i)} b : i \in [k]\}$.*

PROOF. Assume $C \in \mathcal{S}_\pm$ then $C = B \cup B^\gamma = \bigcup_{b\in B}\{b, b^\gamma\}$, where $B \in \mathrm{CNF}_+$. Hence $\mathcal{O}_{G_b(C_b)}(b) = C_b$ where $G_b(C_b) = \{\varnothing, b\} \leq G_b$ meaning that $G_V(C_b) = \mathrm{L}_b(G_b(C_b))$, for every $b \in B$, according to Lemma 11, because $\mathrm{R}_b(G_b(C_b)) = G_b(C_b)$. Let $M := \{X_i :=$

$\bigcup_{b\in V(\mathcal{I}_i)} b : i \in [k]\}$. To verify the assertion, we first show by induction that for any integer $n \geq 1$ and members $X_{i_j} \in M$, $j \in [n]$, one has $\bigoplus_{j=1}^{n} X_{i_j} \in G_V(C)$. So, given $X_i \in M$ and any $b \in B$ then either $X_i \cap b = \varnothing$ iff $b \notin V(\mathcal{I}_i)$. Or $X_i \cap b = b$ iff $b \in V(\mathcal{I}_i)$ hence $X_i \in \mathrm{L}_b(G_b(C_b))$ for every $b \in B$ meaning $M \subseteq G_V(C)$ according to Theorem 6. Next let $X_{i_j} \in M$, $i_j \in [k]$, $j \in [n]$ and assume the assertion holds true for up to $n-1$ members of $M$, $n \geq 2$. Hence, there either is $l \in [n-1]$ such that $i_l = i_n$ hence $X_{i_l} = X_{i_n}$ then $\bigoplus_{j\in[n]} X_{i_j} = \bigoplus_{j\in[n]\setminus\{l,n\}} X_{i_j} \in G_V(C)$. Or all $X_{i_j}$, $j \in [n]$, have pairwise distinct indices, hence are pairwise disjoint by construction meaning $\bigoplus_{j\in[n]} X_{i_j} = \bigcup_{j\in[n]} X_{i_j} = \bigcup_{j\in[n]} \bigcup_{b\in V(\mathcal{I}_{i_j})} b$. Thus given any $b' \in B$ it either follows $\bigoplus_{j\in[n]} X_{i_j} \cap b' = b'$ iff $b' \in \bigcup_{j\in[n]} V(\mathcal{I}_{i_j})$, or this intersection is empty implying $\bigoplus_{j\in[n]} X_{i_j} \in G_V(C)$ according to Theorem 6. So, everything that can be generated by members of $M$ belongs to $G_V(C)$. Reversely, any $X \in G_V(C)$ induces a bipartition $B'(X) \cup B(X) = B = V(\mathcal{I}(\mathcal{H}))$ of the vertex set of $\mathcal{I}(\mathcal{H})$ defined through $X \cap b = \varnothing$, for all $b \in B'(X)$, and $X \cap b = b \neq \varnothing$, for all $b \in B(X)$. Further this bipartition equals an empty cut in $\mathcal{I}(\mathcal{H})$, indeed, otherwise there were $b' \in B'$ and $b \in B$ such that $\varnothing \neq b' \cap b = (b' \cap X) \cap b$ implying $X \cap b' \neq \varnothing$ hence a contradiction. Therefore given $i \in [k]$ one either has $V(\mathcal{I}_i) \subseteq B'(X)$, then set $i \in [k]'(X)$. Or one has $V(\mathcal{I}_i) \subseteq B(X)$, then set $i \in [k](X)$, yielding a bipartition of the index set $[k] =: [k]'(X) \cup [k](X)$ implying $X = \bigoplus_{i\in[k](X)} X_i$. Hence every member of $G_V(C)$ can be generated by elements in $M$ finishing the proof. □

The next result relating the stabilizers of the symmetric and antisymmetric classes is stated in [18] here it is proven:

**Theorem 8** *For $\mathcal{H} = (V, B)$ as above one has:*

*(a) There is an $G_V$-equivariant bijection $\sigma : \mathcal{A}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$.*

*(b) Given $C \in \mathcal{A}(\mathcal{H})$ then $\mathrm{Gn}(G_V(\sigma(C))) = \mathrm{Gn}(G_V(C)) \cup \mathrm{Gn}(G_V(B \cup B^\gamma))$. Moreover for input $\mathrm{Gn}(G_V(C))$, $\mathrm{Gn}(G_V(\sigma(C)))$ can be computed in polynomial time.*

Proof. Given $C \in \mathcal{A}(\mathcal{H})$ then set $\sigma(C) := C \cup C^\gamma \in \mathcal{S}(\mathcal{H})$ which is uniquely determined by $C$. Conversely, given $S \in \mathcal{S}(\mathcal{H})$ then there is the unique subformula $A(S) \in \mathcal{A}(\mathcal{H})$ such that $A(S) \cup [A(S)]^\gamma = S = \sigma(A(S))$ hence $A(S) = \sigma^{-1}(S)$. Now let $X \in G_V$, $C \in \mathcal{A}(\mathcal{H})$ then $[\sigma(C)]^X = [C \cup C^\gamma]^X = C^X \cup (C^X)^\gamma = \sigma(C^X)$, and also $\sigma^{-1}(S^X) = [\sigma^{-1}(S)]^X$ hence $\sigma$, and $\sigma^{-1}$ are equivariant implying (a). Regarding (b) one has $G_V(C) = G_V(C^\gamma)$ because $C^\gamma \in \mathcal{O}(C)$ relying on Lemma 3. Moreover the equivariance of $\sigma$ directly implies $G_V(C) \leq G_V(\sigma(C))$. Hence, $G_V(\sigma(C)) \setminus G_V(C)$ can only consist of such elements $X \in G_V$ bijectively mapping the clauses in $C$ to the clauses in $C^\gamma$. Since $c \in C \Leftrightarrow c^\gamma \in C^\gamma$ these elements are provided by $G_V(B \cup B^\gamma)$, where $B \cup B^\gamma \in \mathcal{S}_\pm(\mathcal{H})$. Finally, the assertion regarding the computational complexity therefore is implied by Lemma 12. □

## 7 Satisfiability Properties of $G$-Orbit Classes

Returning to the $G$-formulas and -classes, some satisfiability aspects shall be addressed here, for which the following notion turns out to be useful.

**Definition 5** *For a base hypergraph $\mathcal{H} = (V, B)$, let $G \leq G_V$ be called a* fibre-wise proper *subgroup if there is a $Y \in G_V$ such that $Y \cap b \notin \mathrm{R}_b(G)$, for every $b \in B$.*

**Proposition 6** *Let $G \leq G_V$ be a fibre-wise subgroup.*

*(a) $\mathcal{O}_G(c) \in \mathrm{SAT}$, for every $c \in K_\mathcal{H}$, and*

*(b) $\mathcal{C}_G(C) \subset \mathrm{SAT}$, for every $\mathcal{H}$-based formula $C$.*

Proof. For fixed $c \in K_\mathcal{H}$ there is a unique $b := V(c) \in B$. As $\mathrm{R}_b(G)$ is a proper subgroup of $G_b$ one has $\mathcal{O}_G(c) \subset W_b$ as a proper subset. Hence the based complement formula $\overline{\mathcal{O}_G(c)} \neq \varnothing$ and due to Theorem 1 it follows that $\mathcal{O}_G(c) \in \mathrm{SAT}$, meaning (a). So, by definition $\mathcal{C}_G(C)$ only consists of satisfiable members yielding a satisfiable class of CNF, hence assertion (b) is verified. □

Even in case $C \in \mathrm{UNSAT}$, the statement (b) above is true. Also showing that in general one has $\mathcal{O}_G(C) \subset \mathrm{UNSAT}$. Moreover, in general, there is no universal model satisfying all members of an orbit-family or -class simultaneously, which would be identical with a model for the union of all member formulas. Thus the question arises for which formulas $C$, respectively non-trivial proper subgroups $G$, such a model could be expected for the corresponding classes $\mathcal{C}_G(C)$, $\mathcal{O}_G(C)$. A first result here is:

**Theorem 9** *Let $F \in \mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})$ then there is a universal model simultaneously satisfying all members of $\mathcal{O}_G(F)$, respectively of $\mathcal{C}_G(F)$, for every fibre-wise proper subgroup $G < G_V$.*

Proof. Let the union of all clauses in the $G$-orbit class of $F$ be $C := \bigcup \mathcal{O}_G(F) \subset K_\mathcal{H}$. As $G$ is fibre-wise proper, $\bar{C}$ also is $\mathcal{H}$-based. Every member of $\mathcal{O}(F)$ is a compatible fibre-transversal according to Theorem 5 (iii), shown in [17], which are mutually distinct. As $G$ is fibre-wise proper, there is $Y \in G_V \setminus G$. Hence the fibre-transversal $F' := F^Y \in \mathcal{O}(F)$ also is compatible but $F' \notin \mathcal{O}_G(F)$. According to Lemma 9 one also has $C := \bigcup \mathcal{C}_G(F)$. Since $G$ is fibre-wise proper it is $F'(b) \notin \mathcal{O}_G(F(b))$ because

$Y \cap b \notin \mathrm{R}_b(G)$, for every $b \in B$, and as $G$ acts transitively on its $G$-formulas of clauses. Therefore $F'$ contains only clauses outside of $C_b$, for every $b \in B$, directly implying $F' \in \mathcal{F}_{\text{comp}}(\bar{C})$. Thus it is $C \in \text{SAT}$ according to Theorem 1. □

Note that modifying the requirements for $F$ would disturb the assertion even for fibre-transversals which are satisfiable but not compatible. Consider the non-Sperner situation with a loop: $V = \{x, y, z\}$ $B = \{b_1, b_2, b_3\}$ where $b_1 = \{x, y\}, b_2 = \{y, z\}, b_3 = \{z\}$ and let $F(b_1) = \{x, y\}$, $F(b_2) = \{y, \bar{z}\}$, $F(b_3) = \{z\}$ obviously yielding $F \in \mathcal{F}(K_{\mathcal{H}}) \cap \text{SAT}$ being not compatible. For $G = \{\varnothing, \{x, y\}\}$, one has $\mathrm{R}_{b_1} = G$, $\mathrm{R}_{b_2} = \{\varnothing, \{y\}\}$, $\mathrm{R}_{b_3} = E$. Taking $Y := \{\{y, z\}\} \in G_V$ one has $Y \cap b_i \notin \mathrm{R}_{b_i}$, $i \in [3]$. Then $F^Y \in \mathcal{F}(\bar{C}) \cap \text{SAT}$, but not compatible, and moreover $C \in \text{UNSAT}$.

As a direct consequence of Lemma 9:

**Corollary 8** *Let $G \leq G_V$ be fibre-wise proper, and $C$ be $\mathcal{H}$-based. Then there is a universal model for $\mathcal{C}_G(C)$ iff there is one for $\mathcal{O}_G(C)$.*

## 8 Concluding Remarks and Open Problems

Theorem 9 states the existence of an universal model for orbit families over a compatible fibre-transversal for fibre-wise proper subgroups of the complementation operation. Here further investigations are necessary to detect other fibre-transversals or linear formulas admitting orbit families of that property. Also the dependence on the structure of the base hypergraph should be clarified in more detail. The structural investigation of the stabilizers of orbit classes and orbit families should be continued in order to generalize the results of Theorem 3. However note that specifically part (a) of this theorem does not hold true in general if the fibre-transversal $F$ is substituted by an arbitrary CNF formula $C$, even a fibre formula. Indeed assume that $C = C_b$ is a fibre formula, only, and let $G \leq G_V$. Suppose that $C$ satisfies the conditions of the first branch of Lemma 11. Then $\mathcal{C}_G(C) = \{\mathcal{O}_G(c) : c \in C\} =: \mathcal{C}_b$ is a $G$-family over the base point $b$. Moreover $\bigcup \mathcal{C}_G(C) = \bigcup \mathcal{C}_b = C$ and by Lemma 11 it follows that $G_V(C) = G_V(\bigcup \mathcal{C}_b) > \mathrm{L}_b(rst(G)) > G$. Observe that given $\mathcal{H} = (V, B)$ and any $F \in \mathcal{F}(K_{\mathcal{H}})$, then Lemma 4 together with Corollary 7 and Lemma 5 imply that the isotropy group jumps from all to trivial, i.e., from $G_{V(C)}$ to $E$ if one switches from $C := K_{\mathcal{H}}$ to $C' := K_{\mathcal{H}} \setminus F$, i.e., when exactly one arbitrary clause is removed from $W_b$, for all $b \in B$. These properties of formulas shall be studied more intensive. To decrease the upper bound for the time complexity in Theorem 4 is a further research task. More generally, Prop. 5 has to be sharpened and moreover it has to be investigated whether the algorithm for fibre formulas can be adapted for computing the stabilizer of a class of fibre formulas over the same fibre, also. Also the computation aspect for the liftings of the stabilizers on the fibre level to the total space has to be investigated further. It would be interesting to find more structured classes of CNF for which an explicit stabilizer can be given. Finally, the FPT-classes as considered in Theorem 2 have to be identified more concretely.

## References


[1] Aspvall, B., Plass, M.R., Tarjan, R.E., "A linear-time algorithm for testing the truth of certain quantified Boolean formulas," *Information Process. Lett.* pp. 121-123, 8/1979.

[2] Berge, C., *Hypergraphs*, North-Holland, Amsterdam, 1989.

[3] Boros, E., Crama, Y., Hammer, P.L., "Polynomial-time inference of all valid implications for Horn and related formulae," *Annals of Math. Artif. Intellig.* pp. 21-32, 1/1990.

[4] Boros, E., Crama, Y., Hammer, P.L., Sun, X., "Recognition of $q$-Horn formulae in linear time," *Discrete Appl. Math.* pp. 1-13, 55/1994.

[5] Clarke, E.M., Grumberg, O., Peled, D.A., *Model Checking*, The MIT Press, 2000.

[6] Cook, S.A., "The Complexity of Theorem Proving Procedures," *3rd ACM Symposium on Theory of Computing* pp. 151-158, 1971.

[7] Downey, R.G., Fellows, M.R., *Parameterized Complexity*, Springer-Verlag, New York, 1999.

[8] Franco, J., VanGelder, A., "A perspective on certain polynomial-time solvable classes of satisfiability," *Discrete Appl. Math.* pp. 177-214, 125/2003.

[9] Garey M.R., Johnson, D.S., *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman and Company, San Francisco, 1979.

[10] Gu, J., Purdom, P.W., Franco, J., Wah, B.W., "Algorithms for the Satisfiability (SAT) Problem: A Survey," in: D. Du, J. Gu, P. M. Pardalos (Eds.), *Satisfiability Problem: Theory and Applications*, DIMACS Workshop, March 11-13, 1996, DIMACS Series, V35, pp. 19-151, American Mathematical Society, Providence, Rhode Island, 1997.

[11] Karp, R.M., "Reducibility Among Combinatorial Problems," in: *Proc. Sympos. IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y.*, Plenum, New York, pp. 85-103, 1972.

[12] Minoux, M., "LTUR: A Simplified Linear-Time Unit Resolution Algorithm for Horn Formulae and Computer Implementation," *Information Process. Lett.* pp. 1-12, 29/1988.

[13] Knuth, D.E., "Nested satisfiability," *Acta Informatica* pp. 1-6, 28/1990.

[14] Kratochvil, J., Krivanek, M., "Satisfiability of co-nested formulas," *Acta Informatica* pp. 397-403, 30/1993.

[15] Lewis, H.R., "Renaming a Set of Clauses as a Horn Set," *J. ACM* pp. 134-135, 25/1978.

[16] Porschen, S., "A CNF Formula Hierarchy over the Hypercube," Proc. AI 2007, *LNAI* pp. 234-243, 4830/2007.

[17] Porschen, S., "Structural Aspects of Propositional SAT," Proc. IMECS 2016, Hong Kong, pp. 126-131.

[18] Porschen, S., "Base Hypergraphs and Orbits of CNF Formulas," Proc. IMECS 2018, Hong Kong, pp. 106-111/2018.

[19] Porschen, S., "Isotropy Groups of CNF Formulas," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2019, 3-5 July, 2019, London, U.K., pp. 84-89.

[20] Porschen, S., Schmidt, T., Speckenmeyer, E., Wotzlaw, A., "XSAT and NAE-SAT of linear CNF classes," *Discrete Appl. Math.* pp. 1-14, 167/2014.

[21] Porschen, S., Speckenmeyer, E., "A CNF class generalizing Exact Linear Formulas," Proc. SAT 2008, *LNCS* pp. 231-245, 2008.

[22] Porschen, S., Speckenmeyer, E., Zhao, X., "Linear CNF formulas and satisfiability," *Discrete Appl. Math.* pp. 1046-1068, 157/2009.

[23] Schlipf, J., Annexstein, F.S., Franco, J., Swaminathan, R.P., "On finding solutions for extended Horn formulas," *Information Process. Lett.* pp. 133-137, 54/1995.

[24] Spannier, E.H., *Algebraic Topology*, McGraw-Hill, New York, 1966.

[25] Stalmarck, G., Säflund, M., "Modeling and verifying systems and software in propositional logic," in Daniels, B.K. (editor), *Safety of Computer Control Systems* (SAFECOMP 90), Pergamon Press, pp. 31-36, 1990.

[26] Tovey, C.A., "A Simplified NP-Complete Satisfiability Problem", *Discrete Appl. Math.* pp. 85-89, 8/1984.