Parameter Estimation for Discretely Observed Stochastic Differential Equations with Small Lévy Noises

Chao Wei, Zhikan Han, Yan Wei, Yingying Zhou, Liting Cao, Ci Yan

***Abstract*—This paper is concerned with parameter estimation for discretely observed stochastic differential equations driven by small Lévy noises. The contrast function is given to obtain the least squares estimator and explicit formula of the estimation error is given. The consistency and asymptotic distribution of the estimator are derived by using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence. The parameter estimation theory of stochastic differential equation driven by Brownian motion is extended to Lévy noises.**

***Index Terms*—Parameter estimation, small Lévy noises, consistency, asymptotic distribution.**

I. Introduction

Itô type stochastic differential equations are widely used in the modeling of stochastic phenomena in the fields of physics, chemistry, medicine([1]–[3]). Recently, they are applied to describe the dynamics of a financial asset, such as Vasicek([4]), Cox-Ingersoll-Ross([5], [6]), Chan-Karloyi-Longstaff-Sanders ([7]) and Hull-White model ([8]). However, part or all of the parameters in stochastic model are always unknown. In the past few decades, some methods have been put forward to estimate the parameters for Itô type stochastic differential equations, such as maximum likelihood estimation([9], [10]), least squares estimation([11], [12]) and Bayes estimation([13]–[15]). But, in fact, non-Gaussian noise can more accurately reflect the practical random perturbation. Lévy noise, as a kind of important non-Gaussian noise, has attracted wide attention in the research and practice in the fields of engineering, economy and society and has been studied by some authors such as Bertoin([16]) and Applebaum([17]). From a practical point of view in parametric inference, it is more realistic and interesting to consider parameter estimation for stochastic differential equations with small Lévy noises. Recently, a number of literatures have been devoted to the parameter estimation for the models driven by small Lévy noises. When the coefficient of the Lévy jump term is constant, drift parameter estimation has been investigated ([18], [19]).

In the past few decades, parameter estimation for nonlinear stochastic differential equations with small Lévy noises has been studied. For example, Long([20]) discussed the consistency and rate of convergence of the least squares estimator of the drift parameter from discrete observations. Clément([21]) proved the local asymptotic mixed normality property of drift parameter estimator from high frequency observations on the fixed time interval [0,1], computed the asymptotic Fisher information and found that the rate in the local asymptotic mixed normality property depends on the behavior of the Lévy measure near zero. Chevallier([22]) estimated the parameter following a two-step procedure. The EM-algorithm was extended to a class of jump-diffusion regime-switching models, simulations were proposed, alongside an empirical application dedicated to the study of financial and commodity time series.

Although parameter estimation for nonlinear stochastic differential equations with small Lévy noises has been investigated by some authors, the diffusion coefficient is constant and the explicit formula of the estimation error has not been given. In this paper, we consider the parameter estimation problem for a class of nonlinear stochastic differential equations with small Lévy noises from discrete observations. The contrast function is given to obtain the least squares estimator and the explicit formula of the estimation error is derived. The consistency and asymptotic distribution of the estimator are analyzed by using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence. The Hyperbolic diffusion is introduced as an example to demonstrate the effectiveness of the methods used in this paper.

This paper is organized as follows. In Section 2, nonlinear stochastic differential equations driven by small Lévy noises is introduced, the contrast function is given and explicit formula of the least squares estimator is obtained. In Section 3, The consistency and asymptotic distribution of the estimator are discussed. An example is given in Section 4. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $(\{\mathscr{F}_t\}_{t\geq 0})$. Let $(L_t, t \geq 0)$ be an $(\{\mathscr{F}_t\})$-adapted Lévy noises with decomposition

$$L_t = B_t + \int_0^t \int_{|z|>1} zN(ds, dz) + \int_0^t \int_{|z|\leq 1} z\widetilde{N}(ds, dz), \tag{1}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion, $N(ds, dz)$ is a Poisson random measure independent of $(B_t, t \geq 0)$ with characteristic measure $dt\nu(dz)$, and $\widetilde{N}(ds, dz) = N(ds, dz) - \nu(dz)$ is a martingale measure. We assume

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that $\nu(dz)$ is a Lévy measure on $\mathbb{R}\backslash 0$ satisfying $\int(|z|^2 \wedge 1)\nu(dz) < \infty$.

In this paper, we study the drift parameter estimation for the following stochastic differential equation driven by small Lévy noises:

$$\begin{cases} dX_t = \alpha f(X_t)dt + \varepsilon g(X_t)dL_t, & t \in [0,1] \\ X_0 = x_0, \end{cases} \tag{2}$$

where α is an unknown parameter. Without loss of generality, it is assumed that $\varepsilon \in (0, 1]$.

Consider the following contrast function

$$\rho_{n,\varepsilon}(\alpha) = \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - \alpha f(X_{t_{i-1}})\Delta t_{i-1}|^2}{\varepsilon^2 g^2(X_{t_{i-1}})\Delta t_{i-1}}, \tag{3}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It is easy to obtain the estimators

$$\widehat{\alpha}_{n,\varepsilon} = \frac{\sum_{i=1}^{n} \frac{(X_{t_i} - X_{t_{i-1}})f(X_{t_{i-1}})}{g^2(X_{t_{i-1}})}}{\sum_{i=1}^{n} \frac{f^2(X_{t_{i-1}})}{g^2(X_{t_{i-1}})}\Delta t_{i-1}}. \tag{4}$$

Before giving the main results, we introduce some assumptions below.

Let $X^0 = (X_t^0, t \geq 0)$ be the solution to the underlying ordinary differential equation under the true value of the parameter:

$$dX_t^0 = \alpha_0 f(X_t^0)dt, \quad X_0^0 = x_0.$$

Assumption 1: $|f(x)| + |g(x)| \leq K_1(1 + |x|)$ and $|f(x) - f(y)| + |g(x) - g(y)| \leq K_2|x - y|$ where K_1 and K_2 are positive constants and $x, y \in \mathbb{R}$, $f(x)$ and $g(x)$ are twice differentiable with respect to x.

Assumption 2: α_0 is the true valve of the parameter.

Assumption 3: $\inf_{0\leq t\leq 1} |g(X_t)| > 0$.

Assumption 4: $|f'(x)| + |g'(x)| \leq K_3(1 + |x|)^\lambda$ where K_3 is a positive constant and $\lambda > 0$.

In the next sections, the consistency and asymptotic distribution of the least squares estimator are discussed.

III. Main Results and Proofs

In the following theorem, the consistency of the least squares estimators are proved by using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence.

Theorem 1: When $\varepsilon \to 0$ and $n \to \infty$, the least squares estimator $\widehat{\alpha}_{n,\varepsilon}$ is consistent, namely

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0.$$

Proof: By using the Euler-Maruyama scheme to discretize equation (2), it follows that

$$X_{t_i} - X_{t_{i-1}} = \alpha_0 f(X_{t_{i-1}})\Delta t_{i-1} + \varepsilon g(X_{t_{i-1}})(L_{t_i} - L_{t_{i-1}}). \tag{5}$$

Then, it is easy to see that

$$\begin{aligned} &\sum_{i=1}^{n} \frac{(X_{t_i} - X_{t_{i-1}})f(X_{t_{i-1}})}{g^2(X_{t_{i-1}})} \\ &= \alpha_0 \sum_{i=1}^{n} \frac{f^2(X_{t_{i-1}})}{g^2(X_{t_{i-1}})}\Delta t_{i-1} \\ &+ \varepsilon \sum_{i=1}^{n} \frac{f(X_{t_{i-1}})}{g(X_{t_{i-1}})}(L_{t_i} - L_{t_{i-1}}). \end{aligned} \tag{6}$$

Substituting Equation (6) into the expression of $\widehat{\alpha}_{n,\varepsilon}$, it follows that

$$\widehat{\alpha}_{n,\varepsilon} - \alpha_0 = \frac{\varepsilon \sum_{i=1}^{n} \frac{f(X_{t_{i-1}})}{g(X_{t_{i-1}})}(L_{t_i} - L_{t_{i-1}})}{\sum_{i=1}^{n} \frac{f^2(X_{t_{i-1}})}{g^2(X_{t_{i-1}})}\Delta t_{i-1}}. \tag{7}$$

Let $M_t^{n,\varepsilon} = X_{[nt]/n}$, in which $[nt]$ denotes the integer part of nt. We will prove that the sequence $\{M_t^{n,\varepsilon}\}$ converges to the deterministic process $\{X_t^0\}$ uniformly in probability as $\varepsilon \to 0$ and $n \to \infty$.

Observe that

$$X_t - X_t^0 = \alpha_0 \int_0^t (f(X_s) - f(X_s^0))ds + \varepsilon \int_0^t g(X_s)dL_s. \tag{8}$$

From the Assumption 1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & |X_t - X_t^0|^2 \\ \leq\ & 2\alpha_0^2 |\int_0^t (f(X_s) - f(X_s^0))ds|^2 + 2\varepsilon^2 |\int_0^t g(X_s)dL_s|^2 \\ \leq\ & 2\alpha_0^2 t \int_0^t |f(X_s) - f(X_s^0|^2 ds + 2\varepsilon^2 |\int_0^t g(X_s)dL_s|^2 \\ \leq\ & 2\alpha_0^2 K_2^2 t \int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2 |\int_0^t g(X_s)dL_s|^2. \end{aligned}$$

According to the Gronwall's inequality, it can be checked that

$$|X_t - X_t^0|^2 \leq 2\varepsilon^2 \sigma^2 e^{2\alpha_0^2 K_2^2 t^2} |\int_0^t g(X_s)dL_s|^2. \tag{9}$$

Then, it follows that

$$\sup_{0\leq t\leq T} |X_t - X_t^0| \leq \sqrt{2}\varepsilon\sigma e^{\alpha_0^2 K_2^2 T^2} \sup_{0\leq t\leq T} |\int_0^t g(X_s)dL_s|. \tag{10}$$

Therefore, for each $T > 0$, when $\varepsilon \to 0$, we have

$$\sup_{0\leq t\leq T} |X_t - X_t^0| \xrightarrow{P} 0. \tag{11}$$

As $[nt]/n \to t$ when $n \to \infty$, we get that the sequence $\{M_t^{n,\varepsilon}\}$ converges to the deterministic process $\{X_t^0\}$ uniformly in probability as $\varepsilon \to 0$ and $n \to \infty$.

Then, we will prove that $\sum_{i=1}^{n} \frac{f(X_{t_{i-1}})}{g(X_{t_{i-1}})}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 \frac{f(X_s^0)}{g(X_s^0)}dL_s$.

Note that

$$\sum_{i=1}^{n} \frac{f(X_{t_{i-1}})}{g(X_{t_{i-1}})}(L_{t_i} - L_{t_{i-1}}) = \int_0^1 \frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})}dL_s. \tag{12}$$

Then, it yields that

$$
\begin{aligned}
& |\int_0^1 \frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})}dL_s - \int_0^1 \frac{f(X_s^0)}{g(X_s^0)}dL_s| \\
= & |\int_0^1 (\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)})dB_s \\
+ & \int_0^1 \int_{|z|>1} (\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)})zN(ds,dz) \\
+ & \int_0^1 \int_{|z|\le 1} (\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)})z\widetilde{N}(ds,dz)| \\
\le & |\int_0^1 (\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)})dB_s| \\
+ & |\int_0^1 \int_{|z|>1} (\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)})zN(ds,dz)| \\
+ & |\int_0^1 \int_{|z|\le 1} (\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)})z\widetilde{N}(ds,dz)|.
\end{aligned}
$$

It can be easily to check that

$$
\begin{aligned}
& |\int_0^1 \int_{|z|>1} (\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)})zN(ds,dz)| \\
\le & \int_0^1 \int_{|z|>1} |\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)}||z|N(ds,dz) \\
\le & \sup_{0\le s\le 1} |\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)}| \int_0^1 \int_{|z|>1} |z|N(ds,dz) \\
\xrightarrow{P} & \; 0,
\end{aligned}
$$

as $\varepsilon \to 0$ and $n \to \infty$.

By using the Markov inequality and dominated convergence, we get

$$
|\int_0^1 (\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)})dB_s| \xrightarrow{P} 0, \tag{13}
$$

and

$$
|\int_0^1 \int_{|z|\le 1} (\frac{f(M_s^{n,\varepsilon})}{g(M_s^{n,\varepsilon})} - \frac{f(X_s^0)}{g(X_s^0)})z\widetilde{N}(ds,dz)| \xrightarrow{P} 0. \tag{14}
$$

Thus, together with the previous results, it follows that

$$
\sum_{i=1}^n \frac{f(X_{t_{i-1}})}{g(X_{t_{i-1}})}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 \frac{f(X_s^0)}{g(X_s^0)}dL_s. \tag{15}
$$

Moreover,

$$
\begin{aligned}
& |\frac{1}{n}\sum_{i=1}^n \frac{f^2(X_{t_{i-1}})}{g^2(X_{t_{i-1}})} - \int_0^1 \frac{f^2(X_s^0)}{g^2(X_s^0)}ds| \\
= & |\int_0^1 \frac{f^2(M_s^{n,\varepsilon})}{g^2(M_s^{n,\varepsilon})}ds - \int_0^1 \frac{f^2(X_s^0)}{g^2(X_s^0)}ds| \\
= & |\int_0^1 \frac{f^2(M_s^{n,\varepsilon})g^2(X_s^0) - f^2(X_s^0)g^2(M_s^{n,\varepsilon})}{g^2(M_s^{n,\varepsilon})g^2(X_s^0)}ds| \\
= & |\int_0^1 \frac{f^2(M_s^{n,\varepsilon})g^2(X_s^0) - f^2(X_s^0)g^2(X_s^0)}{g^2(M_s^{n,\varepsilon})g^2(X_s^0)}ds \\
+ & \int_0^1 \frac{f^2(X_s^0)g^2(X_s^0) - f^2(X_s^0)g^2(M_s^{n,\varepsilon})}{g^2(M_s^{n,\varepsilon})g^2(X_s^0)}ds| \\
= & |\int_0^1 \frac{f^2(M_s^{n,\varepsilon}) - f^2(X_s^0)}{g^2(M_s^{n,\varepsilon})}ds \\
+ & \int_0^1 \frac{f^2(X_s^0)(g^2(X_s^0) - g^2(M_s^{n,\varepsilon}))}{g^2(M_s^{n,\varepsilon})g^2(X_s^0)}ds| \\
= & |\int_0^1 \frac{(f(M_s^{n,\varepsilon}) + f(X_s^0))(f(M_s^{n,\varepsilon}) - f(X_s^0))}{g^2(M_s^{n,\varepsilon})}ds \\
+ & \int_0^1 \frac{f^2(X_s^0)(g(X_s^0) + g(M_s^{n,\varepsilon}))(g(X_s^0) - g(M_s^{n,\varepsilon}))}{g^2(M_s^{n,\varepsilon})g^2(X_s^0)} \\
& ds| \\
= & |\int_0^1 \frac{(f(M_s^{n,\varepsilon}) + f(X_s^0))}{g^2(M_s^{n,\varepsilon})} \\
& \int_0^1 f'(X_s^0 + \theta(M_s^{n,\varepsilon} - X_s^0))(M_s^{n,\varepsilon} - X_s^0)d\theta ds \\
+ & \int_0^1 \frac{f^2(X_s^0)(g(X_s^0) + g(M_s^{n,\varepsilon}))}{g^2(M_s^{n,\varepsilon})g^2(X_s^0)} \\
& \int_0^1 g'(M_s^{n,\varepsilon} + \theta(X_s^0 - M_s^{n,\varepsilon}))(X_s^0 - M_s^{n,\varepsilon})d\theta ds| \\
\le & \int_0^1 |\frac{(f(M_s^{n,\varepsilon}) + f(X_s^0))}{g^2(M_s^{n,\varepsilon})} \\
& |K_3(1 + |X_s^0| + |M_s^{n,\varepsilon}|)^\lambda |M_s^{n,\varepsilon} - X_s^0|ds \\
+ & \int_0^1 |\frac{f^2(X_s^0)(g(X_s^0) + g(M_s^{n,\varepsilon}))}{g^2(M_s^{n,\varepsilon})g^2(X_s^0)} \\
& |K_3(1 + |X_s^0| + |M_s^{n,\varepsilon}|)^\lambda |M_s^{n,\varepsilon} - X_s^0|ds \\
\le & \int_0^1 |\frac{(f(M_s^{n,\varepsilon}) + f(X_s^0))}{g^2(M_s^{n,\varepsilon})} \\
& |K_3(1 + \sup_{0\le s\le 1}|X_s^0| + \sup_{0\le s\le 1}|X_s|)^\lambda \\
& \sup_{0\le s\le 1}|M_s^{n,\varepsilon} - X_s^0|ds \\
+ & \int_0^1 |\frac{f^2(X_s^0)(g(X_s^0) + g(M_s^{n,\varepsilon}))}{g^2(M_s^{n,\varepsilon})g^2(X_s^0)} \\
& |K_3(1 + \sup_{0\le s\le 1}|X_s^0| + \sup_{0\le s\le 1}|X_s|)^\lambda \\
& \sup_{0\le s\le 1}|M_s^{n,\varepsilon} - X_s^0|ds \\
\xrightarrow{P} & \; 0.
\end{aligned}
$$

Thus, we have

$$
\frac{1}{n}\sum_{i=1}^n \frac{f^2(X_{t_{i-1}})}{g^2(X_{t_{i-1}})} \xrightarrow{P} \int_0^1 \frac{f^2(X_s^0)}{g^2(X_s^0)}ds. \tag{16}
$$

With above results, when $\varepsilon \to 0$ and $n \to \infty$, it follows that

$$
\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0. \tag{17}
$$

The proof is complete. ■

Theorem 2: When $\varepsilon \to 0$ and $n \to \infty$,

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \xrightarrow{p} \frac{\int_0^1 \frac{f(X_s^0)}{g(X_s^0)} dL_s}{\int_0^1 \frac{f^2(X_s^0)}{g^2(X_s^0)} ds}.$$

Proof: When $\varepsilon \to 0$ and $n \to \infty$, it is easy to check that

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \xrightarrow{p} \frac{\int_0^1 \frac{f(X_s^0)}{g(X_s^0)} dL_s}{\int_0^1 \frac{f^2(X_s^0)}{g^2(X_s^0)} ds}. \tag{18}$$

■

Remark 1: In this section, we discuss the extension of our main results in Section 3 to the general case when the driving noise is a semi-martingale. Let $Q_t = Q_0 + M_t + A_t$ be a semi-martingale, where M_t is a local martingale and A_t is a finite variation process. Then, we can replace the driving Lévy process L_t by the semi-martingale Q_t to get

$$\begin{cases} dX_t = \alpha f(X_t)dt + \varepsilon g(X_t)dQ_t, \quad t \in [0,1] \\ X_0 = x_0, \end{cases} \tag{19}$$

where α is an unknown parameter. Without loss of generality, it is assumed that $\varepsilon \in (0, 1]$.

All the related information about the least squares estimator of α discussed in this section is same to Section 2. It is easy to check that the consistency and asymptotic behavior of the least squares estimator of α are also hold.

IV. Example

In this section, the Hyperbolic diffusion driven by small Lévy noises is given as an example. Hyperbolic diffusion is widely used to describe the financial phenomenon. The equation has the following expression

$$\begin{cases} dX_t = \alpha \frac{X_t}{\sqrt{1+X_t^2}} dt + \varepsilon\sigma dL_t, \quad t \in [0,1] \\ X_0 = x_0, \end{cases} \tag{20}$$

where α is an unknown parameter.

It is easy to check that Hyperbolic diffusion satisfies the Assumptions 1-4, the contrast function has the following expression

$$\rho_{n,\varepsilon}(\alpha) = \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - \alpha \frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} \Delta t_{i-1}|^2}{\varepsilon^2\sigma^2\Delta t_{i-1}},$$

and the least squares estimator is derived

$$\widehat{\alpha}_{n,\varepsilon} = \frac{\sum_{i=1}^{n} \frac{(X_{t_i} - X_{t_{i-1}})X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}}{\sum_{i=1}^{n} \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2} \Delta t_{i-1}}.$$

We can derive that

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0,$$

and

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \xrightarrow{p} \frac{\int_0^1 \frac{\frac{X_s^0}{\sqrt{1+(X_s^0)^2}}}{\sigma} dL_s}{\int_0^1 \frac{\frac{(X_s^0)^2}{1+(X_s^0)^2}}{\sigma^2} ds}.$$

We generate a discrete sample $(X_{t_i})_{i=0,1,\ldots,n}$ and compute $\widehat{\alpha}_{n,\varepsilon}$ from the sample. We let $\sigma = 0.5$, $x_0 = 0.1$, the character measure υ of Poisson jump satisfies $\upsilon(dz) = \zeta\phi(dz)$, where $\zeta = 1.5$ is the intensity of Poisson distribution and ϕ is the probability intensity of the standard normal distributed variable z. For every given true value of the parameter-α_0, the size of the sample is represented as"Size n" and given in the first column of the table. In Tables 1 and 3, $\varepsilon = 0.05$, the size is increasing from 500 to 3000. In Tables 2 and 4, $\varepsilon = 0.001$, the size is increasing from 5000 to 30000. Tables 1 and 2 list the value of "$\alpha_0 - LSE$" and the absolute errors (AE) of LSE, LSE means least squares estimator. Tables 3 and 4 list the value of "$\alpha_0 - LSE$" and the confidence interval of α_0.

Two tables illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value and the length of the confidence interval is becoming small when the size of the sample is increasing. Therefore, the methods used in this paper are effective and the obtained estimators are good.

TABLE I
LSE simulation results of α_0

True	Aver		AE
(α_0)	Size n	$\alpha_0 - LSE$	α_0
1	500	0.9557	0.0443
	1000	0.9623	0.0377
	3000	0.9775	0.0225
2	500	1.9574	0.0426
	1000	2.0358	0.0358
	3000	2.0162	0.0162
3	500	2.9596	0.0404
	1000	3.0328	0.0328
	3000	3.0183	0.0183

V. Conclusion

In this paper, parameter estimation for nonlinear stochastic differential equations with small Lévy noises has been studied from discrete observations. The explicit formula of the least squares estimator and the estimation error have been obtained. The consistency and asymptotic distribution of the estimator have been derived by using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence. Further research topics will include

TABLE II
LSE SIMULATION RESULTS OF α_0

True		Aver	AE
(α_0)	Size n	$\alpha_0 - LSE$	α_0
1	5000	0.9782	0.0218
	10000	0.9835	0.0165
	30000	0.9946	0.0054
2	5000	2.0285	0.0285
	10000	2.0130	0.0130
	30000	2.0012	0.0012
3	5000	3.0259	0.0259
	10000	3.0126	0.0126
	30000	3.0025	0.0025

TABLE III
SIMULATION RESULTS OF CONFIDENCE INTERVAL FOR α_0

True		Aver	confidence interval of 0.95
(α_0)	Size n	$\alpha_0 - LSE$	α_0
1	500	0.9557	[0.9396,0.9912]
	1000	0.9623	[0.9452,0.9836]
	3000	0.9714	[0.9528,0.9786]
2	500	2.0416	[1.8925,2.0538]
	1000	2.0358	[1.9157,2.0479]
	3000	2.0162	[2.0026,2.0314]
3	500	3.0415	[2.8579,3.0627]
	1000	3.0328	[2.9236,3.0541]
	3000	3.0183	[3.0012,3.0428]

TABLE IV
SIMULATION RESULTS OF CONFIDENCE INTERVAL FOR α_0

True		Aver	confidence interval of 0.95
(α_0)	Size n	$\alpha_0 - LSE$	α_0
1	5000	1.0231	[0.9527,1.0630]
	10000	1.0169	[0.9623,1.0512]
	30000	1.0036	[0.9717,1.0436]
2	5000	2.0228	[1.9633,2.0685]
	10000	2.0106	[1.9721,2.0514]
	30000	2.0010	[1.9813,2.0463]
3	5000	3.0252	[2.9163,3.0710]
	10000	3.0117	[2.9228,3.0632]
	30000	3.0009	[2.9382,3.0519]

the drift and diffusion parameter estimation for nonlinear stochastic differential equations driven by Lévy noises.

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