The Hamiltonicity, Hamiltonian Connectivity, and Longest \((s, t)\)-path of \(L\)-shaped Supergrid Graphs

Fatemeh Keshavarz-Kohjerdi and Ruo-Wei Hung


text continues here...

Fig. 1. (a) A grid graph, (b) a triangular grid graph, and (c) a supergrid graph, respectively. Notice that grid and triangular grid graphs as their subgraphs. The Hamiltonian \((s, t)\)-path of a graph is a Hamiltonian path between any two distinct vertices \(s\) and \(t\) in the graph, and the longest \((s, t)\)-path is a simple path with the maximum number of vertices from \(s\) to \(t\) in the graph. A graph is called Hamiltonian if it contains a Hamiltonian cycle, and is said to be Hamiltonian connected if there exists a Hamiltonian \((s, t)\)-path in it. These problems are known to be NP-complete for general supergrid graphs. As far as we know, their complexities are still unknown for solid supergrid graphs which are supergrid graphs without any hole. In this paper, we will study these problems on \(L\)-shaped supergrid graphs which form a subclass of solid supergrid graphs. First, we prove \(L\)-shaped supergrid graphs to be Hamiltonian except one trivial condition. We then verify the Hamiltonian connectivity of \(L\)-shaped supergrid graphs except few conditions. The Hamiltonicity and Hamiltonian connectivity of \(L\)-shaped supergrid graphs can be applied to compute the minimum trace of computerized embroidery machine and 3D printer when a \(L\)-like object is printed. Finally, we present a linear-time algorithm to compute the longest \((s, t)\)-paths of \(L\)-shaped supergrid graphs. This study can be regarded as the first attempt for solving the Hamiltonian and longest \((s, t)\)-path problems on solid supergrid graphs.

**Abstract**—Supergrid (or called strong grid) graphs contain grid graphs and triangular grid graphs as their subgraphs. The Hamiltonian \((s, t)\)-path of a graph is a Hamiltonian path between any two distinct vertices \(s\) and \(t\) in the graph, and the longest \((s, t)\)-path is a simple path with the maximum number of vertices from \(s\) to \(t\) in the graph. A graph is called Hamiltonian if it contains a Hamiltonian cycle, and is said to be Hamiltonian connected if there exists a Hamiltonian \((s, t)\)-path in it. These problems are known to be NP-complete for general supergrid graphs. As far as we know, their complexities are still unknown for solid supergrid graphs which are supergrid graphs without any hole. In this paper, we will study these problems on \(L\)-shaped supergrid graphs which form a subclass of solid supergrid graphs. First, we prove \(L\)-shaped supergrid graphs to be Hamiltonian except one trivial condition. We then verify the Hamiltonian connectivity of \(L\)-shaped supergrid graphs except few conditions. The Hamiltonicity and Hamiltonian connectivity of \(L\)-shaped supergrid graphs can be applied to compute the minimum trace of computerized embroidery machine and 3D printer when a \(L\)-like object is printed. Finally, we present a linear-time algorithm to compute the longest \((s, t)\)-paths of \(L\)-shaped supergrid graphs. This study can be regarded as the first attempt for solving the Hamiltonian and longest \((s, t)\)-path problems on solid supergrid graphs.

**Index Terms**—Hamiltonicity, Hamiltonian connectivity, longest \((s, t)\)-path, solid supergrid graphs, \(L\)-shaped supergrid graphs, computer embroidery machines, 3D printers.

I. INTRODUCTION

The studied graphs, namely supergrid (or called strong grid) graphs, derive from our industry-university cooperative research project. They can be applied to computerized sewing machines. The process flow of a computerized sewing machine is as follows: The computerized sewing software is given by a colour image. First, it uses the image processing technique to produce some blocks of different colors. Then, the software computes the stitching trace for each block of colors. Finally, the software transmits its computed stitching trace to computerized sewing machine, and the machine then performs the sewing action along the received stitching trace. Since each stitch position of a sewing machine can be moved to its eight neighbor positions (left, right, up, down, up-left, up-right, down-left, and down-right), we define supergrid graphs in [13] as follows: Each lattice of a block of color will be represented by a vertex and each vertex \(v\) is coordinated as \((v_x, v_y)\), denoted by \(v = (v_x, v_y)\), where \(v_x\) and \(v_y\) represent the \(x\) and \(y\) coordinates of node \(v\), respectively. Two vertices \(u\) and \(v\) are adjacent if and only if \(|v_x - u_x| \leq 1\) and \(|v_y - u_y| \leq 1\). Thus, the possible adjacent vertices of a vertex \(u = (u_x, u_y)\) in a supergrid graph contain \((u_x, u_y - 1), (u_x, u_y + 1), (u_x + 1, u_y), (u_x - 1, u_y), (u_x + 1, u_y + 1), (u_x + 1, u_y - 1), (u_x - 1, u_y + 1), (u_x - 1, u_y - 1), and (u_x + 1, u_y + 1). Supergrid graphs contain grid [19] and triangular grid [32] graphs as their subgraphs. For instance, Figs. 1(a)–(c) depict a grid, a triangular grid, and a supergrid graph, respectively. Notice that grid and triangular grid graphs are not subclasses of supergrid graphs, and the converse is also true: these classes of graphs have common elements (nodes) but in general they are distinct since the edge sets of these graphs are different. Obviously, all grid graphs are bipartite [19] but triangular grid graphs and supergrid graphs are not bipartite.

Another intuitive motivation of proposing supergrid graphs is given below. Consider a set of lattices, shown in Fig. 2(a), where each lattice is denoted as a vertex in a graph. For a grid graph, the neighbors of a lattice include its up, down, left, and right lattices, see Fig. 2(b). However, in the real word and other applications, the neighbors of a lattice may also contain its up-left, up-right, down-left, and down-right adjacent lattices. Thus, supergrid graphs can be used in these applications.

A Hamiltonian path (resp., cycle) in a graph is a spanning path (resp., cycle) of the graph. The Hamiltonian path (resp., cycle) problem involves determining whether a graph contains a Hamiltonian path (resp., cycle). A graph is called Hamiltonian if it contains a Hamiltonian cycle. A simple path...
from vertex $s$ to $t$ is denoted by $(s, t)$-path. A graph $G$ is said to be Hamiltonian connected if it contains a Hamiltonian $(s, t)$-path for any two vertices $s$ and $t$ of $G$. The longest $(s, t)$-path of a graph is a simple path with the maximum number of vertices from $s$ to $t$ in the graph. The longest $(s, t)$-path problem is to compute the longest $(s, t)$-path of a graph given any two distinct vertices $s$ and $t$. It is well known that the Hamiltonian and longest $(s, t)$-path problems are NP-complete for general graphs [7], [20]. The same holds true for bipartite graphs [28], split graphs [8], circle graphs [6], undirected path graphs [1], grid graphs [19], triangular grid graphs [9], supergrid graphs [13], and so on. In the literature, there are many studies for the Hamiltonian connectivity of interconnection networks, see [3], [5], [10]–[12], [30], [31].

In [13], we proved the Hamiltonian problems on general supergrid graphs to be NP-complete. A solid supergrid graph is a supergrid graph without any hole. For example, the graph in Fig. 1(c) is a supergrid graph but it is not a solid supergrid graph. The Hamiltonian problems on solid supergrid graphs are still open. In this paper, we will solve the Hamiltonian and longest $(s, t)$-path problems on $L$-shaped supergrid graphs, which form a subclass of solid supergrid graphs, in linear time. Let $R(m, n)$ be a supergrid graph such that its vertex set $V(R(m, n)) = \{v = (v_x, v_y) | 1 \leq v_x \leq m \text{ and } 1 \leq v_y \leq n\}$. A rectangular supergrid graph is a supergrid graph which is isomorphic to $R(m, n)$. Let $L(m, n; k, l)$ be a supergrid graph obtained from a rectangular supergrid graph $R(m, n)$ by removing its subgraph $R(k, l)$ from the upper-right corner. A $L$-shaped supergrid graph is isomorphic to $L(m; n; k, l)$. In this paper, we only consider $L(m; n; k, l)$. Note that the number of vertices in $L(m, n; k, l)$ equals to $mn - kl$. In the figures, we will assume that $(1, 1)$ are coordinates of the vertex located at the upper-left corner of a supergrid graph. For example, Fig. 3(a) indicates the structure of $L(m; n; k, l)$, and Figs. 3(b)–(d) indicate $L(10, 11; 6, 8)$, $L(10, 11; 7, 9)$, and $L(7, 10; 3, 7)$, respectively. The width and height of $L$-shaped supergrid graph $L(m; n; k, l)$ can be adjusted according to the parameters $m$, $n$, $k$, and $l$. The main idea of our strategy is presented as follows. It first separates the input graph into many parts. Then, we verify whether there exists a Hamiltonian $(s, t)$-path in the input graph by checking these separated parts. When there exists no Hamiltonian $(s, t)$-path in the input graph, we then combine the longest paths of these separated parts to get a longest $(s, t)$-path of the input graph. Although this idea does seem to be simple, there are still many issues to be solved. That is, the precise partitions is important. If the separation is done in a wrong way then the result may be wrong. In this paper, we will verify our partition can separate the graph in a correct way.

The possible application of the Hamiltonian connectivity of $L$-shaped supergrid graphs is given below. Consider a computerized embroidery machine for sewing a varied-sized letter $L$ into the object, e.g. clothes. First, we produce a set of lattices to represent the letter. Then, a path is computed to visit the lattices of the set such that each lattice is visited exactly once. Finally, the software transmits the stitching trace of the computed path to the computerized embroidery machine, and the machine then performs the sewing work along the trace on the object. Since each stitch position of an embroidery machine can be moved to its eight neighboring positions, one set of neighboring lattices forms a $L$-shaped supergrid graph. In this case, each lattice will be represented by a vertex of a supergrid graph. The desired sewing trace of the set of adjacent lattices is the Hamiltonian path of the corresponding $L$-shaped supergrid graph. Given a string with varied-sized $L$ letters. By the Hamiltonian connectivity of $L$-shaped supergrid graphs, we can compute the end nodes of Hamiltonian paths in the corresponding $L$-shaped supergrid graphs so that the total length of jump lines connecting two $L$-shaped supergrid graphs is minimum. For instance, given three $L$-shaped supergrid graphs in Figs. 3(b)–(d), in which each $L$-shaped supergrid graph represents a set of lattices. Fig. 3(e) shows such a minimum sewing trace for the sets of lattices.

Another possible application of Hamiltonian connectivity of $L$-shaped supergrid graphs is to compute the minimum printing trace of 3D printers. Consider a 3D printer with a $L$-type object being printed. The software produces a series of thin layers, designs a path for each layer, combines these paths of produced layers, and transmits the above paths to 3D printer. Because 3D printing is performed layer by layer (see Fig. 4(a)), each layer can be considered as a $L$-shaped supergrid graph. Suppose that there are $k$ layers under the above 3D printing. If the Hamiltonian connectivity of $L$-shaped supergrid graphs holds, then we can find a Hamiltonian $(s_i, t_i)$-path of an $L$-shaped supergrid graph $L_i$, where $L_i, 1 \leq i \leq k$, represents a layer under 3D printing. Thus, we can design an optimal trace for the above 3D printing, where $t_i$ is adjacent to $s_{i+1}$ for $1 \leq i < k - 1$. In this application, we restrict the 3D printer nozzle to be located at integer coordinates. For example, Fig. 4(a) shows 4 layers $L_1$–$L_4$ of a 3D printing for a $L$-type object, Fig. 4(b) depicts the Hamiltonian $(s_i, t_i)$-paths of $L_i$ for $1 \leq i \leq 4$, and the result of this 3D printing is shown in Fig. 4(c).

Previous related works on supergrid graphs are summarized as follows. The supergrid graphs were first introduced in [13], in which we proved that the Hamiltonian cycle and path problems on supergrid graphs are NP-complete, and every rectangular supergrid graph is Hamiltonian. Since the Hamiltonian problems on general supergrid graphs are NP-complete, an important investigated direction is to discover...
connectivity of connected properties of rectangular supergrid graphs. These [18]. A preliminary version of this paper has appeared in Hamiltonicity and Hamiltonian connectivity of some shaped ways contain Hamiltonian cycles. In [15], we proved that that \(v_1, v_{i+1}, v_i \in P\) are distinct. If \(v_1 = v_{i+1}\) and \(|P| > 4\), then \(P\) is called a cycle of \(G\). The vertices \(v_1\) and \(v_{i+1}\) are called the path-start and path-end of \(P\), denoted by start\((P)\) and end\((P)\), respectively. We will use \(v_i\) in \(P\) to denote “\(P\) visits vertex \(v_i\)” and use \((v_i, v_{i+1})\) in \(P\) to denote “\(P\) visits edge \((v_i, v_{i+1})\)”. A path from \(v_1\) to \(v_k\) is denoted by \((v_1, v_k)\)-path. For convenience, we will use \(P\) to refer to \(V(P)\) if no ambiguity occurs. Let \(P_1\) and \(P_2\) be two paths (or cycles) in \(G\). If \(V(P_1) \cap V(P_2) = \emptyset\), then they are called vertex-disjoint. When \(P_1\) and \(P_2\) are vertex-disjoint and end\((P_1) \sim \cdots \sim end\((P_2)\), then they can be concatenated into a path, denoted by \(P_1 \sim P_2\).

Rectangular supergrid graphs first appeared in [13], in which we solved the Hamiltonian cycle problem in linear time. A rectangular supergrid graph \(R(m, n)\) is a supergrid graph with \(V(R(m, n)) = \{v = (x, y) | x \leq m \text{ and } 1 \leq y \leq n\}\), and it is called \(n\)-rectangle. In this paper, without loss of generality we will assume that \(m \geq n\). Let \(v = (x, y)\) be a vertex in \(R(m, n)\). The vertex \(v\) is called the upper-left (resp., upper-right, down-left, down-right) corner of \(R(m, n)\) if for any vertex \(u = (w_x, w_y) \in R(m, n)\), \(w_x \geq x\) and \(w_y \geq y\) (resp., \(w_x \leq x\) and \(w_y \geq y\), \(w_x \geq x\) and \(w_y \leq y\), \(w_x \leq x\) and \(w_y \geq y\)). Notice that in the figures we will assume that \((1, 1)\) are coordinates of the upper-left corner of \(R(m, n)\), except we explicitly change this assumption. The edge \((u, v)\) is called horizontal (resp., vertical) if \(y_u = y_v\) (resp., \(x_u = x_v\)), and is said to be crossed if it is neither a horizontal nor a vertical edge. There are four boundaries in a rectangular supergrid graph \(R(m, n)\) with \(m, n \geq 2\). The edge in the boundary of \(R(m, n)\) is called boundary edge. A path is called boundary of \(R(m, n)\) if it visits all vertices of the same boundary in \(R(m, n)\) and its length equals to the number of vertices in the visited boundary. For example, Fig. 5 shows a rectangular supergrid graph \(R(10, 8)\) which is called 8-rectangle and contains \(2 \times (9 + 7) = 32\) boundary edges. Fig. 5 also indicates the types of edges and corners.

A \(L\)-shaped supergrid graph, denoted by \(L(m, n; k, l)\), is a supergrid graph obtained from a rectangular supergrid graph \(R(m, n)\) by removing its subgraph \(R(k, l)\) from the upper-right corner, where \(m, n > 1\) and \(k, l \geq 1\). Then, \(m - k \geq 1\), \(n - l \geq 1\,\text{and}\, |V[L(m, n; k, l)]| = mn - kl\). The structure of \(L(m, n; k, l)\) is depicted in Fig. 3(a).

In proving our results, we need to partition a rectangular or \(L\)-shaped supergrid graph into two disjoint parts. The partition is defined as follows:

**Definition 1.** Let \(G\) be a \(L\)-shaped supergrid graph \(L(m, n; k, l)\) or a rectangular supergrid graph \(R(m, n)\). A separation operation of \(G\) is a partition of \(G\) into two
A rectangular supergrid graph $R(m, n)$, where $m = 10$, $n = 8$, and bold dashed lines indicate vertical and horizontal separations.

Fig. 5. A rectangular supergrid graph $G_1$ and $G_2$, i.e., $V(G) = V(G_1) \cup V(G_2)$ and $V(G_1) \cap V(G_2) = \emptyset$. A separation is called vertical if it consists of a set of horizontal edges, and is called horizontal if it contains a set of vertical edges. For instance, the bold dashed vertical (resp., horizontal) line in Fig. 5 indicates a vertical (resp., horizontal) separation of $R(10, 8)$ which is partitioned into $R(3, 8)$ and $R(7, 8)$ (resp., $R(10, 3)$ and $R(10, 5)$).

Let $R(m, n)$ with $m \geq n$ be a rectangular supergrid graph. In [13], we proved $R(m, n)$ to be Hamiltonian except $n = 1$. Let $C$ be a cycle of $R(m, n)$, and let $B$ be a boundary of $R(m, n)$. The restriction of $C$ to $B$ is denoted by $C_B$. If $|C_B| = 1$, i.e., $C_B$ is a boundary path on $B$, then $C_B$ is called flat face on $B$. If $|C_B| > 1$ and $C_B$ contains at least one boundary edge of $B$, then $C_B$ is called concave face on $B$. A Hamiltonian cycle $HC$ of $R(m, n)$ with $m \geq n \geq 2$ is called canonical if

1. $n = 3$, $HC$ contains three flat faces on two shorter boundaries and one longer boundary, and $HC$ contains one concave face on the other boundary; or
2. $n = 2$ or $n \geq 4$, $HC$ contains three flat faces on three boundaries, and $HC$ contains one concave face on the other boundary.

The following lemma shows the Hamiltonicity of rectangular supergrid graphs and appears in [13].

**Lemma 1.** (See [13].) Let $R(m, n)$ be a rectangular supergrid graph with $m \geq n \geq 2$. Then, $R(m, n)$ contains a canonical Hamiltonian cycle. Moreover, $R(m, n)$ contains four canonical Hamiltonian cycles with concave faces being located on different boundaries when $n \neq 3$.

Fig. 6 shows canonical Hamiltonian cycles for rectangular supergrid graphs found in Lemma 1. Each Hamiltonian cycle found by this lemma contains all the boundary edges on any three sides of the rectangular supergrid graph. This shows that for any rectangular supergrid graph $R(m, n)$ with $m \geq n \geq 4$, we can always construct four canonical Hamiltonian cycles such that their concave faces are placed on different boundaries. For an example, the four distinct canonical Hamiltonian cycles of $R(7, 5)$ are depicted in Figs. 6(b)–(e).

Let $(G, s, t)$ denote the supergrid graph $G$ with two distinct vertices $s$ and $t$. We will assume, without loss of generality, that $s_x \leq t_x$ except we explicitly change this assumption. A Hamiltonian path between $s$ and $t$ in $G$ is denoted by $HP(G, s, t)$. From Lemma 1, $HP(R(m, n), s, t)$ does exist if $m, n \geq 2$ and $(s, t)$ is an edge in the constructed Hamiltonian cycle of $R(m, n)$. In addition, we will use $L(G, s, t)$ to denote the length of longest $(s, t)$-path in $(G, s, t)$. Note that we denote the length of a path by the number of vertices in the path.

Recently, the Hamiltonian connectivity of rectangular supergrid graphs except one condition has been verified in [15]. The forbidden condition for $HP(R(m, n), s, t)$ is satisfied only for 1-rectangle or 2-rectangle. To describe the exception condition, we define the cut vertex and vertex cut of a graph as follows:

**Definition 2.** Let $G$ be a connected graph and let $V_1 \subseteq V(G)$. The set $V_1$ is a vertex cut of $G$ if $G - V_1$ is disconnected. A vertex $v$ of $G$ is a cut vertex of $G$ if $\{v\}$ is a vertex cut of $G$. For instance, in Fig. 7(a) $s$ or $t$ is a cut vertex and in Fig. 7(b) $s$ or $t$ is a vertex cut.

Then, the following condition implies $HP(R(m, 1), s, t)$ and $HP(R(m, 2), s, t)$ do not exist.

(F1) $s$ or $t$ is a cut vertex of $R(m, 1)$, or $\{s, t\}$ is a vertex cut of $R(m, 2)$ (see Figs. 7(a)–(b)).

The following lemma can be easily verified by the same arguments in [24].

**Lemma 2.** Let $R(m, n)$ be a rectangular supergrid graph, and let $s$ and $t$ be its two vertices. If $(R(m, n), s, t)$ satisfies condition (F1), then $HP(R(m, n), s, t)$ does not exist.

In [15], we obtained the following lemma to show the Hamiltonian connectivity of rectangular supergrid graphs.

**Lemma 3.** (See [15].) Let $R(m, n)$ be a rectangular supergrid graph, and let $s$ and $t$ be its two vertices. If $(R(m, n), s, t)$ does not satisfy condition (F1), then
Combining with the above two lemmas, we have the following theorem.

**Theorem 4.** Let \( R(m, n) \) be a rectangular supergrid graph, and let \( s \) and \( t \) be its two vertices. Then, \( HP(R(m, n), s, t) \) does exist if and only if \( (R(m, n), s, t) \) does not satisfy condition (F1).

The Hamiltonian \((s, t)\)-path of \( R(m, n) \) constructed in [15] is to contain at least one boundary edge of each boundary, and is called canonical.

We next give some propositions on the relations among cycle, path, and vertex. These observations will be used in proving our results and are given in [13, 14, 15].

**Proposition 5.** (See [13, 14, 15].) Let \( G \) be a connected graph, \( C_1 \) and \( C_2 \) be two vertex-disjoint cycles of \( G \), \( P_1 \) and \( P_2 \) be a cycle and a path, respectively, of \( G \) such that \( V(C_1) \cap V(P_1) = 0 \), and let \( x \) be a vertex in \( G - V(C_1) \) or \( G - V(P_1) \). Then, the following statements hold:

1. If there exist two edges \( e_1 \in C_1 \) and \( e_2 \in C_2 \) such that \( e_1 \approx e_2 \), then \( C_1 \) and \( C_2 \) can be merged into a cycle of \( G \) (see Fig. 8(a)).
2. If there exist two edges \( e_1 \in C_1 \) and \( e_2 \in P_1 \) such that \( e_1 \approx e_2 \), then \( C_1 \) and \( P_1 \) can be merged into a path of \( G \) (see Fig. 8(b)).
3. If \( x \) and \( P \) are two vertex-disjoint cycles of \( G \) (resp., path of \( G \) (see Fig. 8(c)).
4. If there exist one edge \( e_1 \in C_1 \) such that \( e_1 \approx \text{start}(P_1) \) and \( e_1 \approx \text{end}(P_1) \), then \( C_1 \) and \( P_1 \) can be combined into a cycle of \( C \) (see Fig. 8(d)).

In [15], Hung et al. gave the following formula to compute the length of a longest \((s, t)\)-path in \( R(m, n) \):

\[
L(R(m, n), s, t) = \begin{cases} 
\frac{(t_x - s_x + 1) \cdot \max(2s_x, 2(m - s_x + 1))}{m} & \text{if } n = 2; \\
\frac{(t_x - s_x + 1) \cdot 2m}{mn} & \text{if } n \geq 3.
\end{cases}
\]

**Theorem 6.** (See [15].) Given a rectangular supergrid graph \( R(m, n) \) with \( mn \geq 2 \), and two distinct vertices \( s \) and \( t \) in \( R(m, n) \), a longest \((s, t)\)-path can be computed in \( O(mn) \) - linear time.

In this paper, we will show that a longest \((s, t)\)-path of \( L(m, n; k, l) \), \( s, t \) can be computed in \( O(mn - kl) \) - linear time.

III. TWO HAMILTONIAN CONNECTED PROPERTIES OF RECTANGULAR SUPERGRID GRAPHS

By Theorem 4, rectangular supergrid graph \( R(m, n) \) contains a Hamiltonian \((s, t)\)-path if and only if \( (R(m, n), s, t) \) does not satisfy condition (F1). The Hamiltonian \((s, t)\)-path of \( R(m, n) \) constructed in [15] contains at least one boundary edge of each boundary. In this section, we will prove two additional Hamiltonian connected properties of rectangular supergrid graphs under some conditions. These two properties will be used to prove the Hamiltonian connectivity of L-shaped supergrid graphs. Let \( R(m, n) \) be a rectangular supergrid graph with \( m \geq 3 \) and \( n \geq 2 \), and let \( w = (1, 1), z = (2, 1), \) and \( f = (3, 1) \) be three vertices in \( R(m, n) \). We will prove the following two Hamiltonian connected properties of \( R(m, n) \):

**(P1)** If \( s = w = (1, 1) \) and \( t = z = (2, 1) \), then there exists a Hamiltonian \((s, t)\)-path \( P \) of \( R(m, n) \) such that edge \((z, f) \in P \).

**(P2)** If \( (n = 2) \) and \( (s, t) \not\in \{(w, z), ((1, 1), (2, 2)), ((2, 1), (1, 2)) \} \) or \( n \geq 3 \) and \( (s, t) \neq (w, z) \), then there exists a Hamiltonian \((s, t)\)-path \( Q \) of \( R(m, n) \) such that edge \((w, z) \in Q \), where \( (R(m, n), s, t) \) does not satisfy condition (F1).

First, we verify the first property (P1) as follows:

**Lemma 7.** Let \( R(m, n) \) be a rectangular supergrid graph with \( m \geq 3 \) and \( n \geq 2 \), and let \( s = w = (1, 1), t = z = (2, 1), \) and \( f = (3, 1) \). Then, there exists a Hamiltonian \((s, t)\)-path \( P \) of \( R(m, n) \) such that edge \((z, f) \in P \).

**Proof:** Depending on whether \( m = 3 \), we consider the following two cases:

**Case 1:** \( m = 3 \). In this case, we claim that there exists a Hamiltonian \((s, t)\)-path \( P \) of \( R(m, n) \) such that \((z, f) \in P \) and a boundary path connecting down-left corner and down-right corner is a subpath of \( P \).

We will prove the above claim by induction on \( n \). Initially, let \( n = 2 \). The desired Hamiltonian \((s, t)\)-path \( P \) of \( R(3, 2) \) can be easily constructed and is depicted in Fig. 9(a). Assume that the claim holds true when \( n = k \geq 2 \). Let \( u_1 = (1, k), u_2 = (2, k), \) and \( u_3 = (3, k) \). By induction hypothesis, there exists a Hamiltonian \((s, t)\)-path \( P_k \) of \( R(m, n) \) such that \((z, f) \in P_k \) and \( P_t \) contains the boundary path \( P' = u_1 \rightarrow u_2 \rightarrow u_3 \) as a subpath. Let \( P_k = P_1 \Rightarrow P' \Rightarrow P_2 \). Consider \( n = k + 1 \). Let \( v_1 = (1, k + 1), v_2 = (2, k + 1), \) and \( v_3 = (3, k + 1) \). Then, \( P_1 \Rightarrow v_1 \Rightarrow P \Rightarrow v_2 \Rightarrow u_3 \Rightarrow P_2 \) is the desired Hamiltonian \((s, t)\)-path of \( R(3, k + 1) \). The constructed Hamiltonian \((s, t)\)-path of \( R(3, k + 1) \) is shown in Fig. 9(b). By induction, the claim holds and hence, the lemma holds true in the case of \( m = 3 \).

**Case 2:** \( m > 3 \). In this case, we first make a vertical separation on \( R(m, n) \) to partition it into two disjoint rectangular supergrid subgraphs \( R_{a} = R(2, n) \) and \( R_{b} = R(m - 2, n) \), as depicted in Fig. 9(c). We can easily construct a Hamiltonian \((s, t)\)-path \( P_{a} \) of \( R_{a} \) such that \( P_{a} \) contains a boundary path placed to face \( R_{a} \), as shown in Fig. 9(c). By Lemma 1, \( R_{b} \) contains a canonical Hamiltonian cycle \( C_{b} \). We can place one flat face of \( C_{b} \) to face \( R_{a} \). Then, there exist two edges
containing edge \((z, f)\) such that \(t(=z)\) is a vertex of \(e_1, f\) is a vertex of \(e_2, \) and \(e_1 \approx e_2\). By Statement (2) of Theorem 5, \(P_\alpha\) and \(C_\beta\) can be combined into a Hamiltonian \((s, t)\)-path \(P\) of \(R(m, n)\) such that edge \((z, f) \in P\). The constructed Hamiltonian \((s, t)\)-path of \(R(m, n)\) is depicted in Fig. 9(c). Thus, the lemma holds true when \(m > 4\).

It immediately follows from the above cases that the lemma holds true.

Next, we will verify the second Hamiltonian connected property (P2) of \(R(m, n)\), where \(m > 3\) and \(n > 2\). We first consider the following forbidden condition such that there exists a Hamiltonian \((s, t)\)-path \(Q\) of \(R(m, n)\) with edge \((w, z) \in Q\):

\[(F2) \quad n = 2 \quad \text{and} \quad (s, t) \in \{(w, z), (1, 1), (2, 2), (2, 1), (1, 2)\}, \quad \text{or} \quad n > 3 \quad \text{and} \quad (s, t) = (w, z).

The above condition states that \(R(m, n)\) has no Hamiltonian \((s, t)\)-path containing edge \((w, z)\) if \(R(m, n), s, t\) satisfies condition (F2). We will prove property (P2) by constructing a Hamiltonian \((s, t)\)-path of \(R(m, n)\) visiting edge \((w, z)\) when \((R(m, n), s, t)\) does not satisfy conditions (F1) and (F2). To verify property (P2), we first consider the special case, in Lemma 8, that \(m = 3, n > 2, \) and either \(s = z\) or \(t = z\). This lemma can be proved by similar arguments in proving Case 1 of Lemma 7.

**Lemma 8.** Let \(R(m, n)\) be a rectangular supergrid graph with \(m = 3\) and \(n > 2, \) and \(s\) and \(t\) be its two distinct vertices, and let \(w = (1, 1)\) and \(z = (2, 1)\). If \(R(m, n), s, t\) does not satisfy conditions (F1) and (F2), and either \(s = z\) or \(t = z\), then there exists a Hamiltonian \((s, t)\)-path \(Q\) of \(R(m, n)\) such that edge \((w, z) \in Q\).

**Proof:** Without loss of generality, assume that \(s = z\). Then, \(t_s \leq s_s\) or \(t_z \geq s_z\). That is, \(t\) may be to the left of \(s\). Let \(x = (1, n), y = (2, n), \) and \(r = (3, n)\) be three vertices of \(R(m, n)\). We claim that there exists a Hamiltonian \((s, t)\)-path \(Q\) of \(R(m, n)\) such that \((w, z) \in Q\), and \((x, y) \in Q\) if \(t = r\); and \((x, y) \in Q\) otherwise.

We will prove the above claim by induction on \(n\). Initially, let \(n = 2\). Since \((R(m, n), s, t)\) does not satisfy conditions (F1) and (F2), \(t \not\in \{1, 1, 1, 2, 2, 2\}\). Thus, \(t \in \{3, 1, 3, 2\}\). Then, the desired Hamiltonian \((s, t)\)-path \(Q\) of \(R(3, 2)\) can be easily constructed and is depicted in Fig. 10(a). Assume that the claim holds true when \(n = k \geq 2\). Let \(x_1 = (1, k), y_1 = (2, k), \) and \(r_1 = (3, k)\). By induction hypothesis, there exists Hamiltonian \((s, p)\)-path \(Q_k\) of \(R(3, k)\) such that edge \((w, z) \in Q_k\), and \((x_1, y_1) \in Q_k\), or \((y_1, r_1) \in Q_k\) depending on whether or not \(p = r_1\). Consider that \(n = k + 1\). We first make a horizontal separation on \(R(3, k + 1)\) to obtain two disjoint parts \(R_1 = R(3, k)\) and \(R_2 = R(3, 1)\), as shown in Fig. 10(b). Let \(x_2 = (1, k + 1), y_2 = (2, k + 1), \) and \(r_2 = (3, k + 1)\) be the three vertices of \(R_2\). We will construct a Hamiltonian \((s, t)\)-path \(Q_{k+1}\) of \(R(3, k + 1)\) such that \((w, z) \in Q_{k+1}\), and \((x_2, y_2) \in Q_{k+1}\), or \((y_2, r_2) \in Q_{k+1}\) as follows. Depending on the location of \(t\), there are the following two cases:

**Case 1:** \(t \in R_1\). Let \(P_2 = x_2 \rightarrow y_2 \rightarrow r_2\). By induction hypothesis, there exists Hamiltonian \((s, t)\)-path \(Q_k\) of \(R(3, k)\) such that edge \((w, z) \in Q_k\), and \((x_1, y_1) \in Q_k\) if \(t = r_1\); and \((y_1, r_1) \in Q_k\) otherwise. Thus, there exists an edge \((u_k, v_k) \in Q_k\) such that \(x_k = u_k\) and \(y_k = v_k\) such that \((x_k, y_k) = (x_1, y_1)\) or \((y_1, r_1)\). By Statement (4) of Theorem 5, \(Q_k\) and \(P_2\) can be combined into a Hamiltonian \((s, t)\)-path \(Q_{k+1}\) of \(R(3, k + 1)\) such that edges \((w, z), (x_2, y_2), (y_2, r_2) \in Q_{k+1}\). The construction of such a Hamiltonian path is depicted in Fig. 10(b).

**Case 2:** \(t \in R_2\). In this case, \(t \in \{x_2, y_2, r_2\}\). Then, there are the following three subcases:

1. **Case 2.1:** \(t = x_2\). Let \(p = r_1 \in R_1\) and \(q = r_2 \in R_2\). Then, \(p \approx q\). Let \(P_2 = x_2 = q \rightarrow y_2 \rightarrow x_2 = t\). By induction hypothesis, there exists Hamiltonian \((s, p)\)-path \(Q_k\) of \(R(m, k)\) such that edges \((w, z), (x_1, y_1) \in Q_k\). Then, \(Q_{k+1} = Q_k \Rightarrow P_2\) forms a Hamiltonian \((s, t)\)-path of \(R(m, k + 1)\) with \((w, z), (x_2, y_2), (y_2, r_2) \in Q_{k+1}\). Fig. 10(c) shows the construction of such a Hamiltonian \((s, t)\)-path.

2. **Case 2.2:** \(t = r_2\). Let \(p = x_1 \in R_1\) and \(q = r_2 \in R_2\). Let \(P_2 = x_2 = q \rightarrow y_2 \rightarrow r_2 = t\). By induction hypothesis, there exists Hamiltonian \((s, p)\)-path \(Q_k\) of \(R(m, k)\) such that edges \((w, z), (x_1, y_1) \in Q_k\). Then, \(Q_{k+1} = Q_k \Rightarrow P_2\) forms a Hamiltonian \((s, t)\)-path of \(R(m, k + 1)\) with \((w, z), (x_2, y_2), (y_2, r_2) \in Q_{k+1}\). Fig. 10(d) shows the construction of such a Hamiltonian \((s, t)\)-path.

3. **Case 2.3:** \(t = y_2\). Let \(p = r_1 \in R_1\). Let \(P_2 = r_2 \rightarrow y_2 = t\). By induction hypothesis, there exists Hamiltonian \((s, p)\)-path \(Q_k\) of \(R(m, k)\) such that edges \((w, z), (x_1, y_1) \in Q_k\). Then, \(Q_{k+1} = Q_k \Rightarrow P_2\) is a Hamiltonian \((s, t)\)-path of \(R(m, k + 1)\) such that \((w, z), (x_1, y_1), (y_2, r_2) \in Q_{k+1}\). Since \(x_2 \sim x_1, x_2 \sim y_1, \) and \(y_1 \sim y_2 \in Q_{k+1}\), by Statement (3) of Theorem 5, \(Q_{k+1}\) and \(P_2\) can be combined into a Hamiltonian \((s, t)\)-path \(Q_{k+1}\) of \(R(3, k + 1)\) such that edges \((w, z), (y_2, r_2) \in Q_{k+1}\). Fig. 10(e) depicts such a construction of Hamiltonian \((s, t)\)-path.

It immediately follows from the above cases that the claim holds true when \(n = k + 1\). By induction, the claim holds true and, hence, the lemma is true.

We next verify property (P2) in the following lemma.

**Lemma 9.** Let \(R(m, n)\) be a rectangular supernet graph with \(m = 3\) and \(n \geq 2, \) and \(s\) and \(t\) be its two distinct vertices, and let \(w = (1, 1)\) and \(z = (2, 1)\). If \(R(m, n), s, t\) does not satisfy conditions (F1) and (F2), then there exists a Hamiltonian \((s, t)\)-path \(Q\) of \(R(m, n)\) such that edge \((w, z) \in Q\).
Proof: We will provide a constructive method to prove this lemma. By assumption of this lemma, \( \{s, t\} \neq \{w, z\} \) and, hence, \( 0 \leq |\{s, t\} \cap \{w, z\}| \leq 1 \). Then, there are three cases:

Case 1: \( \{s, t\} \cap \{w, z\} = \emptyset \). In this case, \( s, t \notin \{w, z\} \).

By Lemma 3, \( R(m, n) \) contains a Hamiltonian \((s, t)\)-path \( Q \) if edge \((w, z) \in Q\), then \( Q \) is the desired Hamiltonian \((s, t)\)-path of \( R(m, n) \). Suppose that edge \((w, z) \notin Q\). Let \( x = (1, 2) \) and \( y = (2, 2) \). Then, \( N(w) - \{z\} = \{x, y\} \). Let \( Q = Q^w_1 \Rightarrow w \Rightarrow Q^z_2 \). Since \( N(w) - \{z\} = \{x, y\} \), \( \{\text{end}(Q^w_1), \text{start}(Q^w_2)\} = \{x, y\} \) and, hence, \( \text{end}(Q^w_1) \sim \text{start}(Q^w_2) \).

Then, \( Q^w_1 \Rightarrow w \Rightarrow Q^z_2 \) is a Hamiltonian \((s, t)\)-path of \( R(m, n) \) - \( w \), where edge \((x, y) \) is visited by \( Q^z_2 \). Let \( P' = w \Rightarrow z \). Then, there exist one edge \((x, y) \in Q^z_2 \) such that \( \text{start}(P') \sim x \) and \( \text{end}(P') \sim y \). By Statement (4) of Proposition 5, \( Q^2 \) and \( P' \) can be combined into a Hamiltonian \((s, t)\)-path \( Q \) of \( R(m, n) \) - \( w \), and edge \((x, y) \) is visited by \( Q \). The construction of such a Hamiltonian \((s, t)\)-path is depicted in Fig. 11(a).

Case 2: \( \{s, t\} \cap \{w, z\} = \{x, p\} \). Since \( N(z) - \{w, x\} \) forms a clique, \( x \in \{\text{end}(Q^w_1), \text{start}(Q^z_2)\} \). Then, \( z \sim x \sim y \) is a subpath of \( Q^z_2 \). Let \( Q^z_2 = Q^z_2 \Rightarrow x \Rightarrow Q^z_2 \). Then, \( \{\text{end}(Q^w_1), \text{start}(Q^w_2)\} = \{x, y\} \). Thus, \( Q^z_2 = Q^z_2 \Rightarrow Q^z_2 \) is a Hamiltonian \((s, t)\)-path of \( R(m, n) \) - \( w \), where edge \((y, z) \) is visited by \( Q^z_2 \). Let \( P' = w \Rightarrow x \). Then, there exist one edge \((y, z) \in Q^z_2 \) such that \( \text{start}(P') \sim z \) and \( \text{end}(P') \sim y \). By Statement (4) of Proposition 5, \( Q^2 \) and \( P' \) can be combined into a Hamiltonian \((s, t)\)-path \( Q \) of \( R(m, n) \) such that edge \((w, z) \in Q \). The construction of such a Hamiltonian \((s, t)\)-path is shown in Fig. 11(b).

Case 2: \( s = w \) or \( t = w \). Without loss of generality, assume that \( s = w \). First, consider that \( n = 2 \). Then, \( R(m, n) \) is a 2-rectangle. By assumption of the lemma, \( R(m, n) \) - \( s, t \) does not satisfy condition (F2), and, hence, \( t \notin \{2, 1\} \) or \( \{2, 2\} \). If \( t = (1, 2) \), then a Hamiltonian \((s, t)\)-path \( Q \) of \( R(m, n) \) - \( w \) can be easily constructed by visiting each boundary edge of \( R(m, n) \) except boundary edge \((s, t)\), and, hence, \((w, z) \in Q \). Let \( t = (x, y) \) satisfy that \( x \geq 3 \).

We first make a vertical separation on \( R(m, n) \) to obtain two disjoint parts \( R_\alpha \) and \( R_\beta \), as depicted in Fig. 11(c). Let \( p = (x - 1, 2) \in R_\alpha \) and \( q = (x, y - 1) \) or \((x, y + 1) \in R_\beta \), where \( q \neq t \) and \( q_\beta = t_\beta \).

Then, \( p \sim q \) and we can easily construct Hamiltonian \((s, p)\)-path \( Q_\alpha \) and \((q, t)\)-path \( Q_\beta \) of \( R_\alpha \) and \( R_\beta \), respectively, such that edge \((w, z) \in Q_\alpha \). Thus, \( Q = Q_\alpha \Rightarrow Q_\beta \) is a Hamiltonian \((s, t)\)-path of \( R(m, n) \) such that edge \((w, z) \in Q \). The construction of such a Hamiltonian \((s, t)\)-path is depicted in Fig. 11(c).

Case 2: \( t = 1 \) and \( t = m \). In this subcase, \( t \) is located at the up-right corner of \( R(m, n) \). We first make a horizontal separation on \( R(m, n) \) to obtain two disjoint parts \( R_1 = R(m, 1) \) and \( R_2 = R(m, n - 1) \), as shown in Fig. 11(d). Note that \( m \geq 3 \) and \( n \geq 2 \). By visiting all boundary edges of \( R_1 \) from \( s \) to \( t \), we get a Hamiltonian \((s, t)\)-path \( Q_1 \) of \( R_1 \) with edge \((w, z) \in Q_1 \). By Lemma 1, we can construct a canonical Hamiltonian cycle \( C_2 \) of \( R_2 \) such that its one flat face is placed to face \( R_1 \). Then, there exist two edges \( e_1(= (z, f)) \in Q_1 \) and \( e_2 \in C_2 \) such that \( e_1 \approx e_2 \), where \( z = (2, 1) \) and \( f = (3, 1) \). By Statement (2) of Proposition 5, \( P_1 \) and \( C_2 \) can be merged into a Hamiltonian \((s, t)\)-path \( Q \) of \( R(m, n) \) such that edge \((w, z) \in Q \). The construction of such a Hamiltonian \((s, t)\)-path is shown in Fig. 11(d).

Case 2: \( ty = 1 \) and \( tx < m \). Let \( r = (m, 1) \) be the up-right corner of \( R(m, n) \). Then, \( z_\beta < t_\beta < r_\beta \), i.e., \( 2 < t_\beta < m \), and, hence, \( m \geq 4 \). We first make a vertical separation on \( R(m, n) \) to get two disjoint parts \( R_\alpha = R(2, n) \) and \( R_\beta = R(m - 2, n) \), as depicted in Fig.
11(e), where \( n \geq 3 \) and \( m - 2 \geq 2 \). Let \( p = (2, n) \) be the down-right corner of \( R_m \), and let \( q = (3, n) \) be the down-left corner of \( R_m \). Then, \( p \sim q \) and \( (R_m, s, p) \) and \( (R_m, q, t) \) do not satisfy condition (F1). Since \( R_m \) is a 2-rectangle, we can easily construct a Hamiltonian \((s, p)\)-path \( Q_\alpha \) of \( R_m \) such that edge \((w, z) \in Q_\alpha \), as shown in Fig. 11(e). By Lemma 3, there exists a Hamiltonian \((q, t)\)-path \( Q_\beta \) of \( R_m \). Then, \( Q = Q_\alpha \Rightarrow Q_\beta \) forms a Hamiltonian \((s, t)\)-path of \( R(m, n) \) such that edge \((w, z) \in Q \). Such a Hamiltonian \((s, t)\)-path is depicted in Fig. 11(e).

**Case 2.3:** \( t_y > 1 \). In this subcase, we first make a horizontal separation on \( R(m, n) \) to obtain two disjoint parts \( R_1 = R(m, 1) \) and \( R_2 = R(m, n - 1) \), as shown in Fig. 11(f), where \( m \geq 3 \) and \( n - 1 \geq 2 \). Let \( r = (m, 1) \), then \( r \in R_1 \). Let \( q = (m, 2) \) if \( t \neq (m, 2) \); otherwise \( q = (m - 1, 2) \). A simple check shows that \((R_2, q, t)\) does not satisfy condition (F1). By visiting every vertex of \( R_1 \) from \( s \) to \( r \), we get a Hamiltonian \((s, t)\)-path \( Q_1 \) of \( R_1 \) with edge \((w, z) \in Q_1 \). By Lemma 3, there exists a Hamiltonian \((q, t)\)-path \( Q_2 \) of \( R_2 \). Then, \( Q = Q_1 \Rightarrow Q_2 \) is a Hamiltonian \((s, t)\)-path of \( R(m, n) \) with \((w, z) \in Q \). The constructed Hamiltonian \((s, t)\)-path in this subcase can be found in Fig. 11(f).

**Case 3:** \( s = z \) or \( t = z \). By symmetry, assume that \( s = z \). Then, \( t \) may be to the left of \( s \), i.e., \( t_y < s_y \). When \( n = 2 \), a Hamiltonian \((s, t)\)-path \( Q \) of \( R(m, n) \) with \((w, z) \in Q \) can be constructed by similar arguments in Fig. 11(c). By Lemma 8, the desired Hamiltonian \((s, t)\)-path of \( R(m, n) \) can be constructed when \( m = 3 \). In the following, suppose that \( m \geq 4 \) and \( n \geq 3 \). We then make a horizontal separation on \( R(m, n) \) to obtain two disjoint parts \( R_1 = R(m, 1) \) and \( R_2 = R(m, n - 1) \), as shown in Fig. 11(g), where \( m \geq 4 \) and \( n - 1 \geq 2 \). Then, \( s \in R_1 \). Depending on whether \( t \in R_1 \), we consider the following two subcases:

**Case 3.1:** \( t \in R_1 \). A Hamiltonian \((s, t)\)-path \( Q \) of \( R(m, n) \) with \((w, z) \in Q \) can be constructed by similar arguments in proving Case 2.1 and Case 2.2. Figs. 11(g)–(h) show such constructions of the desired Hamiltonian \((s, t)\)-paths of \( R(m, n) \).

**Case 3.2:** \( t \in R_2 \). In this subcase, we make a vertical separation on \( R(m, n) \) to obtain two disjoint parts \( R_\alpha = R(2, n) \) and \( R_\beta = R(m - 2, n) \), where \( m - 2 \geq 2 \) and \( n \geq 3 \), as shown in Fig. 11(i). Suppose that \( t \in R_\beta \). By similar technique in Fig. 11(c) and Lemma 3, we can easily construct a Hamiltonian \((s, t)\)-path \( Q_\alpha \) of \( R_\alpha \) such that \((w, z) \in Q_\alpha \) and \( Q_\alpha \) contains one boundary edge \( e_\alpha \) that is placed to face \( R_\beta \) as depicted in Fig. 11(i). By Lemma 1, there exists a canonical Hamiltonian cycle \( C_\beta \) of \( R_\beta \) such that its one flat face is placed to face \( R_\alpha \). Then, there exist two edges \( e_\alpha \in Q_\alpha \) and \( e_\beta \in C_\beta \) such that \( e_\alpha \sim e_\beta \). By Statement (2) of Proposition 5, \( Q_\alpha \) and \( C_\beta \) can be combined into a Hamiltonian \((s, t)\)-path \( Q \) of \( R(m, n) \) with edge \((w, z) \in Q \). The construction of such a Hamiltonian \((s, t)\)-path is shown in Fig. 11(i). On the other hand, suppose that \( t \in R_\beta \). Let \( p \in R_\alpha \) and \( q \in R_\beta \) such that \( p \sim q \) and \((R_\alpha, s, p) \) and \((R_\beta, q, t)\) do not satisfy condition (F1). By Lemma 3, there exist Hamiltonian \((s, p)\)-path \( Q_\alpha \) and Hamiltonian \((q, t)\)-path \( Q_\beta \) of \( R_\alpha \) and \( R_\beta \), respectively. Since \( R_\alpha \) is a 2-rectangle, we can easily construct \( Q_\alpha \) to satisfy \((w, z) \in Q_\alpha \). Then, \( Q = Q_\alpha \Rightarrow Q_\beta \) is a Hamiltonian \((s, t)\)-path of \( R(m, n) \) with edge \((w, z) \in Q \).

We have considered any case to construct a Hamiltonian \((s, t)\)-path \( Q \) of \( R(m, n) \) with edge \((w, z) \in Q \). This completes the proof of the lemma.

**IV. THE HAMILTONIAN AND HAMILTONIAN CONNECTED PROPERTIES OF L-SHAPED SUPERGRID GRAPHS**

In this section, we will verify the Hamiltonicity and Hamiltonian connectivity of L-shaped supergrid graphs. Let \( L(m; n; k, l) \) be a L-shaped supergrid graph. We will make a vertical or horizontal separation on \( L(m; n; k, l) \) to obtain two disjoint rectangular supergrid graphs. For an example, the bold dashed vertical (resp., horizontal) line in Fig. 12(a) indicates a vertical (resp., horizontal) separation on \( L(10, 11; 7, 9) \) that is to partition it into \( R(3, 11) \) and \( R(7, 2) \) (resp., \( R(3, 9) \) and \( R(10, 2) \)). The following two subsections will prove the Hamiltonicity and Hamiltonian connectivity of \( L(m; n; k, l) \) respectively.

**A. The Hamiltonian property of L-shaped supergrid graphs**

In this subsection, we will prove the Hamiltonicity of L-shaped supergrid graphs. Obviously, \( L(m; n; k, l) \) contains no Hamiltonian cycle if there exists a vertex \( w \) in \( L(m; n; k, l) \) such that \( deg(w) = 1 \). Thus, \( L(m; n; k, l) \) is not Hamiltonian when the following condition is satisfied:

(F3) there exists a vertex \( w \) in \( L(m; n; k, l) \) such that \( deg(w) = 1 \).

When the above condition is satisfied, we get that \( m - k = 1 \) and \( l > 1 \) or \( n - l = 1 \) and \( k > 1 \). We then show the Hamiltonicity of L-shaped supergrid graphs as follows:
Theorem 10. Let \( L(m, n; k, l) \) be a \( L \)-shaped supergrid graph. Then, \( L(m, n; k, l) \) contains a Hamiltonian cycle if and only if it does not satisfy condition (F3).

Proof: Obviously, \( L(m, n; k, l) \) contains no Hamiltonian cycle if it satisfies condition (F3). In the following, we will prove that \( L(m, n; k, l) \) contains a Hamiltonian cycle if it does not satisfy condition (F3). Assume that \( L(m, n; k, l) \) does not satisfy condition (F3). We prove it by constructing a Hamiltonian cycle of \( L(m, n; k, l) \). First, we make a vertical separation on \( L(m, n; k, l) \) to obtain two disjoint rectangular supergrid subgraphs \( L_\alpha = R(m-k, n) \) and \( L_\beta = R(k, n-l) \), as depicted in Fig. 12(b). Depending on the sizes of \( L_\alpha \) and \( L_\beta \), there are the following two cases:

Case 1: \( m - k = 1 \) or \( n - l = 1 \). By symmetry, we assume that \( m - k = 1 \). Since there exists no vertex \( w \) in \( L(m, n; k, l) \) such that \( \deg(w) = 1 \), we get that \( l = 1 \) (see Fig. 12(c)). Consider that \( n - l = 1 \). Then, \( k = 1 \). Thus, \( L(m, n; k, l) \) consists of only three vertices which forms a cycle. On the other hand, consider that \( n - l \geq 2 \). Let \( w \) be a vertex of \( L_\alpha \), with \( \deg(w) = 2 \), \( L_\alpha^* = L_\alpha - \{w\} \), and let \( L^* = L_\alpha^* \cup L_\beta \). Then, \( L^* = R(k+1, n-l) = R(m, n-1) \), where \( k+1 \geq 2 \) and \( n-l \geq 2 \). By Lemma 1, \( L^* \) contains a canonical Hamiltonian cycle \( HC^* \). We can place one flat face of \( HC^* \) to face \( w \). Thus, there exists an edge \((u, v)\) in \( HC^* \) such that \( w \sim u \) and \( w \sim v \). By Statement (3) of Proposition 5, \( w \) and \( HC^* \) can be combined into a Hamiltonian cycle of \( L(m, n; k, l) \). For example, Fig. 12(c) depicts a such construction of Hamiltonian cycle of \( L(m, n; k, l) \) when \( m-k = 1 \) and \( n-l \geq 2 \). Thus, \( L(m, n; k, l) \) is Hamiltonian if \( m-k = 1 \) or \( n-l = 1 \).

Case 2: \( m - k \geq 2 \) and \( n - l \geq 2 \). In this case, \( \alpha = R(m-k, n) \) and \( \beta = R(k, n-l) \) satisfy that \( m-k \geq 2 \) and \( n-l \geq 2 \). Since \( n-l \geq 2 \) and \( l \geq 1 \), we get that \( n \geq l + 2 \geq 3 \). Thus, \( \alpha = R(m-k, n) \) satisfies that \( m-k \geq 2 \) and \( n \geq 3 \). By Lemma 1, \( \alpha \) contains a canonical Hamiltonian cycle \( HC_\alpha \) whose one flat face is placed to face \( \beta \). Consider that \( k = 1 \). Then, \( \beta = R(k, n-l) \) is a 1-rectangle. Let \( V(L_\beta) = \{v_1, v_2, \ldots, v_{n-l}\} \), where \( v_{i+1} = v_i + 1 \) for \( 1 \leq i \leq n-l - 1 \). Since \( HC_\alpha \) contains a flat face that is placed to face \( \beta \), there exists an edge \((u, v)\) in \( HC_\alpha \) such that \( u \sim v_1 \) and \( v \sim v_1 \). By Statement (3) of Proposition 5, \( v_1 \) and \( HC_\alpha \) can be combined into a cycle \( HC_{\alpha} \). By the same argument, \( v_2, v_3, \ldots, v_{n-l-1} \) can be merged into the cycle to form a Hamiltonian cycle of \( L(m, n; k, l) \). On the other hand, consider that \( k \geq 2 \). Then, \( L_\beta = R(k, n-l) \) satisfies that \( k \geq 2 \) and \( n-1 \geq 2 \). By Lemma 1, \( L_\beta \) contains a canonical Hamiltonian cycle \( HC_\beta \) such that its one flat face is placed to face \( \alpha \). Then, there exist two edges \( e_1 = (u_1, v_1) \in HC_\alpha \) and \( e_2 = (u_2, v_2) \in HC_\beta \) such that \( e_1 \approx e_2 \). By Statement (1) of Proposition 5, \( HC_\alpha \) and \( HC_\beta \) can be combined into a Hamiltonian cycle of \( L(m, n; k, l) \). For instance, Fig. 12(d) shows a Hamiltonian cycle of \( L(m, n; k, l) \) when \( m-k \geq 2 \), \( n-l \geq 2 \), and \( k \geq 2 \). Thus, \( L(m, n; k, l) \) contains a Hamiltonian cycle in this case.

It immediately follows from the above cases that \( L(m, n; k, l) \) contains a Hamiltonian cycle if it does not satisfy condition (F3). Thus, the theorem holds true.

B. The Hamiltonian connected property of \( L \)-shaped supergrid graphs

In this subsection, we will verify the Hamiltonian connectivity of \( L \)-shaped supergrid graphs. Besides condition (F1) (as depicted in Fig. 13(a) and Fig. 13(b)), whenever one of the following conditions is satisfied then \( HP(L(m, n; k, l), s, t) \) does not exist.

(F4) there exists a vertex \( w \) in \( L(m, n; k, l) \) such that \( \deg(w) = 1 \), \( w \neq s \), and \( w \neq t \)

(F5) \( m-k = 1 \), \( n-l = 2 \), \( l = 1 \), \( k \geq 2 \), and \( \{s, t\} = \{(1, 2), (2, 3)\} \) or \( \{1, 3\}, (2, 2) \) (see Fig. 13(d)).

The following lemma shows the necessary condition for that \( HP(L(m, n; k, l), s, t) \) does exist.

Lemma 11. Let \( L(m, n; k, l) \) be a \( L \)-shaped supergrid graph with two vertices \( s \) and \( t \). If \( HP(L(m, n; k, l), s, t) \) does exist, then \( L(m, n; k, l), s, t \) does not satisfy conditions (F1), (F4), and (F5).

Proof: Assume that \( L(m, n; k, l), s, t \) satisfies one of conditions (F1), (F4), and (F5). For condition (F1), the proof is the same as that of Lemma 2. For condition (F4), it is easy to see that \( HP(L(m, n; k, l), s, t) \) does not exist (see Fig. 13(c)). For (F5), we make a horizontal separation on it to obtain two disjoint rectangular supergrid subgraphs \( \alpha = R(m-k, n) \) and \( \beta = R(m, n-1) \), as depicted in Fig. 13(d). Suppose that \( m-k = 1 \), \( n-l = 2 \), \( l = 1 \), and \( k \geq 2 \). Then, \( \alpha \) contains only one vertex \( w \). Let \( s = (1, 2) \), \( t = (2, 3) \), and \( z = (2, 2) \). Then, there exists no Hamiltonian \((s, t)\)-path of \( \alpha \) such that it contains edge \((s, z)\). Thus, \( w \) cannot be combined into the Hamiltonian \((s, t)\)-path of \( \alpha \) and hence \( HP(L(m, n; k, l), s, t) \) does not exist.

We then prove that \( HP(L(m, n; k, l), s, t) \) does exist when \( (L(m, n; k, l), s, t) \) does not satisfy conditions (F1), (F4), and (F5). First, we consider the case that \( m-k = 1 \) or \( n-l = 1 \) in the following lemma.

Lemma 12. Let \( L(m, n; k, l) \) be a \( L \)-shaped supergrid graph, and let \( s \) and \( t \) be its two distinct vertices such...
Lemma 9, where \( n - l \geq 2 \), there exists a Hamiltonian \((s, t)\)-path \( P_3 \) of \( R_\beta \) such that \((w, z) \in P_3\). By Statement (3) of Proposition 5, vertex \( r \) can be combined into path \( P_3 \) to form a Hamiltonian \((s, t)\)-path of \( L(m, n; k, l) \). The construction of a such Hamiltonian \((s, t)\)-path of \( L(m, n; k, l) \) is depicted in Fig. 15. Notice that, in this subcase, we have constructed a Hamiltonian \((s, t)\)-path \( P \) such that an edge \((r, w) \in P\).

Next, we consider the case that \( m - k \geq 2 \) and \( n - l \geq 2 \). Notice that in this case \((L(m, n; k, l), s, t)\) does not satisfy conditions (F4) and (F5).

Lemma 13. Let \( L(m, n; k, l) \) be a \( L \)-shaped supergrid graph with \( m - k \geq 2 \) and \( n - l \geq 2 \), and let \( s \) and \( t \) be its two distinct vertices such that \((L(m, n; k, l), s, t)\) does not satisfy condition (F1). Then, \( L(m, n; k, l) \) contains a Hamiltonian \((s, t)\)-path, i.e., \( HP(L(m, n; k, l), s, t) \) does exist.

Proof: We will provide a constructive method to prove this lemma. That is, a Hamiltonian \((s, t)\)-path of \((L(m, n; k, l), s, t)\) will be constructed. Since \( m - k \geq 2 \), \( n - l \geq 2 \), and \( k, l \geq 1 \), we get that \( m \geq 3 \) and \( n \geq 3 \). Note that \((L(m, n; k, l), s, t)\) is obtained from \((R(m, n))\) by removing \( R(k, l) \) from its upper-right corner. Based on the sizes of \( k \) and \( l \), there are the following two cases:

Case 1: \( k = 1 \) and \( l = 1 \). Let \( z = \) be the only vertex in \( V(R(m, n) - L(m, n; k, l)) \). Then, \((m, 1)\) is the upper-right corner of \( R(m, n) \). By Lemma 3, there exists a Hamiltonian \((s, t)\)-path \( P_3 \) of \((R(m, n))\). Let \( P = P_3 \Rightarrow z \Rightarrow P_2 \). Since \( N(z) \) forms a clique, \( eNd(P_3) \sim start(P_2) \). Thus, \( P_1 \Rightarrow P_2 \) forms a Hamiltonian \((s, t)\)-path of \((L(m, n; k, l))\). The construction of a such Hamiltonian \((s, t)\)-path is depicted in Fig. 16(a).

Case 2: \( k \geq 2 \) or \( l \geq 2 \). By symmetry, we can only consider that \( k \geq 2 \). Depending on the locations of \( s \) and \( t \), we consider the following three subcases:

Case 2.1: \( (m - k \geq 3) \) or \( (m - k = 2 \) and \( (y_0 = t_y, \ (y_0 = t_y = 1) \text{, or } (y_0 = t_y = n)) \)). In this subcase, \((s, t)\) is not a vertex cut of \( \bar{R} \). We make a vertical separation on \((L(m, n; k, l))\) to obtain two disjoint rectangular supergrid graphs \( R_\alpha = R(m - k, n) \) and \( R_\beta = R(k, n - l) \). Consider \((R_\alpha, s, t)\). Condition (F1) holds only if \( m - k = 2 \) and \( 2 \leq y_{\text{min}} = t_y \leq n - 1 \). Since \( y_0 \neq t_y, \ y_0 = t_y = 1, \) or \( y_0 = t_y = n, \) it is clear that \((R_\alpha, s, t)\) does not satisfy
condition (F1). Let \( w = (m - k, n), z = (m - k, n - 1), \) and \( f = (m - k, n - 2). \) Also, assume \((1, 1)\) is the down-right corner of \( R_\alpha. \) Since \((R_\alpha, s, t)\) does not satisfy condition (F1), by Lemma 3 (when \((R_\alpha, s, t)\) satisfies condition (F2), Lemma 7, and Lemma 9, we can construct a Hamiltonian \((s, t)\)-path \( P_{st} \) of \( R_\alpha \) such that \( s, t \) is in \( V(R_\alpha) \) and \( s \) is not a vertex cut of \( R_\alpha \). By Lemma 1, there exists a Hamiltonian cycle \( C_\beta \) of \( R_\beta \) such that its one flat face is placed to face \( R_\alpha. \) Then, there exist two edges \( e_1 \in C_\beta \) and \((w, z)\) or \((z, f)\) in \( P_{st} \). By (2) of Proposition 5, \( P_{st} \) and \( C_\beta \) can be combined into a Hamiltonian \((s, t)\)-path of \( L(m, n; k, l). \) The construction of a such Hamiltonian path is shown in Figs. 16(b)–(c).

Case 2.1.2: \( m - k = 2 \) and \( 2 \leq s_y = t_y < n - 1. \) In this subcase, \([s, t]\) is a vertex cut of \( R_\beta. \) If \( s_y = t_y \leq l, \) then \((L(m, n; k, l), s, t)\) satisfies condition (F1), a contradiction. Thus, \( s_y = t_y \geq l. \) Let \( w = (1, l + 1), z = (2, l + 1), \) and \( f = (3, l + 1). \) We make a horizontal separation on \( L(m, n; k, l) \) to obtain two disjoint rectangular supergrid graphs \( R_\beta = R(m, k - l) \) and \( R_\alpha = R(m, n - l). \) A simple check shows that \((R_\alpha, s, t)\) does not satisfy condition (F1). Since \((R_\alpha, s, t)\) does not satisfy conditions (F1), by Lemma 7 and Lemma 9, we can construct a Hamiltonian \((s, t)\)-path \( P_{st} \) of \( R_\alpha \) such that \((w, z)\) or \((z, f)\) in \( P_{st} \) depending on whether \([s, t]\) is \((1, l + 1), (2, l + 1)\). First, let \( l = 1. \) By Lemma 1, there exists a Hamiltonian cycle \( C_\beta \) of \( R_\beta \) such that its one flat face is placed to face \( R_\alpha. \) Then, there exist two edges \( e_1 \in C_\beta \) and \((w, z)\) or \((z, f)\) in \( P_{st} \) such that \( e_1 \approx (w, z) \) or \( e_1 = (z, f). \) By (2) of Proposition 5, \( P_{st} \) and \( C_\beta \) can be combined into a Hamiltonian \((s, t)\)-path of \( L(m, n; k, l). \) The construction of a such Hamiltonian path is shown in Figs. 16(d).

Case 2.2: \( s_x, t_x > m - k. \) Based on the size of \( l, \) we consider the following two subcases:

Case 2.2.1: \( l > 1 \) or \( l = 1 \) and \( m - k = 2. \) A Hamiltonian \((s, t)\)-path of \( L(m, n; k, l) \) can be constructed by similar arguments in proving Case 2.1.2. Figs. 17(a)–(b) depict the construction of a such Hamiltonian \((s, t)\)-path of \( L(m, n; k, l) \) in this subcase.

Case 2.2.2: \( l = 1 \) and \( m - k > 2. \) Let \( r = (m - k, 1) \) and \( w = (m - k, 2) \) be two vertices in \( L(m, n; k, l). \) We make a vertical separation on \( L(m, n; k, l) \) to obtain two disjoint supergrid subgraphs \( R_\beta = R(m, n) \) and \( R_\alpha = (m - m, n; k, l), \) where \( m' = m - k - 1; \) as depicted in Fig. 17(c). Clearly, \( m - m' = 1 \) and \( (R_\alpha, s, t) \) lies on Case 2 of Lemma 12. By Lemma 12, we can construct a Hamiltonian \((s, t)\)-path \( P_{st} \) of \( R_\alpha \) such that \((r, w)\) in \( P_{st} \). By Lemma 1, there exists a Hamiltonian cycle \( C_\beta \) of \( R_\beta \) such that its one flat face is placed to face \( R_\alpha. \) Then, there exist two edges \( e_1 \in C_\beta \) and \((r, w)\) in \( P_{st} \) such that \( e_1 \approx (r, w). \) By (2) of Proposition 5, \( P_{st} \) and \( C_\beta \) can be combined into a Hamiltonian \((s, t)\)-path of \( L(m, n; k, l). \) The construction of a such Hamiltonian path is shown in Fig. 17(c).
Theorem 14. Let $L(m, n; k, l)$ be a $L$-shaped supergrid graph with vertices $s$ and $t$. Then, $L(m, n; k, l)$ contains a Hamiltonian $(s, t)$-path if and only if $(L(m, n; k, l), s, t)$ does not satisfy conditions $(F1)$, $(F4)$, and $(F5)$.

V. The Longest $(s, t)$-Path Algorithm

It follows from Theorem 14 that if $(L(m, n; k, l), s, t)$ satisfies one of conditions $(F1)$, $(F4)$, and $(F5)$, then $(L(m, n; k, l), s, t)$ contains no Hamiltonian $(s, t)$-path. So in this section, first for these cases we give upper bounds on the lengths of longest paths between $s$ and $t$. Then, we show that these upper bounds equal to the lengths of longest paths between $s$ and $t$. Recall that $\hat{L}(G, s, t)$ denotes the length of longest $(s, t)$-path in $G$, and the length of a path is the number of vertices in the path. In the following, we will use $\hat{U}(G, s, t)$ to indicate the upper bound on the length of longest $(s, t)$-paths in $G$, where $G$ is a rectangular or $L$-shaped supergrid graph. Notice that the isomorphic cases are omitted. Depending on the sizes of $m - k$ and $n - l$, we provide the following two lemmas to compute the upper bounds when $(L(m, n; k, l), s, t)$ satisfies either condition $(F1)$ or $(F4)$.

Lemma 15. Let $m - k = n - l = 1$ and $l > 1$. Then, the following implications (conditions) hold:

- $(UB1)$ If $s_y, t_y \leq l$, then the length of any path between $s$ and $t$ cannot exceed $|t_y - s_y| + 1$ (see Fig. 18(a)).
- $(UB2)$ If $s_y < l$ and $t_y > 1$, then the length of any path between $s$ and $t$ cannot exceed $n - s_y + t_x$ (see Fig. 18(b)).
- $(UB3)$ If $s_x = t_x = 1, \max\{s_y, t_y\} = n$, and $[(k > 1) or (k = 1 and \min\{s_y, t_y\} > 1)]$, then the length of any path between $s$ and $t$ cannot exceed $|t_y - s_y| + 2$ (see Fig. 18(c)).

Proof: Since $n - l = m - k = 1$, there is only one single path between $s$ and $t$ that has the specified.

Lemma 16. Let $n - l > 1$. Then, the following implications (conditions) hold:

- $(UB4)$ If $m = k = 1$, $l > 1$, and $\{s_y, t_y > l and \{s, t\} is not a vertex cut\}$, $s_y \leq l$ and $t_y > 1$, or $(t_y \leq l$ and $s_y > 1)$, then the length of any path between $s$ and $t$ cannot exceed $\hat{L}(G', s, t)$; where $G' = L(m, n - n'; k, l') and l' = l - n'$, and $n' = l - 1$ if $s_y, t_y \geq l$; otherwise $n' = \min\{s_y, t_y\} - 1$ (see Figs. 18(d)–(f)).

Proof: For $(UB4)$, let $w = (1, l)$ if $s_y, t_y \geq l$; otherwise $w = \min\{s_y, t_y\}$. Since $w$ is a cut vertex, hence removing $w$ clearly disconnects $L(m, n; k, l)$ into two components, and a simple path between $s$ and $t$ can only go through a component that contains $s$ and $t$, let this component be $G'$. Therefore, its length cannot exceed $\hat{L}(G', s, t)$. For $(UB5)$, consider Fig. 19(a). Since $(s, t)$ is a vertex cut of $L(m, n; k, l)$, the length of any path between $s$ and $t$ cannot exceed $\max\{L(G_1, s, t), L(G_2, s, t)\}$, where $G_1$ and $G_2$ are defined in Figs. 19(b)–(g).

We have computed the upper bounds of the longest $(s, t)$-paths when $(L(m, n; k, l), s, t)$ satisfies condition $(F1)$ or $(F4)$. The following lemma shows the upper bound when $(L(m, n; k, l), s, t)$ satisfies condition $(F5)$.

Lemma 17. If $(L(m, n; k, l), s, t)$ satisfies condition $(F5)$, then the length of any path between $s$ and $t$ cannot exceed $mn - kl - 1$.

Proof: Consider Fig. 20. We can easily check that the length of any path between $s$ and $t$ cannot exceed $\hat{L}(G_1, s, p) + \hat{L}(G_2, q, t) = mn - kl - 1$.

It is easy to show that any $(L(m, n; k, l), s, t)$ must satisfy one of conditions $(L0)$, $(UB1)$, $(UB2)$, $(UB3)$, $(UB4)$, $(UB5)$, $(UB6)$, and $(F5)$, where $(L0)$ is defined as follows:

![Fig. 18](image1)

![Fig. 19](image2)
(L0) \((L(m, n; k, l), s, t)\) does not satisfy any of conditions (F1), (F4), and (F5).

If \((L(m, n; k, l), s, t)\) satisfies (L0), then \(\hat{U}(L(m, n; k, l), s, t)\) is not a Hamiltonian graph. Otherwise, \(\hat{U}(L(m, n; k, l), s, t)\) can be computed by using Lemmas 15–17. So, we have the following formula of upper bounds:

\[
\hat{U}(L(m, n; k, l), s, t) = \begin{cases} 
|t_y - s_y| + 1, & \text{if (UB1) holds;} \\
|n - s_y| + t_x, & \text{if (UB2) holds;} \\
|s_y - s| + 2, & \text{if (UB3) holds;} \\
L(G', s, t), & \text{if (UB4) or (UB5) holds;} \\
\max\{\hat{L}(G_1, s, t), \hat{L}(G_2, s, t)\}, & \text{if (UB6) holds;} \\
mn - kl - 1, & \text{if (F5) holds;} \\
mn - kl, & \text{if (L0) holds.}
\end{cases}
\]

Now, we show how to obtain a longest \((s, t)\)-path for \(L\)-shaped supergrid graphs. Notice that if \((L(m, n; k, l), s, t)\) satisfies conditions (F1), (F4), and (F5). Then, let \(s, p, t\) be a vertex cut of \(L(m, n; k, l)\) and see Fig. 13(d). Consider Fig. 20. By Lemma 17, \(\hat{U}(L(m, n; k, l), s, t) = \hat{L}(G_1, s, p) + \hat{L}(G_2, q, t)\). By Theorem 6, there exist a longest \((s, p)\)-path \(P_1\) and a longest \((q, t)\)-path \(P_2\) of \(G_1\) and \(G_2\), respectively. Then, \(P_1 \Rightarrow P_2\) forms a Hamiltonian \((s, t)\)-path of \(L(m, n; k, l)\).

It follows from Theorem 14 and Lemmas 15–18 that the following theorem concludes our result.

**Theorem 19.** Given a \(L\)-shaped supergrid \(L(m, n; k, l)\) and two distinct vertices \(s\) and \(t\) in \(L(m, n; k, l)\), a longest \((s, t)\)-path can be computed in \(O(nm - kl)\)-linear time.

The linear-time algorithm is formally presented as Algorithm 1.

**VI. CONCLUDING REMARKS**

Based on the Hamiltonicity and Hamiltonian connectivity of rectangular supergrid graphs, we first obtain two Hamiltonian connected properties of rectangular supergrid graphs. Using the Hamiltonicity and Hamiltonian connectivity of rectangular supergrid graphs, we prove \(L\)-shaped supergrid graphs to be Hamiltonian and Hamiltonian connected except one or three conditions. Furthermore, we present a linear-time algorithm to compute the longest \((s, t)\)-path of a \(L\)-shaped supergrid graph. The Hamiltonian cycle problem on solid grid graphs was known to be polynomial solvable. However, it remains open for solid supergrid graphs in which there exists no hole. This result can be regarded as the first attempt for solving the Hamiltonian and longest \((s, t)\)-path problems on solid supergrid graphs, where \(L\)-shaped supergrid graphs form a subclass of solid supergrid graphs.
Algorithm 1: The longest \((s, t)\)-path algorithm

**Input:** A \(L\)-shaped supergrid graph \(L(m, n; k, l)\) with \(mn \geq 2\), and two distinct vertices \(s\) and \(t\) in \(L(m, n; k, l)\).

**Output:** The longest \((s, t)\)-path.

**Method:**

1. If \((m - k = 1)\) or \((n - l = 1)\) and \((L(m, n; k, l), s, t)\) does not satisfy conditions \((F1)\), \((F4)\), and \((F5)\) then output \(HP(L(m, n; k, l), s, t) \) constructed from Lemma 12;
   // construct Hamiltonian \((s, t)\)-path when \(m - k = 1\) or \(n - l = 1\)

2. If \((m - k \geq 2)\) and \((n - l \geq 2)\) and \((L(m, n; k, l), s, t)\) does not satisfy conditions \((F1)\), \((F4)\), and \((F5)\) then output \(HP(L(m, n; k, l), s, t) \) constructed from Lemma 13;
   // construct Hamiltonian \((s, t)\)-path when \(m - k \geq 2\) and \(n - l \geq 2\)

3. If \((L(m, n; k, l), s, t)\) satisfies one of conditions \((F1)\), \((F4)\), and \((F5)\), then output the longest \((s, t)\)-path based on Lemma 18;
   // construct the longest \((s, t)\)-path when \(L(m, n; k, l)\) contains no Hamiltonian \((s, t)\)-path

**REFERENCES**


