The Bounded Projector in Weighted Mixed Lebesgue Spaces

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Abstract—The approximation theory for non-decaying signals in $L^{p,q}(\mathbb{R}^{d+1})$ is studied recently. In this paper, we prove that $P_{\varphi,h}$ is a bounded projector with the norm estimation from $L^{p,q}(\mathbb{R}^{d+1})$ onto $V_{\varphi,h}^{p,q}(\mathbb{R})$ if $\varphi$ belongs to an appropriate mixed Wiener amalgam space $W(\mathbb{L}_{k}^{1})(\mathbb{R}^{d+1})$. This gives the reconstruction of signals which belong to $V_{\varphi,h}^{p,q}(\mathbb{R})$.

Index Terms—Mixed Lebesgue spaces, Bounded projector, Sampling, Approximation, Wiener amalgam spaces.

I. INTRODUCTION

The sampling and reconstruction theory plays an important role in signal processing since it bridges the modern digital world and the analog world of continuous functions. The sampling states that during conversion of signals, it need to take values at some discrete points. The standard problem in sampling is to recover a signal $f \in V \subset L^2(\mathbb{R})$ from a sequence of sample values $\{f(x_i) : i \in \Lambda\}$, where $\Lambda$ is a countable indexing set. In other words, sampling converts the continuous signal $f(x)$ into discrete signal $c(k)$.

Reconstruction is the inverse process of sampling. It refers to the process of converting the sampled discrete-time signals into the continuous-time signals. In practical application, we always consider the reconstructed signal which has the translation invariant formula

$$\tilde{f}(x) = \sum_{k \in \mathbb{Z}^d} c(k)\varphi(x/h - k).$$

Here $\varphi$ is the generating kernel that satisfies some certain conditions. This form is very popular in spline theory [1], [2], [3], [4]. Recently, many scholars have done more in-depth research work on sampling and reconstruction [5], [6], [7].

In 1970s, Strang and Fix extended the work of Schoenberg [2] by considering the compactly supported function on $\mathbb{R}^d$ and its multiple integer transformations. In [8], [9], Strang, Fix and Jia gave the concept of controlled approximation. Then they successfully proved that the Strang-Fix condition (SF-condition, for brief) of order $k$ is equivalent to the controlled $L^2$-approximation property of order $k$. Strang and Fix gave several equivalent forms of this condition in [9], and their results have been extended in different directions [10], [11], [12], [13]. In [14], [15], Nguyen and Unser extended the classical Strang-Fix theory to two common types: projection and (direct) interpolation in shift-invariant spaces. They proved that if $\varphi$ has the SF-condition of order $k$, then the weighted $L^p$-norm of the error function is bounded when $\varphi$ belongs to a suitable hybrid-norm space.

The mixed Lebesgue spaces (MLS for brief) are the natural extension of classical Lebesgue spaces. They were first introduced by Benedek and Panzone in [16]. When a function which depends on independent quantities with different properties, it may be belong to the MLS. Later in [17], [18] further research was done by Robio de Francia et al. The integrability of each variable can be considered separately when a function belongs the MLS [19]. Under this property assumption, the multi-dimensional non-decaying functions in weighted $L^{p,q}(\mathbb{R}^{d+1})$ can be studied well. Wiener amalgam spaces (WAS for brief) [20] and mixed WAS [21] both were introduced for controlling the local-analytical property of a signal.

In this paper, we mainly give the approximation properties for non-decaying functions in MLS $L^{p,q}(\mathbb{R}^{d+1})$.

We prove that $P_{\varphi,h}$ is a bounded surjective projector from $L^{p,q}(\mathbb{R}^{d+1})$ to $V_{\varphi,h}^{p,q}(\mathbb{R})$ if $\varphi$ belongs to an appropriate mixed WAS $W(\mathbb{L}_{k}^{1})(\mathbb{R}^{d+1})$. This gives reconstruction formula for signals which belong to $V_{\varphi,h}^{p,q}(\mathbb{R})$.

II. PRELIMINARIES

First we introduce the definitions of the important MLS $L^{p,q}(\mathbb{R}^{d+1})$ and its discrete version $p^{q}(\mathbb{Z}^{d+1})$ ([22]).

Definition 2.1: Let $p, q \in [1, +\infty)$. Then $f \in L^{p,q}(\mathbb{R}^{d+1})$ if

$$\|f\|_{L^{p,q}} = \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{d}} |f(t_1,t_2)|^q \, dt_2 \right)^{\frac{p}{q}} \, dt_1 \right]^{\frac{1}{p}} < +\infty.$$

The discrete version $p^{q}(\mathbb{Z}^{d+1})$ is defined as following

$$\|c\|_{p^{q}} = \left\{ c : \|c\|_{p^{q}} = \sum_{n_1 \in \mathbb{Z}} \left( \sum_{n_2 \in \mathbb{Z}^d} |c(n_1, n_2)|^q \right)^{\frac{1}{q}} < +\infty \right\}.$$ 

The weighting function is given in the following.

Definition 2.2: Let a weighting function $\omega$ on $\mathbb{R}^{d+1}$ be continuous, symmetric and positive. If there is a constant $C_{\omega}$ satisfying that for any $s_1, t_1 \in \mathbb{R}, s_2, t_2 \in \mathbb{R}^d$, we have

$$\omega(s_1 + t_1, s_2 + t_2) \leq C_{\omega} \omega(s_1, s_2) \omega(t_1, t_2). \quad (1)$$

then it is called (weakly) submultiplicative.

The definitions of weighted MLS $L^{p,q}(\mathbb{R}^{d+1})$ and $p^{q}(\mathbb{Z}^{d+1})$ are shown in the below.
Definition 2.3: For \( p, q \in [1, +\infty) \), a function \( f \) and a weighting function \( \omega \), if \( f, \omega \in L^{p,q}(\mathbb{R}^{d+1}) \), then we call \( f \in L^{p,q}(\mathbb{R}^{d+1}) \). For a sequence \( \{c(n_1, n_2)\}_{n_1, n_2 \in \mathbb{Z}^d} \), if \( \{c(n_1, n_2)\}_{n_1, n_2 \in \mathbb{Z}^d} \in L^{p,q}(\mathbb{Z}^{d+1}) \), then we call \( \{c(n_1, n_2)\}_{n_1, n_2 \in \mathbb{Z}^d} \in L^{p,q}(\mathbb{Z}^{d+1}) \). Their weighted norms are defined as follows

\[
\|f\|_{L^{p,q}} := \|f\omega\|_{L^{p,q}};
\]

\[
\|\omega\|_{L^{p,q}} := \|\omega\|_{L^{p,q}}.
\]

Definition 2.4: [21] For \( p, q \in [1, +\infty) \), the mixed WAS \( W(L^{p,q})(\mathbb{R}^{d+1}) \) consists of all functions \( f \) which satisfy

\[
\|f\|_{W(L^{p,q})} := \left[ \sum_{k \in \mathbb{Z}^d} \left( \sum_{t \in \mathbb{Z}^d} |f(t_1 + k, t_2 + l)|^q \right)^{p/q} \right]^{1/p} < \infty.
\]

Its weighted norm is defined by

\[
\|f\|_{W(L^{p,q})} := \|f\omega\|_{W(L^{p,q})}.
\]

We write \( \langle \cdot \rangle \) as Sobolev weighting function \( \left( 1 + \| \cdot \|_2 \right)^{1/2} \). When \( \omega = \langle t \rangle^\alpha \) for some \( \alpha \in \mathbb{R} \), we write \( L^{p,q}(\mathbb{R}^{d+1}) \) for \( L^{p,q}(\mathbb{R}^{d+1}) \), \( L^{p,q}(\mathbb{Z}^{d+1}) \) for \( L^{p,q}(\mathbb{Z}^{d+1}) \), and \( W(L^{p,q})(\mathbb{R}^{d+1}) \) for \( W(L^{p,q})(\mathbb{R}^{d+1}) \). Now we introduce two important properties of this weighting function. When \( \alpha \geq 0 \), according to [14], the weighting function \( \omega = \langle t \rangle^\alpha \) satisfies

\[
\langle s + t \rangle^\alpha \leq C_\alpha \langle s \rangle^\alpha \langle t \rangle^\alpha, \quad \forall s, t \in \mathbb{R}^{d+1},
\]

where \( C_\alpha \) is a constant. This condition is equivalent to

\[
\langle s + t \rangle^{-\alpha} \leq C_\alpha \langle s \rangle^{-\alpha} \langle t \rangle^{-\alpha}, \quad \forall s, t \in \mathbb{R}^{d+1}.
\]

The other property of this weighting function \( \omega \) is when \( \alpha \geq 0 \), it has the Gelfand-Raikov-Shilov(GRS) condition [23]

\[
\lim_{n \to \infty} \omega(ntl)^{1/\alpha} = 1, \quad \forall l \in \mathbb{Z}^{d+1}.
\]

Definition 2.5: Let \( \alpha \geq 0 \), and \( h > 0 \) as a varying scale. The weighted non-decaying shift-invariant subspaces(SIS for brief) \( V_{\alpha,h}^{p,q}(\varphi) \) in mixed WLS are defined by

\[
V_{\alpha,h}^{p,q}(\varphi) := \left\{ f = \sum_{n_1, n_2 \in \mathbb{Z}^d} c(n_1, n_2) \varphi \left( \frac{\cdot}{h} - n_1, \frac{\cdot}{h} - n_2 \right) \in L^{p,q}(\mathbb{Z}^{d+1}) \right\}.
\]

III. THE MAIN RESULT

We assume that the kernel \( \varphi \in W(L^{1,1})(\mathbb{R}^{d+1}) \) and its shifts \( \{\varphi(\cdot - n_1, \cdot - n_2)\}_{n_1, n_2 \in \mathbb{Z}^d} \) is a Riesz basis of \( V^2(\varphi) \). According to [26], the dual kernel \( \varphi_{\langle \rangle} \) exists and is determined by the Fourier domain as following

\[
\hat{\varphi}_d(\omega) = \frac{\hat{\varphi}(\omega + 2\pi k)}{\sum_{k \in \mathbb{Z}^d+1} |\hat{\varphi}(\omega + 2\pi k)|^2}.
\]

Let us define the operator

\[
P_{\varphi,h} f := \sum_{n_1, n_2 \in \mathbb{Z}^d} c(n_1, n_2) \varphi(\frac{\cdot}{h} - n_1, \frac{\cdot}{h} - n_2),
\]

where \( c(k_1, k_2) \) is given by

\[
c(k_1, k_2) = \frac{1}{h^{d+1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y_1, y_2) \varphi_d \left( \frac{y_1}{h} - k_1, \frac{y_2}{h} - k_2 \right) dy_2 dy_1.
\]

Note that, we write \( P_{\varphi} \) for \( P_{\varphi,1} \).

The main result proves that the projector \( P_{\varphi,h} \) is bounded and subjective from \( L^{p,q}_{\alpha,h}(\varphi) \) to \( V_{\alpha,h}^{p,q}(\varphi) \), then we can approximate \( f \) by \( P_{\varphi,h} f \).

Theorem 3.1: Let \( p, q \in [1, +\infty) \) and \( \alpha \geq 0 \). If \( \varphi \in W(L^{1,1})(\mathbb{R}^{d+1}) \) and \( \{\varphi(\cdot - n_1, \cdot - n_2)\}_{n_1, n_2 \in \mathbb{Z}^d} \) is a Riesz basis for \( V^2(\varphi) \), then for each \( h > 0 \), \( V_{\alpha,h}^{p,q}(\varphi) \) is a closed subspace of \( L^{p,q}_{\alpha,h}(\varphi) \) and \( P_{\varphi,h} \) is a projector from \( L^{p,q}_{\alpha,h}(\varphi) \) onto \( V_{\alpha,h}^{p,q}(\varphi) \). Furthermore, there is a constant \( C_{\varphi,\alpha} \) such that for any \( f \in L^{p,q}_{\alpha,h}(\mathbb{R}^{d+1}) \) and \( h \in (0, 1) \),

\[
\|P_{\varphi,h} f\|_{L^{p,q}_{\alpha,h}} \leq C_{\varphi,\alpha} \|f\|_{L^{p,q}_{\alpha,h}}.
\]

Proof: See section VI.

IV. EXAMPLE

In this section, we give an example to show the reconstruction of function.

Let \( x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \),

\[
\varphi(x) = \varphi(x_1, \ldots, x_{d+1}) = \chi_{[0,1]}(x_1)\chi_{[0,1]}(x_2)\cdots\chi_{[0,1]}(x_{d+1}) = \chi_{[0,1]}(x_1),
\]

then \( \varphi \in W(L^{1,1}_{\alpha})(\mathbb{R}^{d+1}) \) and its shift \( \{\varphi(\cdot - n_1, \cdot - n_2)\}_{n_1, n_2 \in \mathbb{Z}^d} \) is a Riesz basis of \( V^2(\varphi) \). Since

\[
\sum_{k \in \mathbb{Z}^{d+1}} |\hat{\varphi}(\omega + 2\pi k)|^2
\]

\[
= \sum_{k \in \mathbb{Z}^{d+1}} \left| \int_{\mathbb{R}^{d+1}} \varphi(x)e^{-ix.(\omega+2\pi k)}dx \right|^2
\]

\[
= \sum_{i=1}^{d+1} \prod_{k \in \mathbb{Z}^{d+1}} \left( \int_{\mathbb{R}} \varphi(x_l)e^{-ix_l.((\omega+2\pi k)_l)}dx_l \right)^2
\]

\[
= \sum_{i=1}^{d+1} \prod_{k \in \mathbb{Z}^{d+1}} \int_{\mathbb{R}} e^{-ix_l.((\omega+2\pi k)_l)}dx_l
\]

\[
= \sum_{i=1}^{d+1} \prod_{k \in \mathbb{Z}^{d+1}} e^{-i(\omega_l + 2\pi k_l)} - 1 \omega_l + 2\pi k_l
\]

\[
= \prod_{i=1}^{d+1} \sum_{k \in \mathbb{Z}} e^{-i(\omega_i + 2\pi k_i)} - 1 \omega_i + 2\pi k_i
\]
we have the dual kernel
\( \hat{\varphi}_d(\omega) = \frac{\hat{\varphi}(\omega)}{\sum_{k \in \mathbb{Z}^{d+1}} |\hat{\varphi}(\omega + 2\pi k)|^2} = \hat{\varphi}(\omega). \)

Let \( f = e^{-\frac{|x|^2}{2}}, \) then \( f \in L^{p,q}_{\alpha}(\mathbb{R}^{d+1}) \) when \( \alpha \geq 0. \)

\[
c(k) = \frac{1}{h^{d+1}} \int_{\mathbb{R}^{d+1}} f(x) \hat{\varphi}_d(\frac{x}{h} - k) \, dx
= \frac{1}{h^{d+1}} \int_{\mathbb{R}^{d+1}} e^{-\frac{|x|^2}{2}} \chi_{[0,1]^{d+1}}(\frac{x}{h} - k) \, dx.
\]

Let \( d = 1. \) Figure 1 and Figure 2 are the graphs of the functions \( f = e^{-\frac{|x|^2}{2}}, P_{\varphi,0.1} f, P_{\varphi,0.05} f \) and \( P_{\varphi,0.01} f. \)

V. CONCLUSION

The reconstruction of a signal from the sampling is very important in signal processing. As a result, the sampling is studied in various spaces and the approximation theory of sampling is also concerned. In this paper, we consider the approximation property for non-decaying signals in MLS. We prove that \( P_{\varphi,h} \) is a bounded projector from the MLS onto mixed shift-invariant space when the generator belongs to an appropriate mixed WAS. This gives the reconstruction of signals which belong to mixed shift-invariant space.

VI. PROOFS OF THEOREM 3.1

Proof: First, we prove that \( V_{\varphi,h}^{p,q}(\mathbb{R}) \) is a subspace of \( L^{p,q}_{\alpha}(\mathbb{R}^{d+1}) \), for each \( h > 0. \) Let \( f \in V_{\varphi,h}^{p,q}(\varphi), \) then

\[
\sigma_{1/h} f = \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}^d} c(n_1, n_2) \varphi(\cdot - n_1, \cdot - n_2) \in V_{\varphi,h}^{p,q}(\varphi).
\]

According to Theorem 3.7 in [24], \( V_{\varphi,h}^{p,q}(\varphi) \) is a closed subspace of \( L^{p,q}_{\alpha}(\mathbb{R}^{d+1}) \), Thus, \( \sigma_{1/h} f \in L^{p,q}_{\alpha}(\mathbb{R}^{d+1}). \) For the convenience of writing, let \( (x)^\alpha = \langle x_1, x_2 \rangle^\alpha, \) where \( x_1 \in \mathbb{R}, x_2 \in \mathbb{R}^d, \)

\[
\|f\|_{L^{p,q}_{\alpha}}^2
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |(x_1, x_2)^\alpha f(x_1, x_2)|^q \, dx_2 \right)^{\frac{p}{q}} \, dx_1
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |(h x_1, h x_2)^\alpha f(h x_1, h x_2)|^q \, dx_2 \right)^{\frac{p}{q}} \, dx_1
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |(h x_1, h x_2)^\alpha (\sigma_{1/h} f)(x_1, x_2)|^q \, dx_2 \right)^{\frac{p}{q}} \, dx_1
= h^{d+1} \times \frac{d}{h}
\int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |(h x_1, h x_2)^\alpha (\sigma_{1/h} f)(x_1, x_2)|^q \, dx_2 \right)^{\frac{p}{q}} \, dx_1
\leq h^{d+1} \cdot \max (1, h^{-\alpha}) \times \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (x_1, x_2)^\alpha (\sigma_{1/h} f)(x_1, x_2)|^q \, dx_2 \right)^{\frac{p}{q}} \, dx_1
= h^{d+1} \cdot \max (1, h^{-\alpha}) \cdot \|\sigma_{1/h} f\|_{L^{p,q}_{\alpha}}^2.
\]

Thus, \( f \in L^{p,q}_{\alpha}(\mathbb{R}^{d+1}). \) This implies that \( V_{\varphi,h}^{p,q}(\varphi) \) is a subspace of \( L^{p,q}_{\alpha}(\mathbb{R}^{d+1}). \)

Second, we prove that \( V_{\varphi,h}^{p,q}(\varphi) \) is closed under the norm of \( L^{p,q}_{\alpha}(\mathbb{R}^{d+1}), \) when \( h > 0. \) Assume that \( \{f_n\} \) is a sequence in \( V_{\varphi,h}^{p,q}(\varphi) \) which satisfies \( f_n \rightarrow f \) in \( L^{p,q}_{\alpha}(\mathbb{R}^{d+1}) \) as \( n \rightarrow \infty. \) Then

\[
\|\sigma_{1/h} f_n - \sigma_{1/h} f\|_{L^{p,q}_{\alpha}}^2
= \int_{\mathbb{R}} \int_{\mathbb{R}^d} (x_1, x_2)^{-\alpha} \times (f_n(h x_1, h x_2) - f(h x_1, h x_2)) \, dx_2 \, dx_1
\leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} (x_1, x_2)^{-\alpha} \times (f_n(h x_1, h x_2) - f(h x_1, h x_2)) \, dx_2 \, dx_1
= h^{d+1} \cdot \max (1, h^{-\alpha}) \cdot \|\sigma_{1/h} f\|_{L^{p,q}_{\alpha}}^2.
\]
\[ n \to \infty. \] Since \( \{\sigma_{1/h, f_n}\} \) is a sequence in \( V^{p,q}_{\alpha,h}(\varphi) \), it is known from the Theorem 3.7 in [24] that \( \sigma_{1/h, f} \in V^{p,q}_{\alpha,h}(\varphi) \) (i.e. \( f \in V^{p,q}_{\alpha,h}(\varphi) \)). This implies that \( V^{p,q}_{\alpha,h}(\varphi) \) is closed in \( L^p_{\alpha,h}(\mathbb{R}^{d+1}) \) norm.

Third, we prove that \( P_{\varphi,h} \) is a projector mapping \( L^p_{\alpha,h}(\mathbb{R}^{d+1}) \) to \( V^{p,q}_{\alpha,h}(\varphi) \), for each \( h > 0 \). Inequality (4) implies that \( \sigma_{1/h,f} \) maps \( L^p_{\alpha,h}(\mathbb{R}^{d+1}) \) to itself. According to the definition of \( V^{p,q}_{\alpha,h}(\varphi) \), \( \sigma_{h} \) maps \( V^{p,q}_{\alpha,h}(\varphi) \) to \( V^{p,q}_{\alpha,h}(\varphi) \).

We can known from Theorem 3.5 in [24] that \( P_{\varphi} \) maps \( L^p_{\alpha,h}(\mathbb{R}^{d+1}) \) to \( V^{p,q}_{\alpha,h}(\varphi) \). Since \( P_{\varphi,h} = \sigma_{h} P_{\varphi} \sigma_{1/h} \), we have \( P_{\varphi,h} \) maps \( L^p_{\alpha,h}(\mathbb{R}^{d+1}) \) to \( V^{p,q}_{\alpha,h}(\varphi) \), then we get

\[
P_{\varphi,h}^2 = \sigma_{h} P_{\varphi} \sigma_{1/h} \sigma_{h} P_{\varphi} \sigma_{1/h} = \sigma_{h} P_{\varphi}^2 \sigma_{1/h} = \sigma_{h} P_{\varphi} \sigma_{1/h} = P_{\varphi,h}.
\]

Finally, we prove the bound (3). Let

\[
\omega_h(x_1, x_2) := (h x_1, h x_2)^\alpha,
\]

using the submultiplicative of \( \omega_h \), one has for each \( x_1, y_1 \in \mathbb{R}, x_2, y_2 \in \mathbb{R}^d \) and \( h > 0 \),

\[
\omega_h(x_1 + y_1, x_2 + y_2) \leq C_\alpha \omega_h(x_1, x_2) \omega_h(y_1, y_2).
\]

Then

\[
\|P_{\varphi,h} f\|_{L^p_{\alpha,h}}^\alpha = \|\sigma_{h} P_{\varphi} \sigma_{1/h} f\|_{L^p_{\alpha,h}}^\alpha = \left[ \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \varphi \left( \frac{x_1}{h} \right) \varphi \left( \frac{x_2}{h} \right) \right)^\alpha d_{\mathbb{R}^d} dx_1 \right]^\frac{1}{\alpha} \\
\leq h^{-d/\alpha} \left[ \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \varphi \left( \frac{x_1}{h} \right) \varphi \left( \frac{x_2}{h} \right) \right)^\alpha d_{\mathbb{R}^d} dx_1 \right]^\frac{1}{\alpha} \\
= h^{-d/\alpha} \varphi \left( \frac{x_1}{h}, \frac{x_2}{h} \right) \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \varphi \left( \frac{x_1}{h} \right) \varphi \left( \frac{x_2}{h} \right) \right)^\alpha d_{\mathbb{R}^d} dx_1 \\
\leq C_\alpha \cdot \|\sigma_{1/h,f}\|_{L^p_{\alpha,h}}^\alpha = C_\alpha \cdot \|\sigma_{h} P_{\varphi} \sigma_{1/h} f\|_{L^p_{\alpha,h}}^\alpha
\]

which shows that \( \sigma_{1/h,f_n} \to \sigma_{1/h,f} \) in \( L^p_{\alpha,h}(\mathbb{R}^{d+1}) \) as

\[ n \to \infty. \]
Then and where (W and (7) follows by the proof of Theorem 3.5 in [24]. Combining (8)-(10), one has

\[
\|\varphi\|_{W(L^1_{\alpha})} \leq \|\varphi\|_{W(L^1_{\alpha})} < \infty,
\]
and

\[
\|\varphi\|_{W(L^1_{\alpha})} \leq \|\varphi\|_{W(L^1_{\alpha})} < \infty.
\]

Combining (8)-(10), one has

\[
\|P_{p,h} f\|_{L^p_{\alpha}} \leq C^2 \|\varphi\|_{W(L^1_{\alpha})} \|\varphi\|_{W(L^1_{\alpha})} = C^2 \|\varphi\|_{L^p_{\alpha}},
\]
Therefore, one completes this proof.

REFERENCES