Robust Multivariable Adaptive Control of Time-varying Systems

Felipe Osorio-Arteaga, Jesser James Marulanda-Durango, Eduardo Giraldo

***Abstract*—In this article a robust multivariable adaptive control technique is proposed for time varying AutoRegressive Moving Average systems with exogenous inputs considering identification and auto-tuning of the proposed model reference controller. The proposed method is validated for linear multivariable systems with time-varying parameters under noise conditions. The results are evaluated in terms of tracking error, and robustness to time-varying parameters. The proposed method is compared with an adaptive approach named the multivariable model reference adaptive control based on an AutoRegressive Moving Average model where the proposed robust multivariable approach improved the performance of the adaptive method. In addition, the proposed estimation methods for ARMA and ARMAX structures are evaluated over an electric arc model and a real system implemented with an analog computer.**

***Index Terms*—Adaptive, Multivariable, Control, Nonlinear, Time-varying.**

I. Introduction

THE control of multivariable nonlinear systems is a task that requires the application of high mathematical techniques, specially if the multivariable system has inherent time-varying parameters. AutoRegressive Moving Average (ARMA) models can be used to describe the systems under a deterministic domain. In the other hand, AutoRegressive Moving Average models with exogenous inputs (ARMAX) can include the robustness of the system into the identification approach [1] which turn the adaptive controllers into robust controllers. For example, in [2] a variable structure approach is applied for a multivariable nonlinear system but considering only time invariant parameters. Several systems that involve electric and mechanical elements can be modeled as systems with time-varying parameters where the nominal value change during the normal operation. That systems required the application of time-varying control techniques, as proposed in [3]. Several techniques can also be applied to fulfill the requirement of complex multivariable nonlinear systems by using nonlinear [4] and deep learning techniques [5], [6].

In this work, a robust multivariable adaptive control technique is proposed for time varying ARMAX model considering identification and auto-tuning of the proposed model reference controller. The model is compared with an ARMA model by using a model reference adaptive controller. The results are evaluated for estimation and control of simulated systems under ARMA and ARMAX structures. In addition, the proposed estimation methods for ARMA and ARMAX structures are evaluated over an electric arc model and a real system implemented with an analog computer. The paper is organized as follows: in section II is shown the identification and adaptive control of multivariable systems under ARMA and ARMAX approaches and in section III the experimental results based on simulated and real models under several operational conditions are presented and discussed.

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II. Theoretical framework

A. Multivariable identification

A discrete multivariable (multiple-input multiple-output MIMO) system with m outputs and p inputs with delay operator q can be defined as [1]:

$$\mathbf{A}\left(q^{-1}\right)\mathbf{y}\left(k\right)=\mathbf{B}\left(q^{-1}\right)\mathbf{u}\left(k\right) \tag{1}$$

where $\mathbf{A}$ is given by

$$\mathbf{A}\left(q^{-1}\right)=\mathbf{A}_0+\mathbf{A}_1\left(q^{-1}\right)+\cdots+\mathbf{A}_{n_1}\left(q^{-n_1}\right) \tag{2}$$

and $\mathbf{B}$ is given by

$$\mathbf{B}\left(q^{-1}\right)=\left(\mathbf{B}_1\left(q^{-1}\right)+\cdots\mathbf{B}_{n_2}\left(q^{-n_2}\right)\right) \tag{3}$$

with $n_1 \geq n_2$ and where $\mathbf{A}_i$ is $m\times m$ and $\mathbf{B}_i$ is $p\times p$, and the input vector $\mathbf{u}$ is $p\times 1$ and output vector $\mathbf{y}$ is $m\times 1$ as follows:

$$\mathbf{y}\left(k\right)=\begin{bmatrix} y_1\left(k\right)\\ \vdots \\ y_m\left(k\right)\end{bmatrix},\quad \mathbf{u}\left(k\right)=\begin{bmatrix} u_1\left(k\right)\\ \vdots \\ u_p\left(k\right)\end{bmatrix} \tag{4}$$

By defining $\mathbf{A}_0=\mathbf{I}$ with $\mathbf{I}$ the identity matrix, the following equation is obtained:

$$\begin{aligned}\mathbf{y}\left(k\right)=&\mathbf{B}_1\mathbf{u}\left(k-1\right)+\cdots+\mathbf{B}_{n_2}\mathbf{u}\left(k-n_2\right)\\ &-\mathbf{A}_1\mathbf{y}\left(k-1\right)-\cdots-\mathbf{A}_{n_1}\mathbf{y}\left(k-n_1\right)\end{aligned} \tag{5}$$

where $\mathbf{A}_i$ and $\mathbf{B}_i$ are given by

$$\mathbf{A}_i=\begin{bmatrix} a^i_{11} & \cdots & a^i_{1m}\\ \vdots & \ddots & \vdots\\ a^i_{m1} & \cdots & a^i_{mm}\end{bmatrix},\quad \mathbf{B}_i=\begin{bmatrix} b^i_{11} & \cdots & b^i_{1p}\\ \vdots & \ddots & \vdots\\ b^i_{m1} & \cdots & b^i_{mp}\end{bmatrix} \tag{6}$$

From (5) and (6) the output y_i can be described in terms of past inputs and outputs as:

$$\begin{aligned}y_i(k)= \quad & b^1_{i1}u_1\left(k-1\right)+\cdots+b^1_{ip}u_p\left(k-1\right)\\ &+\cdots\\ &+b^{n_2}_{i1}u_1\left(k-n_2\right)+\cdots+b^{n_2}_{ip}u_p\left(k-n_2\right)\\ &-a^1_{i1}y_1\left(k-1\right)-\cdots-a^1_{im}y_m\left(k-1\right)\\ &-\cdots\\ &-a^{n_1}_{i1}y_1\left(k-n_1\right)-\cdots-a^{n_1}_{im}y_m\left(k-n_1\right)\end{aligned} \tag{7}$$

By considering (7), and (1), the following equation can be obtained:

$$\mathbf{y}(k) = \boldsymbol{\theta}^T \boldsymbol{\phi}(k-1); \quad k \geq 0 \tag{8}$$

being $\boldsymbol{\theta}^T$ the transpose of $\boldsymbol{\theta}$, and with $\boldsymbol{\theta}$ a matrix of dimension $(mn_1 + pn_2) \times m$ that holds the matrix parameters $\mathbf{A}_i$ and $\mathbf{B}_i$ as follows:

$$\boldsymbol{\theta}^T = \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{B}_{n_2} & \mathbf{A}_1 & \cdots & \mathbf{A}_{n_1} \end{bmatrix} \tag{9}$$

and being $\boldsymbol{\phi}(k-1)$ a vector of dimension $(mn_1 + pn_2) \times 1$ that holds the past inputs and outputs as follows:

$$\boldsymbol{\phi}(k-1) = \begin{bmatrix} \mathbf{u}(k-1) \\ \vdots \\ \mathbf{u}(k-n_2) \\ -\mathbf{y}(k-1) \\ \vdots \\ -\mathbf{y}(k-n_1) \end{bmatrix} \tag{10}$$

In [1] a class of identification algorithms is presented, where $\hat{\boldsymbol{\theta}}(k)$ is computed from $\hat{\boldsymbol{\theta}}(k-1)$, as follows:

$$\hat{\boldsymbol{\theta}}(k) = \hat{\boldsymbol{\theta}}(k-1) + \mathbf{M}(k-1)\boldsymbol{\phi}(k-1)\boldsymbol{e}(k) \tag{11}$$

being $\hat{\boldsymbol{\theta}}(k)$ the estimated parameter matrix at sample k, $\mathbf{M}(k-1)$ the gain matrix, $\boldsymbol{\phi}(k-1)$ the vector with past inputs and outputs, and $\boldsymbol{e}(k)$ the estimation error, as follows:

$$\boldsymbol{e}(k) = \mathbf{y}(k)^T - \hat{\mathbf{y}}(k)^T \tag{12}$$

being $\hat{\mathbf{y}}(k)$ given by:

$$\hat{\mathbf{y}}(k) = \hat{\boldsymbol{\theta}}(k-1)^T \boldsymbol{\phi}(k-1) \tag{13}$$

As described in [1] the multivariable least squares algorithm can be defined as:

$$\begin{aligned} \mathbf{e}(k) &= \mathbf{y}(k)^T - \boldsymbol{\phi}(k-1)^T \hat{\boldsymbol{\theta}}(k-1) \\ \mathbf{M}(k) &= \frac{\mathbf{P}(k-2)}{1+\boldsymbol{\phi}(k-1)^T \mathbf{P}(k-2)\boldsymbol{\phi}(k-1)} \\ \hat{\boldsymbol{\theta}}(k) &= \hat{\boldsymbol{\theta}}(k-1) + \mathbf{M}(k)\boldsymbol{\phi}(k-1)\mathbf{e}(k) \end{aligned} \tag{14}$$

and

$$\mathbf{P}(k-1) = \mathbf{P}(k-2) - \frac{\mathbf{P}(k-2)\boldsymbol{\phi}(k-1)\boldsymbol{\phi}(k-1)^T \mathbf{P}(k-2)}{1+\boldsymbol{\phi}(k-1)^T \mathbf{P}(k-2)\boldsymbol{\phi}(k-1)} \tag{15}$$

with initial estimate $\hat{\boldsymbol{\theta}}$ given and $\mathbf{P}(0)$ a positive diagonal matrix.

1) MIMO 2×2 *system:* Consider a system of second order ($n_1 = n_2 = 2$) with 2 inputs and 2 outputs, where the estimated output can be defined by

$$\hat{\mathbf{y}}(k) = \hat{\boldsymbol{\theta}}(k-1)^T \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \\ u_1(k-2) \\ u_2(k-2) \\ -y_1(k-1) \\ -y_2(k-1) \\ -y_1(k-2) \\ -y_2(k-2) \end{bmatrix} \tag{16}$$

and where the least squares algorithm of (14) results in:

$$[\mathbf{e}(k)]_{1\times 2} = \left[\mathbf{y}(k)^T - \boldsymbol{\phi}(k-1)^T \hat{\boldsymbol{\theta}}(k-1)\right]_{1\times 2} \tag{17}$$

$$[\mathbf{C}(k)]_{8\times 1} = \frac{[\mathbf{P}(k-2)\boldsymbol{\phi}(k-1)]_{8\times 1}}{1+\left[\boldsymbol{\phi}(k-1)^T \mathbf{P}(k-2)\boldsymbol{\phi}(k-1)\right]_{1\times 1}} \tag{18}$$

$$\left[\hat{\boldsymbol{\theta}}(k)\right]_{8\times 2} = \left[\hat{\boldsymbol{\theta}}(k-1)\right]_{8\times 2} + [\mathbf{C}(k)]_{8\times 1}[\mathbf{e}(k)]_{1\times 2} \tag{19}$$

where $\mathbf{P}$ is matrix of 8×8.

By using $\mathbf{A}_i$ and $\mathbf{B}_i$ it is possible to obtain the transfer matrix as follows:

$$\begin{aligned} \mathbf{y}(k) = & \left[\mathbf{I} + \mathbf{A}_1\left(q^{-1}\right) + \cdots + \mathbf{A}_{n_1}\left(q^{-n_1}\right)\right]_{m\times m}^{-1} \\ & \left[\mathbf{B}_1\left(q^{-1}\right) + \cdots + \mathbf{B}_{n_2}\left(q^{-n_2}\right)\right]_{m\times p}\mathbf{u}(k) \end{aligned} \tag{20}$$

where the transfer matrix $\mathbf{T}(z)$ is $m \times p$ and is defined as

$$\begin{aligned} \mathbf{T}(z) = & \left[\mathbf{I} + \mathbf{A}_1\left(z^{-1}\right) + \cdots + \mathbf{A}_{n_1}\left(z^{-n_1}\right)\right]_{m\times m}^{-1} \\ & \left[\mathbf{B}_1\left(z^{-1}\right) + \cdots + \mathbf{B}_{n_2}\left(z^{-n_2}\right)\right]_{m\times p} \end{aligned} \tag{21}$$

B. Robust ARMAX Identification

Consider an ARMAX model with m outputs, p inputs and r exogenous inputs with delay operator q can be defined as [1]:

$$\mathbf{A}\left(q^{-1}\right)\mathbf{y}(k) = \mathbf{B}\left(q^{-1}\right)\mathbf{u}(k) + \mathbf{C}\left(q^{-1}\right)\mathbf{w}(k) \tag{22}$$

where $\mathbf{C}$ is given by

$$\mathbf{C}\left(q^{-1}\right) = \mathbf{C}_0 + \mathbf{C}_1\left(q^{-1}\right) + \cdots + \mathbf{C}_{n_3}\left(q^{-n_3}\right) \tag{23}$$

with $n_1 \geq n_2$, $n_1 \geq n_3$, and where $\mathbf{A}_i$ is $m \times m$, $\mathbf{B}_i$ is $p \times p$, $\mathbf{C}_i$ is $m \times m$, and the input vector $\mathbf{u}$ is $p \times 1$, the output vector $\mathbf{y}$ is $m \times 1$, and exogenous vector $\mathbf{w}$ is $r \times 1$ as follows:

$$\mathbf{y}(k) = \begin{bmatrix} y_1(k) \\ \vdots \\ y_m(k) \end{bmatrix}, \quad \mathbf{u}(k) = \begin{bmatrix} u_1(k) \\ \vdots \\ u_p(k) \end{bmatrix}$$
$$\mathbf{w}(k) = \begin{bmatrix} w_1(k) \\ \vdots \\ w_r(k) \end{bmatrix} \tag{24}$$

By defining $\mathbf{A}_0 = \mathbf{I}$ with $\mathbf{I}$ the identity matrix, the following equation is obtained:

$$\begin{aligned} \mathbf{y}(k) = & \mathbf{B}_1\mathbf{u}(k-1) + \cdots + \mathbf{B}_{n_2}\mathbf{u}(k-n_2) \\ & -\mathbf{A}_1\mathbf{y}(k-1) - \cdots - \mathbf{A}_{n_1}\mathbf{y}(k-n_1) \\ & +\mathbf{C}_0\mathbf{w}(k) + \mathbf{C}_1\mathbf{w}(k-1) + \cdots + \mathbf{C}_{n_3}\mathbf{w}(k-n_3) \end{aligned} \tag{25}$$

where $\mathbf{C}_i$ is given by

$$\mathbf{C}_i = \begin{bmatrix} c_{11}^i & \cdots & c_{1r}^i \\ \vdots & \ddots & \vdots \\ c_{r1}^i & \cdots & c_{rr}^i \end{bmatrix} \tag{26}$$

From (25) and by defining $\mathbf{C}_0 = \mathbf{I}$ the input $\mathbf{w}$ can be estimated in terms of past inputs and outputs as:

$$\begin{aligned} \hat{\mathbf{w}}(k) = & \mathbf{y}(k) - \mathbf{B}_1\mathbf{u}(k-1) - \cdots - \mathbf{B}_{n_2}\mathbf{u}(k-n_2) \\ & +\mathbf{A}_1\mathbf{y}(k-1) + \cdots + \mathbf{A}_{n_1}\mathbf{y}(k-n_1) \\ & -\mathbf{C}_1\mathbf{w}(k-1) - \cdots - \mathbf{C}_{n_3}\mathbf{w}(k-n_3) \end{aligned} \tag{27}$$

By considering (27), and (22), the following equation can be obtained:

$$\hat{\mathbf{w}}(k) = \mathbf{y}(k) - \hat{\mathbf{y}}(k) = \mathbf{y}(k) - \boldsymbol{\theta}^T \boldsymbol{\phi}(k-1) \tag{28}$$

being $\boldsymbol{\theta}^T$ the transpose of $\boldsymbol{\theta}$, and with $\boldsymbol{\theta}$ a matrix of dimension $(mn_1 + pn_2 + rn_3) \times m$ that holds the matrix parameters $\mathbf{A}_i$, $\mathbf{B}_i$, and $\mathbf{C}_i$ as follows:

$$\boldsymbol{\theta}^T = \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{B}_{n_2} & \mathbf{A}_1 & \cdots & \mathbf{A}_{n_1} & \mathbf{C}_1 & \cdots & \mathbf{C}_{n_1} \end{bmatrix} \tag{29}$$

and being $\boldsymbol{\phi}(k-1)$ a vector of dimension $(mn_1 + pn_2 + rn_3) \times 1$ that holds the past inputs and outputs as follows:

$$\boldsymbol{\phi}(k-1) = \begin{bmatrix} \mathbf{u}(k-1) \\ \vdots \\ \mathbf{u}(k-n_2) \\ -\mathbf{y}(k-1) \\ \vdots \\ -\mathbf{y}(k-n_1) \\ \hat{\mathbf{w}}(k-1) \\ \vdots \\ \hat{\mathbf{w}}(k-n_3) \end{bmatrix} \tag{30}$$

In [1] a class of identification algorithms is presented, where $\hat{\boldsymbol{\theta}}(k)$ is computed from $\hat{\boldsymbol{\theta}}(k-1)$, as follows:

$$\hat{\boldsymbol{\theta}}(k) = \hat{\boldsymbol{\theta}}(k-1) + \mathbf{M}(k-1)\boldsymbol{\phi}(k-1)\boldsymbol{e}(k) \tag{31}$$

being $\hat{\boldsymbol{\theta}}(k)$ the estimated parameter matrix at sample k, $\mathbf{M}(k-1)$ the gain matrix, $\boldsymbol{\phi}(k-1)$ the vector with past inputs and outputs, and $\boldsymbol{e}(k)$ the estimation error, as follows:

$$\boldsymbol{e}(k) = \mathbf{y}(k)^T - \hat{\mathbf{y}}(k)^T \tag{32}$$

being $\hat{\mathbf{y}}(k)$ given by:

$$\hat{\mathbf{y}}(k) = \hat{\boldsymbol{\theta}}(k-1)^T \boldsymbol{\phi}(k-1) \tag{33}$$

As described in [1] the multivariable least squares algorithm can be defined as:

$$\begin{aligned} \mathbf{e}(k) &= \mathbf{y}(k)^T - \boldsymbol{\phi}(k-1)^T \hat{\boldsymbol{\theta}}(k-1) \\ \mathbf{M}(k) &= \mathbf{P}(k-1) \\ \hat{\boldsymbol{\theta}}(k) &= \hat{\boldsymbol{\theta}}(k-1) + \mathbf{M}(k)\boldsymbol{\phi}(k-1)\mathbf{e}(k) \end{aligned} \tag{34}$$

and

$$\mathbf{P}(k-1) = \mathbf{P}(k-2) - \frac{\mathbf{P}(k-2)\boldsymbol{\phi}(k-1)\boldsymbol{\phi}(k-1)^T\mathbf{P}(k-2)}{1+\boldsymbol{\phi}(k-1)^T\mathbf{P}(k-2)\boldsymbol{\phi}(k-1)} \tag{35}$$

with initial estimate $\hat{\boldsymbol{\theta}}$ given and $\mathbf{P}(0)$ a positive diagonal matrix.

C. Multivariable Adaptive Control

A model reference adaptive control structure is selected for control of the multivariable system as described in [1]. Consider a system with a reference model defined by:

$$\mathbf{y}_r(k) = \mathbf{E}^{-1}(q^{-1})\mathbf{H}(q^{-1})\mathbf{r}(k) \tag{36}$$

being E and H defined as

$$\mathbf{E}(q^{-1}) = E_0 + E_1 q^{-1} + E_2 q^{-2} + \cdots \tag{37}$$

$$\mathbf{H}(q^{-1}) = H_1 q^{-1} + H_2 q^{-2} + \cdots \tag{38}$$

where $\mathbf{E}(q^{-1})$ is stable and the transfer matrix in steady state is $\mathbf{E}^{-1}(1)\mathbf{H}(1) = \mathbf{I}$ the identity matrix. Therefore, the closed loop system can be defined by the following equations:

$$\mathbf{E}(q^{-1})\mathbf{y}_r(k) = \alpha(q^{-1})\mathbf{y}(k) + \beta(q^{-1})u(k) \tag{39}$$

$$\mathbf{H}(q^{-1})\mathbf{r}(k) = \alpha(q^{-1})\mathbf{y}(k) + \beta(q^{-1})u(k) \tag{40}$$

$$\tag{41}$$

being $\mathbf{F}$ and $\mathbf{G}$ the polynomials of order 0 and $n_1 - 1$ defined as

$$\alpha(q^{-1}) = \mathbf{G}(q^{-1})\beta(q^{-1}) = \mathbf{F}(q^{-1})\mathbf{B}'(q^{-1}) \tag{42}$$

$$\tag{43}$$

that satisfy

$$\mathbf{E}(q^{-1}) = \mathbf{F}(q^{-1})\mathbf{A}(q^{-1}) + q^{-1}\mathbf{G}(q^{-1}) \tag{44}$$

1) MIMO 2×2 system: Consider a system of second order ($n_1 = n_2 = 2$) with 2 inputs and 2 outputs as follows:

$$A(q^{-1}) = I + A_1 q^{-1} + A_2 q^{-2}$$

$$B(q^{-1}) = q^{-1}(B_1 + B_2 q^{-1}) \tag{45}$$

$$E(q^{-1}) = I + E_1 q^{-1} + E_2 q^{-2} \tag{46}$$

$$H(q^{-1}) = q^{-1}(H_1 + H_2 q^{-1}) \tag{47}$$

$$\tag{48}$$

with polynomials F and G defined as

$$F(q^{-1}) = F_0$$

$$G(q^{-1}) = (G_1 + G_2 q^{-1}) \tag{49}$$

$$\tag{50}$$

where the resulting control signal is defined as:

$$\mathbf{u}(k) = B_1^{-1}(H(q^{-1})r(k) - G(q^{-1})y(k) - B_2 q^{-1} u(k)) \tag{51}$$

A pole placement approach can also be defined as an error driven controller:

$$E(z) = K_g R(z) - Y(z) \tag{52}$$

$$U(z) = C(z)E(z) \tag{53}$$

being $C(z)$ the transfer matrix of the controller defined as follows:

$$C(z) = P^-1(z)L(z) \tag{54}$$

and K_g the direct matrix defined as

$$K_g = (B(1)L(1))^1(A(1)P(1) + B(1)L(1)) \tag{55}$$

being the closed loop representative equation defined as

$$P_{CL}(z) = A(z)P(z) + B(z)L(z) \tag{56}$$

By defining a desired closed loop set of roots $P_d(z)$ the following equation can be defined

$$P_d(z) = P_{LC}(z) \tag{57}$$

2) MIMO 2×2 *system:* Consider a system of second order ($n_1 = n_2 = 2$) with 2 inputs and 2 outputs as follows:

$$\begin{aligned} A(q^{-1}) &= I + A_1 q^{-1} + A_2 q^{-2} \\ B(q^{-1}) &= q^{-1}(B_1 + B_2 q^{-1}) && (58) \\ P(q^{-1}) &= I + P_1 q^{-1} + P_2 q^{-2} && (59) \\ L(q^{-1}) &= q^{-1}(L_1 + L_2 q^{-1}) && (60) \\ & && (61) \end{aligned}$$

with

$$P_d(z) = I z^4 + \alpha_1 I z^3 + \alpha_2 I z^2 + \alpha_3 I z + \alpha_4 I \tag{62}$$

The following equation can be defined

$$\begin{bmatrix} I & 0 & 0 & 0 \\ A_1 & I & B_1 & 0 \\ A_2 & A_1 & B_2 & B_1 \\ 0 & A_2 & 0 & B_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ L_0 \\ L_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 I - A_1 \\ \alpha_2 I - A_2 \\ \alpha_3 I \\ \alpha_4 I \end{bmatrix} \tag{63}$$

III. Results and Discussions

The performance of the described adaptive multivariable model reference control are evaluated for simulated and real systems in order to evaluate the tracking performance and also the effectiveness of the estimation multivariable technique.

A. Simulation results

A simulated system defined by

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} \frac{0.2z}{z^2-1.6z+0.64} & \frac{0.3z}{z^2-1.6z+0.64} \\ \frac{0.1z}{z^2-1.6z+0.64} & \frac{0.5z}{z^2-1.6z+0.64} \end{bmatrix} \begin{bmatrix} U_1(z) \\ U_2(z) \end{bmatrix} \tag{64}$$

is used to validate the results of the robust multivariable identification and control.

In this case, the parameter matrix $\boldsymbol{\theta}$ is given by:

$$\boldsymbol{\theta}^T = \begin{bmatrix} B_1 & A_1 & A_2 \end{bmatrix} \tag{65}$$

$$\boldsymbol{\theta}^T = \begin{bmatrix} b_{11}^1 & b_{12}^1 & a_{11}^1 & a_{12}^1 & a_{11}^2 & a_{12}^2 \\ b_{21}^1 & b_{22}^1 & a_{21}^1 & a_{22}^1 & a_{21}^2 & a_{22}^2 \end{bmatrix} \tag{66}$$

By considering the input signals shown in Fig. 1, the estimated output signals of Fig. 2 are obtained.

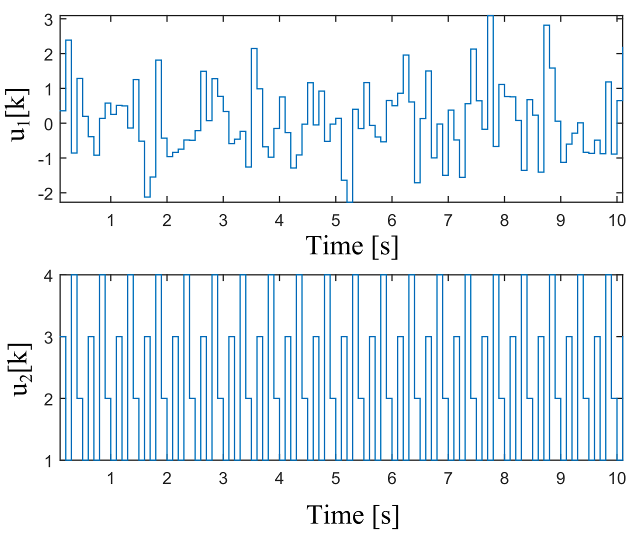

Fig. 1. Inputs for a 2×2 simulated multivariable system

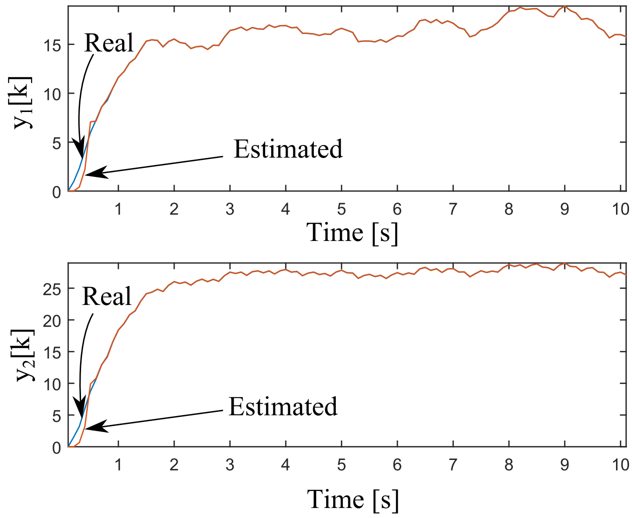

Fig. 2. Real and estimated outputs for a 2×2 simulated multivariable system

The estimated parameters $\boldsymbol{\theta}$ after 10 seconds are:

$$\boldsymbol{\theta} = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.5 \\ 1.599 & 0.0004176 \\ 0.000412 & 1.6 \\ -0.6394 & -0.0003961 \\ -0.0004024 & -0.6397 \end{bmatrix} \tag{67}$$

It is noticeable that the elements of B_1 from (67) are the same coefficients of each numerator of (64). Also, the elements of A_1 and A_2 are the coefficients of the denominator (64) multiplied by identity 2×2 matrices. In Fig. 3 is presented the evolution of the estimated parameters $\boldsymbol{\theta}$.

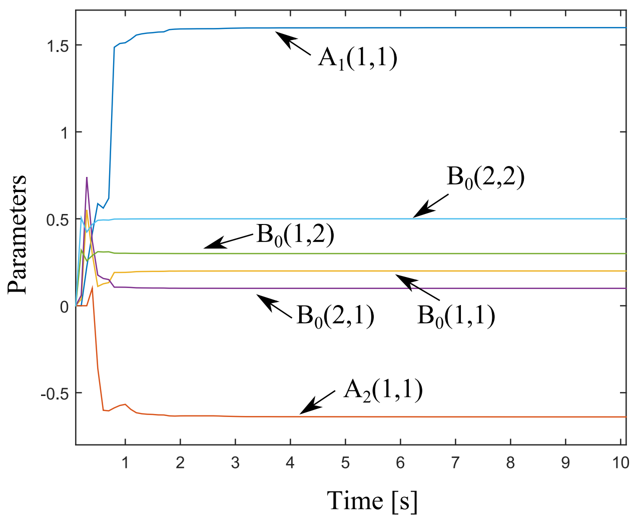

Fig. 3. Evolution of estimated parameters for the simulated 2×2 system

By using the model reference control defined in (51) combined with the identification stage of (17), a model reference adaptive control is obtained for multivariable systems. A reference model defined by the following equations is selected:

$$\mathbf{E}(q^{-1}) = I - 1.8 I q^{-1} + 0.81 I q^{-2} \tag{68}$$

$$\mathbf{H}(q^{-1}) = 0.01 I q^{-1} \tag{69}$$

which is stable (two poles at $z = 0.9$) and has steady unity state gain.

In Fig.4 are presented the outputs and their corresponding references. The initial value of parameters θ is set to zero, and the initial value of $\mathbf{P}$ is set to the identity matrix. The simulation is performed during 50 seconds. It can be seen that the first reference change is not followed since the identification system is still tracking the parameters, but the second reference change is followed according to the reference model with zero tracking error. That phenomenon can be observed for each output.

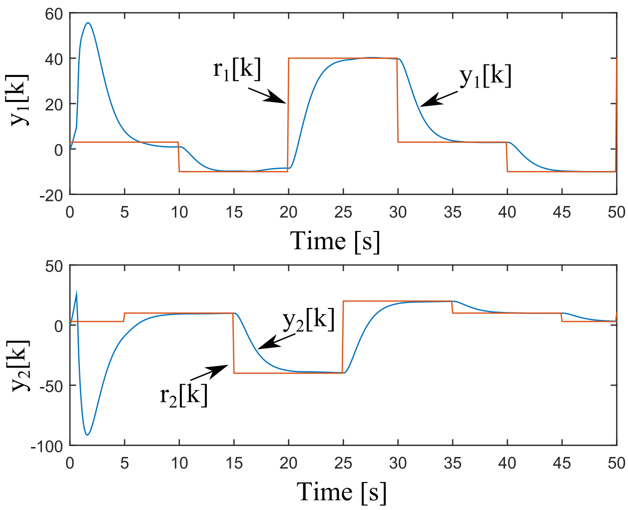

Fig. 4. Outputs and their corresponding references for the simulated 2×2 system using the model reference approach

The corresponding control signals for the simulation presented in Fig. 4 are shown in Fig. 5. It is noticeable that during the first samples the control signals for both channels shown high values since the parameters of the system θ are still been estimated. However, after the 2 seconds, it can be seen that the control signals are in an adequate range, which correspond to a successful tracking, as shown in Fig. 4.

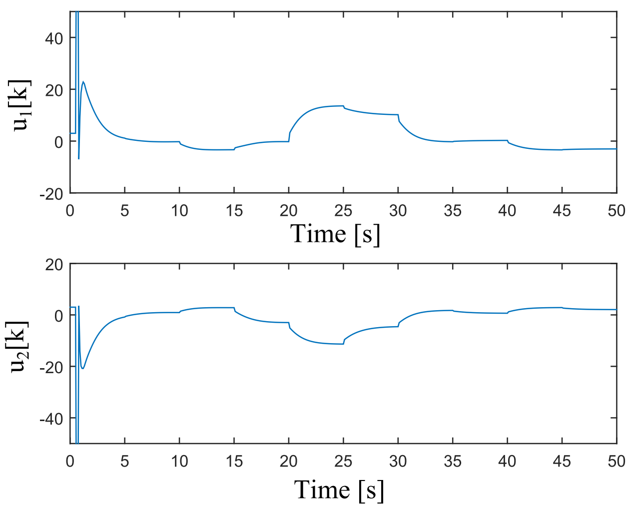

Fig. 5. Control signals for the simulated 2×2 system

In Fig. 6 are presented the corresponding evolution of the estimated parameters. It is noticeable that the parameters are consistent with the estimation shown in (67).

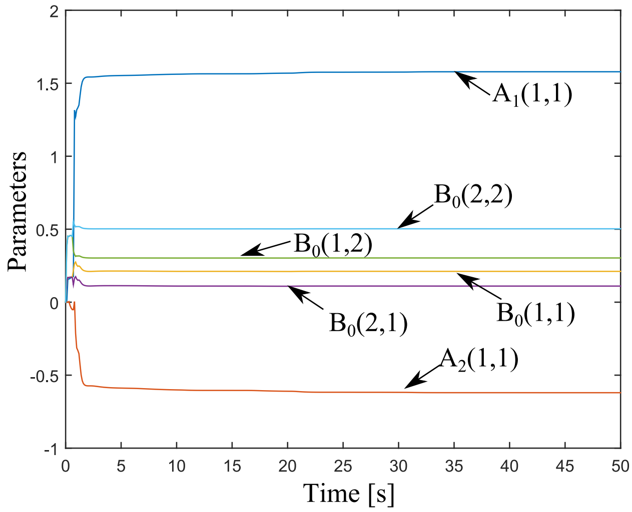

Fig. 6. Evolution of estimated parameter for the simulated 2×2 system

In order to evaluate the pole placement method, a desired closed loop roots equal to zero are designed $P_{LC}(z) = z^4$. Therefore, in Fig. 7 are presented the outputs and their corresponding references, and in Fig. 8 are presented the control signals. It can be seen that the system has a tracking error equal to zero in almost four samples. However, it can be seen that the control signal effort is higher than the one required for the model reference approach.

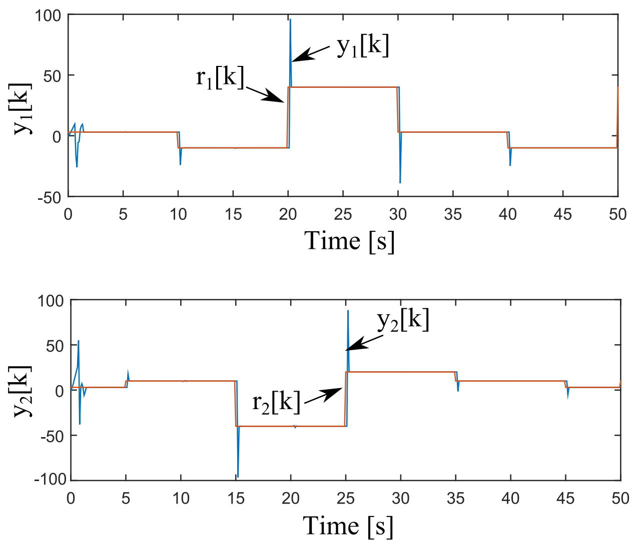

Fig. 7. Outputs and their corresponding references for the simulated 2×2 system using the pole placement approach

In addition, the direct gain matrix K_g is computed as:

$$K_g = \begin{bmatrix} 1.1161 & 0.0 \\ 0.0 & 1.1161 \end{bmatrix} \tag{70}$$

B. Robust identification and control

In this case, the parameter matrix $\boldsymbol{\theta}$ is given by:

$$\boldsymbol{\theta}^T = \begin{bmatrix} B_1 & A_1 & A_2 & C_1 & C_2 \end{bmatrix} \tag{71}$$

By using the pole placement approach combined with the ARMAX identification stage of (34), a desired closed loop roots equal to zero are designed $P_{LC}(z) = z^4$ and a white noise at the output system with variance of 0.5. Therefore,

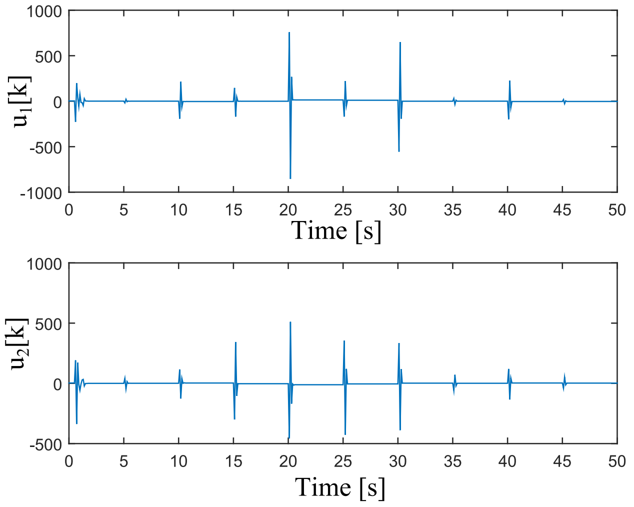

Fig. 8. Control signals for the simulated 2×2 system using the pole placement approach

in Fig. 9 are presented the outputs and their corresponding references, and in Fig. 10 are presented the control signals. The initial value of parameters θ is set to zero, and the initial value of $\mathbf{P}$ is set to the identity matrix. The simulation is performed during 50 seconds. Only the parameters θ that are involved in pole placement approach is taken. It can be seen that the first reference change is not followed since the identification system is still tracking the parameters, but few samples after the reference is followed according to the closed loop roots in presence of white noise. That phenomenon can be observed for each output.

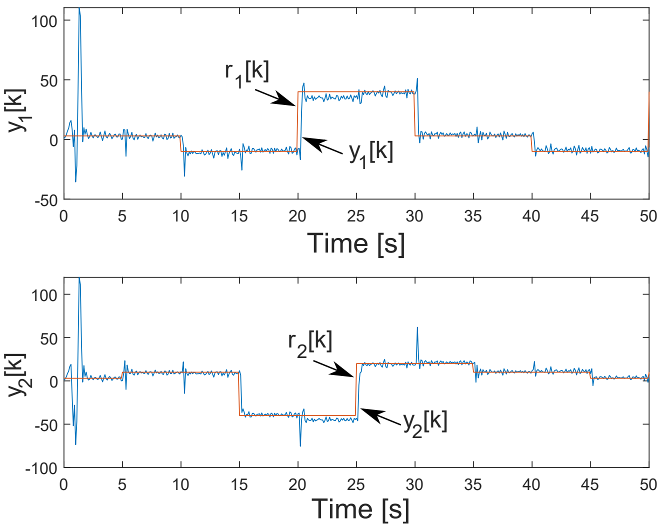

Fig. 9. Outputs and their corresponding references for the simulated 2×2 system using the pole placement approach

In Fig. 10, it is noticeable that during the first samples the control signals for both channels shown high oscillate values since the parameters of the system θ are still been estimated. However, after the 2 seconds, it can be seen that the control signals are in an adequate range, with peak values associated to the reference changes, as shown in Fig. 9.

C. Real system evaluation

A real system of second order with 2 inputs and 2 outputs is simulated with an analog computer COMDYNA GP-6

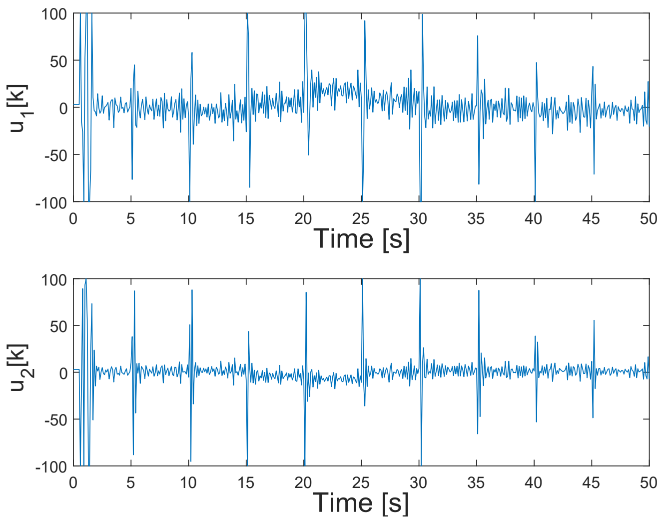

Fig. 10. Control signals for the simulated 2×2 system

according to the following model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} \mathbf{u}(t)$$
$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{72}$$

The system is sampled at $h = 0.1$ seconds.

In this case, the parameter matrix $\boldsymbol{\theta}$ is given by:

$$\boldsymbol{\theta}^T = \begin{bmatrix} b^1_{11} & b^1_{12} & b^2_{11} & b^2_{12} & a^1_{11} & a^1_{12} & a^2_{11} & a^2_{12} \\ b^1_{21} & b^1_{22} & b^2_{21} & b^2_{22} & a^1_{21} & a^1_{22} & a^2_{21} & a^2_{22} \end{bmatrix} \tag{73}$$

Estimation results are presented in Fig. 11 and Fig. 12.

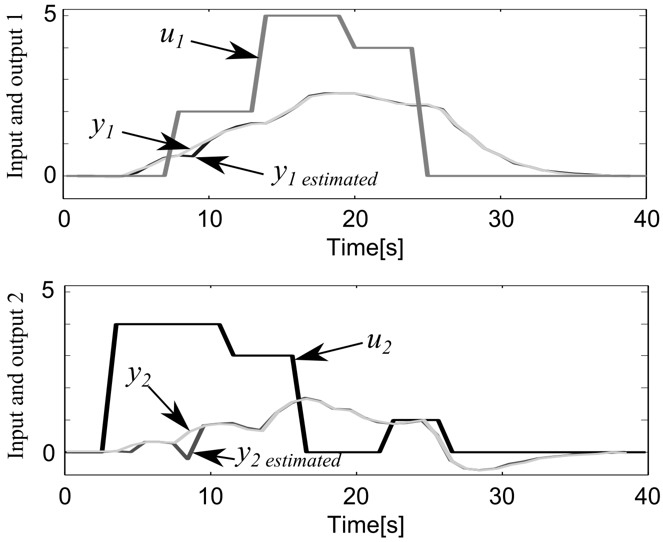

Fig. 11. Inputs and outputs for a 2×2 real multivariable system

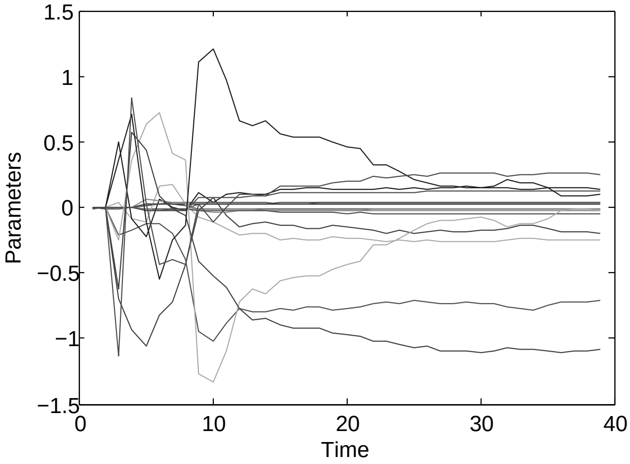

Fig. 12. Parameter estimation of $\boldsymbol{\theta}$ using MIMO least squares algorithm

D. Electric arc model

The electric arc model is described by the following non-linear differential equation [7], based on the energy conservation principle,

$$vi = k_1 r^2 + k_2 r \frac{dr}{dt} \tag{74}$$

where the state variable r represent the radius of the electric arc. The relation between the voltage and the current is given by

$$v = \frac{k_3}{r^2} i \tag{75}$$

The parameters k_1, k_2, and k_3, can be estimated as follow. From the above equation, it is possible to demonstrate that

$$r\frac{dr}{dt} = k_3 \frac{i'v - v'i}{v^2}\sqrt{\frac{v}{4i}} \tag{76}$$

replacing (76) in (74), and multiplying both sides of the resulting equation by v^2, we have

$$v^3 i = k_1 k_3 (vi) + k_2 k_3 (i'v - v'i)\sqrt{\frac{v}{4i}} \tag{77}$$

the above equation can be written in the form

$$y = a x_1 + b x_2 \tag{78}$$

where

$$\begin{cases} a = vi \\ b = (i'v - v'i)\sqrt{\frac{v}{4i}} \\ y = v^3 i \\ x_1 = k_1 k_3 \\ x_2 = k_2 k_3 \end{cases} \tag{79}$$

By considering N samples of the real voltage and current, the parameters of the linear model given by (78) can be estimated each half cycle of the real measurements, using the least squares method. In this paper, the value of the parameter k_3 has been set in 30, then, once the vector $\mathbf{x}$ is calculated, it is possible to determine the values of parameters k_1 and k_2. In Fig. 13 are shown the estimated parameters by using the least squares method. It is noticeable that the parameters are oscillating and do not tend to a fixed value.

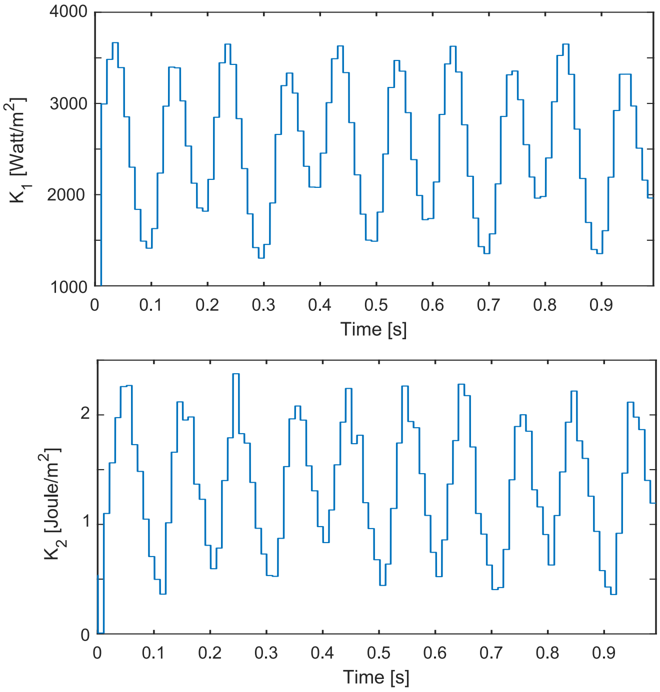

Fig. 13. Estimated parameter of the electric arc using least squares method

Based on the estimated parameters of Fig. 13 it is possible to reconstruct the voltage and current signals. The currents and voltage of the electric arc system and their corresponding estimated or reconstructed signals are shown in Fig. 14.

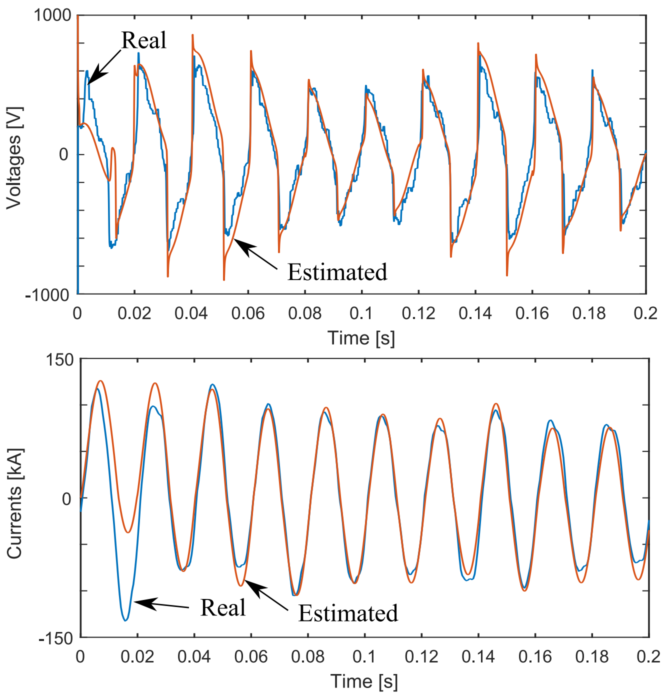

Fig. 14. Real and estimated signals of the electric arc system by using the least squares

By considering the ARMA representation of (5) and the ARMAX representation of (25), an estimation of the electric arc model of (78) can be obtained. In Fig. 15 is presented

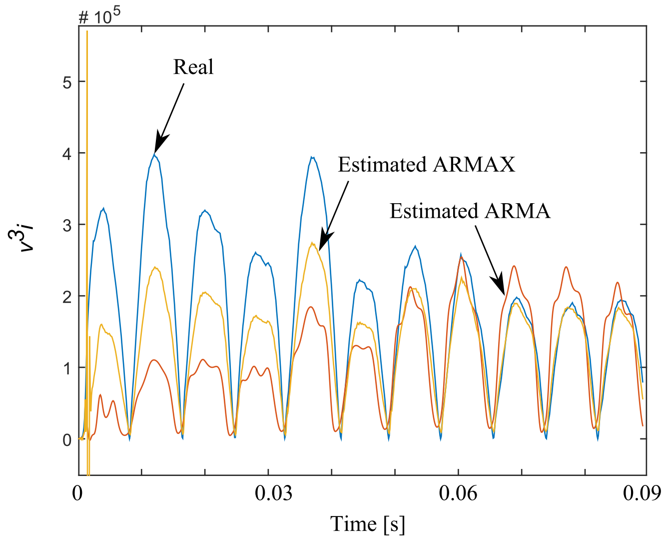

Fig. 15. Real and estimated signals for ARMA and ARMAX models

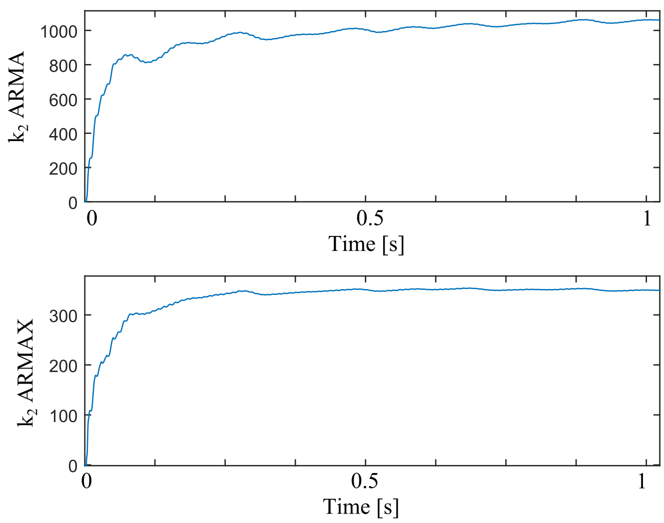

Fig. 16. Estimation of k_2 parameters by using ARMA and ARMAX models

a comparison of the real and estimated solutions for (78). It is noticeable that the ARMAX estimated signal describes a better approximation of the real signal than the ARMA estimated signal.

An additional result is presented in Fig. 16 related to the estimation of the k_2 parameter. As shown in Fig. 16 the k_2 parameter estimated by using an ARMA model describes small oscillations. In the other hand, the k_2 parameter estimated by using an ARMAX model tends to steady state without oscillations which implies that the exogenous input of the ARMAX model allows a better estimation of the parameters.

IV. Conclusions

In this paper, a robust multivariable adaptive model reference control is presented for linear systems with time varying parameters based on an ARMAX model. It can be seen that the model parameters are continuously adjusted which increase the robustness of the method to time varying parameters in comparison with the ARMA model. Also, it can be seen that a tracking performance is obtained effectively for the proposed method for simulated and real systems. In addition, since the selected reference model is decoupled, a decoupled response is obtained which reduce the variability of the system to the internal coupling.

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