Stability and Bifurcation Analysis of A Delayed Worm Propagation Model in Mobile Internet

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***Abstract*—This paper is concerned with the Hopf bifurcation of a delayed SEIQR (Susceptible-Exposed-Infectious-Quarantined-Recovered) worm propagation model in mobile internet. Firstly, a distinct set of conditions that ensures existence of the Hopf bifurcation are derived by analyzing the roots of the associated characteristic equation and regarding the delay as a bifurcation parameter. Secondly, the normal form theory and the center manifold theorem are employed to determine direction of the Hopf bifurcation and stability of the bifurcating periodic solutions. Finally, some numerical simulations are presented to illustrate the theoretical results.**

***Index Terms*—Delayed SEIQR model, Hopf bifurcation, Periodic solutions, Worm.**

I. Introduction

THE cyber security threats have increased sharply in recent years as a result of propagation of computer viruses in the Internet [1-3]. Worms that can lead to large scale of network congestion are one of the most harmful computer viruses in the Internet. Due to the high similarity between computer worms and biological viruses, many mathematical models [4-8] have been proposed to explore dynamics of computer worms in the Internet since the pioneering work of Kephart and White [9, 10].

With the rapid development of network communication technology, mobile devices have become increasingly pervasive, which attracts attackers to propagate worm programs among these mobile equipments. However, the above worms propagation models can not be used directly in the mobile environment owing to the differences between computers and smartphones, especially in the Wi-Fi scenario. Chameleon, a new Wi-Fi worm with high performance, appeared in 2014 and it could be spread in a manner similar to that of airborne diseases [11]. In order to prevent worms attacks in mobile internet, Xiao et al. [12] proposed an SEIQR worm propagation model and explored the dynamic behavior of worms in mobile internet.

However, Xiao et al. [12] assume that the exposed nodes, the infected nodes and the quarantined nodes revert to the recovered ones instantaneously in the proposed SEIQR worm propagation model in mobile internet, which is not reasonable as the anti-virus software needs a period of time to clean the worms between these nodes. Thus, when the worms in the exposed nodes, the infected nodes and the quarantined nodes are being cleaned by the anti-virus software, there is a time delay before those nodes develop themselves into the recovered ones. In addition, a stability switch and Hopf bifurcation will occur even if the ignored delay is small for a dynamical model. Realistic examples can be found in computer virus models [13-16], predator-prey models [17-22], and neural networks [23-25]. The occurrence of Hopf bifurcation means that the state of worm prevalence changes from an equilibrium point to a limit cycle, which is not welcomed in networks. Motivated by the work above, the time delay representing the period that anti-virus software uses to clean the worms in the nodes, is incorporated into the SEIQR model in [12]. And Hopf bifurcation of the delayed SEIQR propagation model is also investigated in this paper.

The remaining materials of the present paper are organized as follows. Section 2 formulates the SEIQR worm propagation model with time delay. Section 3 shows the local stability of the viral equilibrium and existence of Hopf bifurcation, and analyzes the properties of the Hopf bifurcation. Section 4 examines the results by using some numerical simulations. A conclusion is given in Section 5.

II. Model formulation

The proposed SEIQR worm propagation model in [12] is as follows:

$$\begin{cases} \frac{dS(t)}{dt} &= -\beta S(t)I(t) + \mu N - \mu S(t), \\ \frac{dE(t)}{dt} &= \beta S(t)I(t) - \eta E(t) - \varepsilon E(t) - \mu E(t), \\ \frac{dI(t)}{dt} &= \eta E(t) - \mu I(t) - \sigma I(t) - \gamma I(t), \\ \frac{dQ(t)}{dt} &= \sigma I(t) - \varphi Q(t) - \mu Q(t), \\ \frac{dR(t)}{dt} &= \varepsilon E(t) + \gamma I(t) + \varphi Q(t) - \mu R(t), \end{cases} \tag{1}$$

where $S(t)$, $E(t)$, $I(t)$, $Q(t)$ and $R(t)$ represent the numbers of the susceptible nodes, the exposed nodes, the infected nodes, the quarantined nodes and the recovered nodes at time t, respectively. N is the total number of nodes in system (1); μ is the natural birth rate of the nodes and it is also the death rate of the nodes; β is the infection rate of the susceptible nodes; η, ε, σ, γ and φ are the transition rates. Xiao et al. [12] studied stability of the worm-free equilibrium and the viral equilibrium. Based on system (1), we consider the delayed model:

$$\begin{cases} \frac{dS(t)}{dt} &= -\beta S(t)I(t) + \mu N - \mu S(t), \\ \frac{dE(t)}{dt} &= \beta S(t)I(t) - \eta E(t) - \varepsilon E(t-\tau) - \mu E(t), \\ \frac{dI(t)}{dt} &= \eta E(t) - \mu I(t) - \sigma I(t) - \gamma I(t-\tau), \\ \frac{dQ(t)}{dt} &= \sigma I(t) - \varphi Q(t-\tau) - \mu Q(t), \\ \frac{dR(t)}{dt} &= \varepsilon E(t-\tau) + \gamma I(t-\tau) + \varphi Q(t-\tau) - \mu R(t), \end{cases} \tag{2}$$

Manuscript received November 18, 2019; revised March 21, 2020. This work was supported by Natural Science Foundation of Anhui Province (Nos. 1608085QF145), Natural Science Foundation of the Higher Education Institutions of Anhui Province(No. KJ2014A006) and ”Qinglan Project” of Jiangsu Province.

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where τ is the time delay, during which anti-virus software will clean the worms in the nodes.

Assumption 1.

$$D_2 = \det\begin{pmatrix} m_{04} & 1 \\ m_{02} & m_{03} \end{pmatrix} > 0, \quad (3)$$

$$D_3 = \det\begin{pmatrix} m_{04} & 1 & 0 \\ m_{02} & m_{03} & m_{04} \\ 0 & m_{01} & m_{02} \end{pmatrix} > 0, \quad (4)$$

$$D_4 = \det\begin{pmatrix} m_{04} & 1 & 0 & 0 \\ m_{02} & m_{03} & m_{04} & 1 \\ m_{00} & m_{01} & m_{02} & m_{03} \\ 0 & 0 & m_{00} & m_{01} \end{pmatrix} > 0, \quad (5)$$

$$D_5 = \det\begin{pmatrix} m_{04} & 1 & 0 & 0 & 0 \\ m_{02} & m_{03} & m_{04} & 1 & 0 \\ m_0 & m_{01} & m_{02} & m_{03} & m_{04} \\ 0 & 0 & m_{00} & m_{01} & m_{02} \\ 0 & 0 & 0 & 0 & m_{00} \end{pmatrix} > 0, \quad (6)$$

where

$$\begin{aligned}
m_{00} &= m_0 + n_0 + p_0 + q_0, m_{01} = m_1 + n_1 + p_1 + q_1, \\
m_{02} &= m_2 + n_2 + p_2 + q_2, m_{03} = m_3 + n_3 + p_3, \\
m_{04} &= m_4 + n_4,
\end{aligned}$$

and

$$\begin{aligned}
m_0 &= -a_{44}a_{55}(a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32}), \\
m_1 &= a_{11}a_{22}a_{55}(a_{33} + a_{44}) + a_{33}a_{44}a_{55}(a_{11} + a_{22}) \\
&\quad + a_{13}a_{21}a_{32}(a_{44} + a_{55}) + a_{11}a_{22}a_{33}a_{44}, \\
m_2 &= -a_{55}(a_{11} + a_{22})(a_{33} + a_{44}) \\
&\quad -(a_{11}a_{22}a_{55} + a_{33}a_{44}a_{55} + a_{13}a_{21}a_{32}) \\
&\quad -(a_{11}a_{22}(a_{33} + a_{44}) + a_{33}a_{44}(a_{11} + a_{22})), \\
m_3 &= a_{55}(a_{11} + a_{22} + a_{33} + a_{44}) + a_{11}a_{22} + a_{33}a_{44} \\
&\quad + (a_{11} + a_{22})(a_{33} + a_{44}), \\
m_4 &= -(a_{11} + a_{22})(a_{33} + a_{44} + a_{55}), \\
n_0 &= -a_{55}b_{44}(a_{11}a_{22}(a_{33} + a_{13}a_{21}a_{32}) \\
&\quad -a_{11}a_{44}a_{55}(a_{33}b_{22} + a_{22}b_{33}), \\
n_1 &= a_{55}(a_{11} + a_{22})(a_{33}b_{44} + a_{44}b_{33}) - a_{11}(a_{44} + a_{55}) \\
&\quad (a_{33}b_{22} + a_{22}b_{33}) + a_{44}a_{55}b_{22}(a_{11} + a_{33}) \\
&\quad + a_{11}a_{22}b_{44}(a_{33} + a_{55}) + a_{13}a_{21}a_{32}b_{44}, \\
n_2 &= -(b_{22}(a_{11}a_{33} + a_{44}a_{55}) + b_{33}(a_{11}a_{22} + a_{44}a_{55}) \\
&\quad + b_{44}(a_{11}a_{22} + a_{33}a_{55}) \\
&\quad + b_{22}(a_{11} + a_{33})(a_{44} + a_{55}) \\
&\quad + b_{33}(a_{11} + a_{22})(a_{44} + a_{55}) \\
&\quad + b_{44}(a_{11} + a_{22})(a_{33} + a_{55})), \\
n_3 &= b_{22}(a_{11} + a_{33} + a_{44} + a_{55}) \\
&\quad + b_{33}(a_{11} + a_{22} + a_{44} + a_{55}) \\
&\quad + b_{44}(a_{11} + a_{22} + a_{33} + a_{55}), \\
n_4 &= -(b_{22} + b_{33} + b_{44}), \\
p_0 &= a_{11}a_{55}b_{44}(a_{22}b_{33} + a_{33}b_{22}) + a_{11}a_{22}a_{44}b_{22}b_{33}, \\
p_1 &= b_{22}b_{33}(a_{11}a_{22} + a_{11}a_{44} + a_{22}a_{44}) \\
&\quad + b_{22}b_{44}(a_{11}a_{33} + a_{11}a_{55} + a_{33}a_{55}) \\
&\quad + b_{33}b_{44}(a_{11}a_{22} + a_{11}a_{55} + a_{22}a_{55}), \\
p_2 &= -(b_{33}b_{44}(a_{11} + a_{22} + a_{55}) + b_{22}b_{33}(a_{11} + a_{22} + a_4) \\
&\quad + b_{22}b_{44}(a_{11} + a_{33} + a_{55})), \\
p_3 &= b_{22}b_{33} + b_{22}b_{44} + b_{33}b_{44}, \\
q_0 &= -a_{11}a_{55}b_{22}b_{33}b_{44}, \\
q_1 &= (a_{11} + a_{55})b_{22}b_{33}b_{44}, \\
q_2 &= -b_{22}b_{33}b_{44},
\end{aligned}$$

and

$$\begin{aligned}
a_{11} &= -(\beta I_* + \mu), a_{13} = \beta S_*, a_{21} = \beta I_*, \\
a_{22} &= -(\eta + \mu), a_{23} = \beta S_*, a_{32} = \eta, \\
a_{33} &= -(\mu + \sigma), a_{43} = \sigma, a_{44} = -\mu, a_{55} = -\mu, \\
b_{22} &= -\varepsilon, b_{33} = -\gamma, b_{44} = -\varphi, \\
b_{52} &= \varepsilon, b_{53} = \gamma, b_{54} = \varphi.
\end{aligned}$$

Assumption 2. Eq.(7) has at least one positive root ω_0.

$$f_1^2(\omega) + f_2^2(\omega) = 1, \quad (7)$$

where

$$f_1(\omega) = \cos\tau\omega, f_2(\omega) = \sin\tau\omega, \quad (8)$$

and $\cos\tau\omega$ is the root of the following equation

$$\begin{aligned}
&c_4(\omega)\cos^4\tau\omega + c_3(\omega)\cos^3\tau\omega + c_2(\omega)\cos^2\tau\omega \\
&+ c_1(\omega)\cos\tau\omega + c_0(\omega) = 0, \quad (9)
\end{aligned}$$

with

$$\begin{aligned}
c_0(\omega) &= r_0^2(\omega) - r_1^2(\omega), \\
c_1(\omega) &= 2(r_0(\omega) \times r_3(\omega) - r_1(\omega) \times r_2(\omega)), \\
c_2(\omega) &= r_3^2(\omega) + 2r_0(\omega) \times r_4(\omega) + r_1^2(\omega) - r_2^2(\omega), \\
c_3(\omega) &= 2(r_3(\omega) \times r_4(\omega) + r_1(\omega) \times r_2(\omega)), \\
c_4(\omega) &= r_4^2(\omega) + r_2^2(\omega),
\end{aligned}$$

and

$$\begin{aligned}
r_0(\omega) &= \omega^{10} + (m_4^2 + n_4^2 - 2(m_3 + p_3))\omega^8 \\
&\quad + ((m_3 + p_3)^2 + n_3^2 + 2m_1 - 2p_1 \\
&\quad - 2m_2m_4 - 2m_4p_2 - 2n_2n_4)\omega^6 \\
&\quad + (m_2^2 + n_2^2 + p_2^2 - q_2^2 + 2m_0m_4 + 2m_2p_2 + 2m_4p_0 \\
&\quad + 2n_0n_4 - 2n_1n_3 - 2(m_1 + p_1)(m_3 + p_3))\omega^4 \\
&\quad + (m_1^2 + n_1^2 + p_1^2 - q_1^2 - 2m_0m_2 \\
&\quad - 2n_0n_2 - 2p_0p_2 - 2m_1p_1)\omega^2 \\
&\quad + m_0^2 + n_0^2 + p_0^2 - q_0^2 + 2m_0p_0, \\
r_1(\omega) &= -2n_4\omega^9 + 2(n_2 - m_4n_3 + n_4(m_3 + p_3))\omega^7 \\
&\quad + 2(m_4n_1 - n_0 + n_3(m_2 - p_2) \\
&\quad - n_2(m_3 + p_3) - n_4(m_1 - p_1))\omega^5 \\
&\quad + 2(n_0(m_3 + p_3) + n_2(m_1 - p_1) \\
&\quad - n_1(m_2 - p_2) - n_3(m_0 - p_0))\omega^3 \\
&\quad + 2(n_1(m_0 - p_0) - n_0(m_1 - p_1))\omega,
\end{aligned}$$

$$\begin{aligned} r_2(\omega) &= -4p_2\omega^7 + 4(p_0 - m_4p_1 + p_2(m_3+p_3))\omega^5 \\ &\quad -4(p_0(m_3+p_3) + m_1p_2 + m_2p_1)\omega^3 \\ &\quad +4(m_1p_0 - m_0p_1)\omega, \\ r_3(\omega) &= 2(m_4n_4 - n_3)\omega^8 + 2(n_1 - m_4n_2 \\ &\quad +n_3(m_3+p_3) - n_4(m_2+p_2))\omega^6 \\ &\quad +2(m_4n_0 + n_2(m_2+p_2) + n_4(m_0+p_0) \\ &\quad -n_1(m_3+p_3) - n_3(m_1+p_1))\omega^4 \\ &\quad +2(n_1(m_1+p_1) - n_2(m_0+p_0))\omega^2 \\ &\quad +2n_0(m_0+p_0), \\ r_4(\omega) &= 4(p_1 + m_4p_2)\omega^6 \\ &\quad +4(m_2p_2 + m_4p_0 - p_1(m_3+p_3))\omega^4 \\ &\quad +4m_1p_1\omega^2 - 4m_0p_0. \end{aligned}$$

Assumption 3.

$$G_{1R} \times G_{2R} + G_{1I} \times G_{2I} \neq 0,$$

where

$$\begin{aligned} G_{1R} &= (5\omega_0^4 - 3(m_3-p_3)\omega_0^2 + m_1 + p_1)\cos\tau_0\omega_0 \\ &\quad -2((m_2-p_2)\omega_0 - 2m_4\omega_0^3)\sin\tau_0\omega_0 \\ &\quad +2q_2\omega_0\sin 2\tau_0\omega_0 + q_1\cos 2\tau_0\omega_0 + n_1 - 3n_3\omega_0^2, \\ G_{1I} &= (5\omega_0^4 - 3(m_3-p_3)\omega_0^2 + m_1 - p_1)\sin\tau_0\omega_0 \\ &\quad +2((m_2+p_2)\omega_0 - 2m_4\omega_0^3)\cos\tau_0\omega_0 \\ &\quad +2q_2\omega_0\cos 2\tau_0\omega_0 - q_1\sin 2\tau_0\omega_0 \\ &\quad +2n_2\omega_0 - 4n_4\omega_0^3, \\ G_{2R} &= (\omega_0^4 - (m_3-p_3)\omega_0^2 + m_1 - p_1)\omega_0^2\cos\tau_0\omega_0 \\ &\quad -((m_2+p_2)\omega_0^2 - m_4\omega_0^4 - m_0 - p_0)\sin\tau_0\omega_0 \\ &\quad +(q_0\omega_0 - q_2\omega_0^3)\sin 2\tau_0\omega_0 - 2q_1\omega_0\cos 2\tau_0\omega_0, \\ G_{2I} &= (\omega_0^4 - (m_3+p_3)\omega_0^2 + m_1 + p_1)\omega_0^2\sin\tau_0\omega_0 \\ &\quad +((m_2-p_2)\omega_0^2 - m_4\omega_0^4 - m_0 + p_0)\cos\tau_0\omega_0 \\ &\quad +(q_0\omega_0 - q_2\omega_0^3)\cos 2\tau_0\omega_0 + 2q_1\omega_0\sin 2\tau_0\omega_0. \end{aligned}$$

III. MAIN RESULTS

Remark 1. It is easy to see that $m_{04} = \beta I_* + \eta + \sigma + \varepsilon + \gamma + \varphi + 5\mu > 0$. Therefore, under *Assumption* **1**, system (2) without delay is locally asymptotically stable based on the Routh-Hurwitz criterion.

Theorem 1. Under *Assumptions* **1**, **2** and **3**, the viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*)$ is locally asymptotically stable when $\tau \in [0, \tau_0)$; a Hopf bifurcation occurs at the viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*)$ when $\tau = \tau_0$ and a family of periodic solutions bifurcate from the viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*)$, where $\tau_0 = \frac{1}{\omega_0}\{\arccos f_1(\omega_0)\}$.

Proof. By direction computation, we obtain the unique viral equilibrium $P_*(S_*, E_*, I_*, Q_*, R_*)$ of system (2), where

$$\begin{aligned} S_* &= \frac{(\eta+\varepsilon+\mu)(\sigma+\gamma+\mu)}{\beta\eta}, \\ E_* &= \frac{\mu(N-S_*)}{\eta+\varepsilon+\mu}, I_* = \frac{\mu(N-S_*)}{\beta S_*}, \\ Q_* &= \frac{\sigma I_*}{\varphi+\mu}, R_* = \frac{\varepsilon E_* + \gamma I_* + \varphi Q_*}{\mu}. \end{aligned}$$

Straightforward calculations show that the characteristic equation of system (2) at the viral equilibrium P_* is as follows:

$$\begin{vmatrix} \lambda - a_{11} & 0 & -a_{13} & 0 & 0 \\ -a_{21} & \lambda - a_{22} - b_{22}e^{-\lambda\tau} & -a_{23} & 0 & 0 \\ 0 & -a_{32} & \lambda - a_{33} - b_{33}e^{-\lambda\tau} & 0 & 0 \\ 0 & 0 & -a_{43} & \lambda - a_{44} - b_{44}e^{-\lambda\tau} & 0 \\ 0 & -b_{52}e^{-\lambda\tau} & -b_{53}e^{-\lambda\tau} & -b_{54}e^{-\lambda\tau} & \lambda - a_{55} \end{vmatrix} = 0. \tag{10}$$

Eq.(10) equals

$$\begin{aligned} &\lambda^5 + m_4\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 \\ &+(n_4\lambda^4 + n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} \\ &+(p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)e^{-2\lambda\tau} \\ &+(q_2\lambda^2 + q_1\lambda + q_0)e^{-3\lambda\tau} = 0, \end{aligned} \tag{11}$$

Multiplying $e^{\lambda\tau}$ on both sides of Eq.(11), Eq.(11) becomes

$$\begin{aligned} &n_4\lambda^4 + n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0 \\ &+(\lambda^5 + m_4\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0)e^{\lambda\tau} \\ &+(p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)e^{-\lambda\tau} \\ &+(q_2\lambda^2 + q_1\lambda + q_0)e^{-2\lambda\tau} = 0, \end{aligned} \tag{12}$$

Let $\lambda = i\omega(\omega > 0)$ be a root of Eq.(12) when $\tau > 0$. Then,

$$\begin{cases} q_1\omega\sin 2\tau\omega + (q_0 - q_2\omega^2)\cos 2\tau\omega = P_1(\omega), \\ q_1\omega\cos 2\tau\omega - (q_0 - q_2\omega^2)\sin 2\tau\omega = P_2(\omega), \end{cases}$$

where

$$\begin{aligned} P_1(\omega) &= (\omega^5 - (m_3+p_3)\omega^3 + (m_1-p_1)\omega)\sin\tau\omega \\ &\quad -(m_4\omega^4 - (m_2+p_2)\omega^2 + m_0 + p_0)\cos\tau\omega \\ &\quad +n_2\omega^2 - n_4\omega^4 - n_0, \\ P_2(\omega) &= -(\omega^5 - (m_3+p_3)\omega^3 + (m_1+p_1)\omega)\cos\tau\omega \\ &\quad -(m_4\omega^4 - (m_2-p_2)\omega^2 + m_0 - p_0)\sin\tau\omega \\ &\quad +n_3\omega^3 - n_1\omega. \end{aligned}$$

Then, the following equation can be obtained

$$\begin{aligned} &r_0(\omega) + r_1(\omega)\sin\tau\omega + r_2(\omega)\sin\tau\omega\cos\tau\omega \\ &+r_3(\omega)\cos\tau\omega + r_4(\omega)\cos^2\tau\omega = 0, \end{aligned} \tag{13}$$

It is known to all, $\sin\tau\omega = \pm\sqrt{1-\cos^2\tau\omega}$. Thus, Eq.(13) can be transformed into

$$\begin{aligned} &r_0(\omega) + r_3(\omega)\cos\tau\omega + r_4(\omega)\cos^2\tau\omega = \\ &\pm(r_1(\omega)\sqrt{1-\cos^2\tau\omega} + r_2(\omega)\sqrt{1-\cos^2\tau\omega}\cos\tau\omega). \end{aligned} \tag{14}$$

It is equivalent to Eq.(9). According to the discussion about Eq.(9) in [26, 27], we can obtain the expression of $\cos\tau\omega$, say $f_1(\omega)$. Also, we can obtain the expression of $\sin\tau\omega$, say $f_2(\omega)$. Further we can obtain Eq.(7). Under *Assumption* **2**, we have

$$\tau_0 = \frac{1}{\omega_0}\{\arccos f_1(\omega_0)\}.$$

Taking the derivative of λ with respect to τ in Eq.(12), we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{G_1(\lambda)}{G_2(\lambda)} - \frac{\tau}{\lambda},$$

with

$$\begin{aligned} G_1(\lambda) &= 4n_4\lambda^3 + 3n_3\lambda^2 + 2n_2\lambda + n_1 + \\ &\quad (5\lambda^4 + 4m_4\lambda^3 + 3m_3\lambda^2 + 2m_2\lambda + m_1)e^{\lambda\tau} \\ &\quad +(3p_3\lambda^2 + 2p_2\lambda + p_1)e^{-\lambda\tau} \\ &\quad +(2q_2\lambda + q_1)e^{-2\lambda\tau}, \\ G_2(\lambda) &= 2(q_2\lambda^3 + q_1\lambda^2 + q_0\lambda)e^{-2\lambda\tau} \\ &\quad +(p_3\lambda^4 + p_2\lambda^3 + p_1\lambda^2 + p_0\lambda)e^{-\lambda\tau} \\ &\quad -(\lambda^6 + m_4\lambda^5 + m_3\lambda^4 \\ &\quad +m_2\lambda^3 + m_1\lambda^2 + m_0\lambda)e^{\lambda\tau}. \end{aligned}$$

Thus,

$$\text{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\lambda=i\omega_0} = \frac{G_{1R}\times G_{2R} + G_{1I}\times G_{2I}}{G_{2R}^2 + G_{2I}^2}.$$

Clearly, under *Assumption* **3**, $\text{Re}[\frac{d\lambda}{d\tau}]_{\lambda=i\omega_0} \neq 0$. Therefore, based on the Hopf bifurcation theorem from Hassard et al. [28], **Theorem 1** is obtained and the proof is completed.

Theorem 2. The sign of μ_2 determines direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical); The sign of β_2 determines stability of the bifurcating periodic solutions: if $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable); The sign of T_2 determines period of the bifurcating solutions: if $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solutions increases (decreases). The expressions of μ_2, $\beta_2 < 0$ and $T_2 > 0$ are as follows

$$\begin{aligned} &C_1(0) = \tfrac{i}{2\tau_0\omega_0}(g_{11}g_{20} - 2|g_{11}|^2 - \tfrac{|g_{02}|^2}{3}) + \tfrac{g_{21}}{2} \\ &\mu_2 = -\tfrac{\text{Re}\{C_1(0)\}}{\text{Re}\{\lambda'(\tau_0)\}}, \\ &\beta_2 = 2\text{Re}\{C_1(0)\}, \\ &T_2 = -\tfrac{\text{Im}\{C_1(0)\} + \mu_2\text{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0}. \end{aligned} \tag{15}$$

Proof.

Letting $u_1(t) = S(t) - S_*$, $u_2(t) = E(t) - E_*$, $u_3(t) = I(t) - I_*$, $u_4(t) = Q(t) - Q_*$, $u_5(t) = R(t) - R_*$, and rescaling the delay by $t \to (t/\tau)$. Let $\tau = \tau_0 + \varrho$, $\varrho \in R$, and the Hopf bifurcation occurs at $\varrho = 0$. Then, system (2) becomes

$$\dot{u}(t) = L_\varrho u_t + F(\varrho, u_t), \tag{16}$$

where $u_t = (u_1(t), u_2(t), u_3(t), u_4(t), u_5(t))^T = (S, E, I, Q, R)^T \in R^5$, $u_t(\theta) = u(t+\theta) \in C = C([-1,0], R^5)$ and $L_\varrho : C \to R^5$, $F(\varrho, u_t) \to R^5$ are given respectively by

$$L_\varrho\phi = (\tau_0 + \varrho)(A_{max}\phi(0) + B_{max}\phi(-1)),$$

and

$$F(\varrho, \phi) = \begin{pmatrix} -\beta\phi_1(0)\phi_3(0) \\ \beta\phi_1(0)\phi_3(0) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

with

$$A_{max} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix},$$

$$B_{max} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 \\ 0 & b_{52} & b_{53} & b_{54} & 0 \end{pmatrix}.$$

Based on Riesz representation theorem, there is $\eta(\theta, \varrho)$ in $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^0 d\eta(\theta, \varrho)\phi(\theta), for \phi \in C. \tag{17}$$

In fact, we choose

$$\eta(\theta, \varrho) = (\tau_0 + \varrho)(A_{max}\delta(\theta) + B_{max}\delta(\theta + 1)),$$

where $\delta(\theta)$ is the Dirac delta function.

For $\phi \in C([-1, 0], R^5)$, define

$$A(\varrho)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 d\eta(\theta, \varrho)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\varrho)\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\varrho, \phi), & \theta = 0. \end{cases}$$

Then system (16) is equivalent to

$$\dot{u}(t) = A(\varrho)u_t + R(\varrho)u_t. \tag{18}$$

For $\varphi \in C^1([0, 1]), (R^5)^*$, the adjoint operator A^* of $A(0)$ is defined as following

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \leq 1, \\ \int_{-1}^0 d\eta^T(s, 0)\varphi(-s), & s = 0, \end{cases}$$

Next, we define the bilinear inner form for A and A^*

$$\langle\varphi(s), \phi(\theta)\rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^0\int_{\xi=0}^\theta \bar{\varphi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \tag{19}$$

where $\eta(\theta) = \eta(\theta, 0)$.

Let $\rho(\theta) = (1, \rho_2, \rho_3, \rho_4, \rho_5)^T e^{i\tau_0\omega_0\theta}$ and $\rho^*(s) = (1, \rho_2^*, \rho_3^*, \rho_4^*, \rho_5^*)^T e^{i\tau_0\omega_0 s}$ be the eigenvectors for $A(0)$ and $A^*(0)$ corresponding to $+i\tau_0\omega_0$ and $-i\tau_0\omega_0$. Then, it is not difficult to show that

$$\begin{aligned} \rho_2 &= \frac{a_{21} + a_{23}\rho_3}{i\omega_0 - a_{22} - b_{22}e^{-i\tau_0\omega_0}}, \rho_3 = \frac{i\omega_0 - a_{11}}{a_{13}}, \\ \rho_4 &= \frac{a_{43}\rho_3}{i\omega_0 - a_{44} - b_{44}e^{-i\tau_0\omega_0}}, \rho_5 = \frac{b_{52}\rho_2 + b_{53}\rho_3 + b_{54}\rho_4}{i\omega_0 - a_{55}e^{i\tau_0\omega_0}}, \\ \rho_2^* &= -\frac{i\omega_0 + a_{11}}{a_{21}}, \\ \rho_3^* &= -\frac{(i\omega_0 + a_{22} + b_{22}e^{i\tau_0\omega_0})\rho_2^* - b_{52}e^{i\tau_0\omega_0} + \rho_5^*}{a_{32}}, \\ \rho_4^* &= -\frac{b_{44} + b_{54}\rho_5^*}{(i\omega_0 + a_{44})e^{-i\tau_0\omega_0}}, \rho_5^* = \frac{\rho_{51}^*}{\rho_{52}^*}, \end{aligned} \tag{20}$$

where

$$\rho_{51}^* = (i\omega_0 + a_{22} + b_{22}e^{i\tau_0\omega_0})(i\omega_0 + a_{33} + b_{33}e^{i\tau_0\omega_0})\rho_2^* -a_{13}a_{32} + a_{23}a_{32}\rho_2^* + \frac{a_{32}a_{43}b_{44}e^{i\tau_0\omega_0}}{i\omega_0 + a_{44}},$$

$$\rho_{52}^* = a_{32}b_{53}e^{i\tau_0\omega_0} - \frac{a_{32}a_{43}b_{54}e^{i\tau_0\omega_0}}{i\omega_0 + a_{44}} -b_{52}e^{i\tau_0\omega_0}(i\omega_0 + a_{33} + b_{33}e^{i\tau_0\omega_0}).$$

According to Eq.(19), we have

$$\bar{D} = [1 + \rho_2\bar{\rho}_2^* + \rho_3\bar{\rho}_3^* + \rho_4\bar{\rho}_4^* + \rho_5\bar{\rho}_5^* + \tau_0 e^{-i\tau_0\omega_0}(b_{22}\rho_2\bar{\rho}_2^* +b_{33}\rho_3\bar{\rho}_3^* + b_{53}\rho_3\bar{\rho}_5^* + b_{44}\rho_4\bar{\rho}_4^* + b_{54}\rho_4\bar{\rho}_5^*]^{-1}$$

such that $\langle \rho^*, \rho \rangle = 1$ and $\langle \rho^*, \bar{\rho} \rangle = 0$.

Next, using the algorithms given in [28] and following the similar computation process as in [29, 30], we get the expressions of g_{20}, g_{11}, g_{02} and g_{21} as follows

$$g_{20} = 2\beta\tau_0\bar{D}\rho_3(\bar{\rho}_2^* - 1), g_{11} = \beta\tau_0\bar{D}(\rho_3 + \bar{\rho}_3)(\bar{\rho}_2^* - 1),$$
$$g_{02} = 2\beta\tau_0\bar{D}\bar{\rho}_3(\bar{\rho}_2^* - 1),$$
$$g_{21} = 2\beta\tau_0\bar{D}(\bar{\rho}_2^* - 1)(W_{11}^{(1)}(0)\rho_3 + \frac{1}{2}W_{20}^{(1)}(0)\bar{\rho}_3 +W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)),$$

with

$$W_{20}(\theta) = \frac{ig_{20}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{02}\bar{\rho}(0)}{3\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_1 e^{2i\tau_0\omega_0\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{11}\bar{\rho}(0)}{\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_2.$$

E_1 and E_2 can be obtained by the following two equations

$$E_1 = \begin{pmatrix} i\omega_0 - a_{11} & 0 & -a_{13} & 0 & 0 \\ -a_{21} & a'_{22} & -a_{23} & 0 & 0 \\ 0 & -a_{32} & a'_{33} & 0 & 0 \\ 0 & 0 & -a_{43} & a'_{44} & 0 \\ 0 & -b_{52}e^{-2i\tau_0\omega_0} & -b_{53}e^{-2i\tau_0\omega_0} & -b_{54}e^{-2i\tau_0\omega_0} & 2i\omega_0 - a_{55} \end{pmatrix}^{-1} \times \begin{pmatrix} -\beta\rho_3 \\ \beta\rho_3 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$E_2 = -\begin{pmatrix} a_{11} & 0 & a_{13} & 0 & 0 \\ a_{21} & a_{22} + b_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} + b_{33} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} + b_{44} & 0 \\ 0 & b_{52} & b_{53} & b_{54} & a_{55} \end{pmatrix}^{-1} \times \begin{pmatrix} -\beta(\rho_3 + \bar{\rho}_3) \\ \beta(\rho_3 + \bar{\rho}_3) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$a'_{22} = 2i\omega_0 - a_{22} - b_{22}e^{-2i\tau_0\omega_0},$$
$$a'_{33} = 2i\omega_0 - a_{33} - b_{33}e^{-2i\tau_0\omega_0},$$
$$a'_{44} = 2i\omega_0 - a_{44} - b_{44}e^{-2i\tau_0\omega_0}.$$

Then, the expressions of μ_2, $\beta_2 < 0$ and $T_2 > 0$ are obtained. Based on the properties of the Hopf bifurcation stated in [28], we have Theorem 2 and the proof is completed.

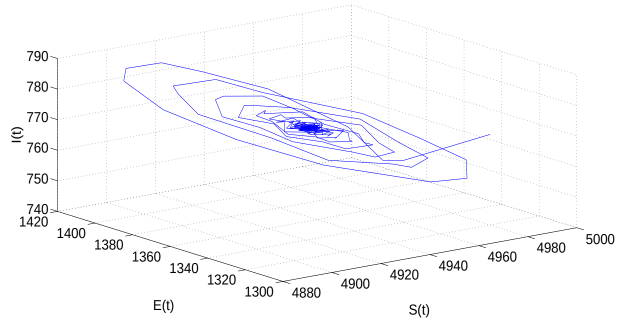

Fig. 1. Dynamic behavior of system (21): projection on S-E-I with $\tau = 40.4628$.

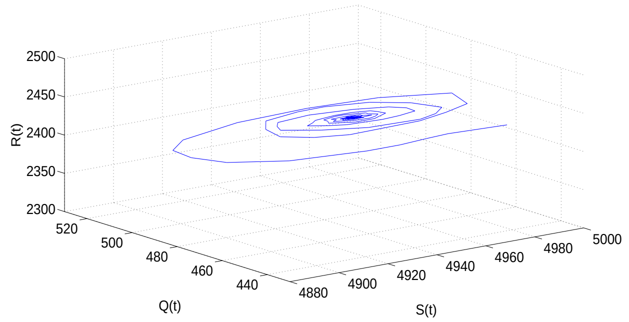

Fig. 2. Dynamic behavior of system (21): projection on S-Q-R with $\tau = 40.4628$.

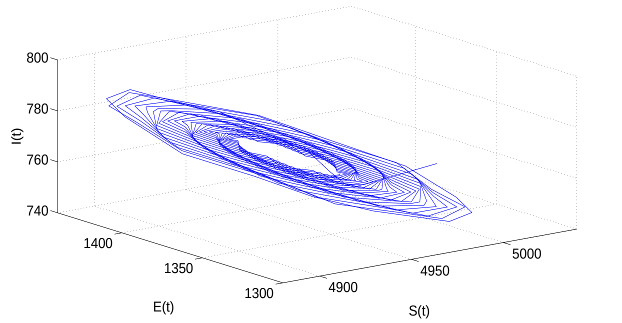

Fig. 3. Dynamic behavior of system (21): projection on S-E-I with $\tau = 48.3616$.

IV. Numerical simulation

In order to validate the previous main theorems, some numerical simulations are presented in this section. By extracting part of values from [12] and considering the conditions for existence of Hopf bifurcation, we choose the following set of parameters: $\beta = 0.00004$, $\mu = 0.03$, $\eta = 0.05$, $\varepsilon = 0.03$, $\sigma = 0.05$, $\gamma = 0.01$, $\varphi = 0.05$ and $N = 10000$. Then, the following specific case of system (2)

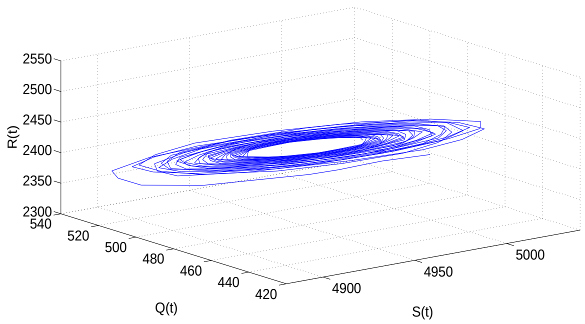

Fig. 4. Dynamic behavior of system (21): projection on S-Q-R with $\tau = 48.3616$

is aquired:

$$\begin{cases} \frac{dS(t)}{dt} &= -0.00004S(t)I(t) + 300 - 0.03S(t), \\ \frac{dE(t)}{dt} &= 0.00004S(t)I(t) - 0.08E(t) - 0.03E(t-\tau), \\ \frac{dI(t)}{dt} &= 0.05E(t) - 0.08I(t) - 0.01I(t-\tau), \\ \frac{dQ(t)}{dt} &= 0.05I(t) - 0.05Q(t-\tau) - 0.03Q(t), \\ \frac{dR(t)}{dt} &= 0.03E(t-\tau) + 0.01I(t-\tau) \\ & \quad +0.05Q(t-\tau) - 0.03R(t). \end{cases} \tag{21}$$

By using Matlab software package, the unique viral equilibrium $P_*(4950, 1377.3, 765.1515, 478.2197, 2429.4)$ is obtained. Also, $\lambda'(\tau_0) = 1.7349 - 0.8215i$, $\omega_0 = 0.3153$ and $\tau_0 = 44.4792$ are acquired.

Then, $\tau = 40.4682 \in [0, \tau_0)$ is set. The viral equilibrium $P_*(4950, 1377.3, 765.1515, 478.2197, 2429.4)$ is asymptotically stable as depicted in Figure 1 and Figure 2. This means that the worms can be controlled easily. Figure 3 and Figure 4 show that bifurcating periodic solutions occur when τ is larger than $\tau_0 = 44.4792$ such as $\tau = 48.3616$. Thus, we can conclude that the time delay is vital to the solutions of system (21). It is shown that if we shorten the period that anti-virus software uses to clean the worms in the exposed nodes, the infected nodes and the quarantined nodes, the propagation of worms in nodes can be controlled.

In addition, by some complex computations, we obtain $C_1(0) = -0.7656 + 0.2960i$. Then, according to Eq.(15), we get $\mu_2 = 0.4413 > 0$, $\beta_2 = -1.5312 < 0$ and $T_2 = 0.0047 > 0$. Thus, it is known that the Hopf bifurcation at τ_0 is supercritical. Also, the bifurcating periodic solutions are stable and the period of the bifurcating periodic solutions increases based on Theorem 2. Since the bifurcating periodic solutions are stable, it can be shown that the five classes of nodes in system (21) can coexist in an oscillatory mode from the view of ecology. Therefore, it is concluded that the time delay is harmful for system (21).

V. Conclusions

In this paper, the time delay is introduced into an SEIQR worm propagation model in mobile internet. Stability of the viral equilibrium and existence of Hopf bifurcation are studied by analyzing the associated characteristic equation and regarding the time delay as a bifurcation parameter. Furthermore, with the help of normal form theory and center manifold theorem from Hassard et al.[28], direction and stability of the Hopf bifurcation have been derived. Finally, through numerical simulations, it can be shown that the time delay that the anti-virus software uses to clean worms in the exposed nodes, the infected nodes and the quarantined nodes plays an important role in the worms propagation. And the worms may be controlled by shortening the period that the anti-virus software uses to clean worms. These results obtained in the present paper may help understand the laws governing the spread of worms in mobile internet.

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