Real-time implementation of a Discrete Fractional-Order PID Control

Carlos A. Ramírez-Vanegas, Eduardo Giraldo

Abstract—**In this article, a novel implementation in discrete time of a fractional-order PID control is presented. The system is evaluated over a real third-order single input single output system where the proposed implementation of the fractional-order PID is compared with a classical integer-order PID. The fractional-order operator's response for positive and negative constants is evaluated for a simulated square signal. Additional results are obtained by comparing the closed-loop response of integer and fractional order PIDs over the simulated system. The closed-loop response of the real system is evaluated under two types of reference signals: square signal and sinusoidal signal. It can be seen that the proposed implementation results in a more efficient response by using the same parameters in terms of steady-state error and settling-time.**

Index Terms—**Fractional order control, discrete PID, real-time.**

I. Introduction

THE fractional-order calculus is a generalization of the integer calculus for derivatives and integrals of non-integer order [1]. Fractional-order controllers has been widely used to control linear and nonlinear systems where they have proved their effectiveness by increasing the controllers' flexibility. Several fractional-order controllers have been designed from integer-order controllers, which included PID, lead-lag compensator, state feedback, among others and where the improvement of fractional-order controllers in comparison to integer-order controllers is verified for several applications [2], [3], [4], [5], [6], [7].

It is noticeable that the PID controller has been improved by using the fractional order calculus in several linear and nonlinear applications [8], where optimization techniques have been used for tuning the controller parameters, as described in [9], [10], [11]. Moreover, variations of PID controllers such as Fuzzy PIDs have also been modified by using fractional calculus [12].

On the other hand, it is noticeable that control strategies over real systems require discrete-time versions to be implemented in computers and micro-controllers. Several controllers can be applied over discrete-time systems by using robust control techniques [13] and also sliding mode controllers [14]. Intelligent controllers have also been applied over discrete-time multivariable systems in discrete time [15], [16]. In addition, fractional-order controllers in discrete time are also implemented over real systems for several applications, as described in [17].

In this work, a novel implementation in discrete-time fractional-order PID control is presented. The system is evaluated over a real third-order single input single output system where the proposed implementation of the fractional-order PID is compared with a classical integer-order PID. The response of the fractional-order operator for integral and derivative cases is evaluated for a simulated square signal. A simulated system's closed-loop response is also considered by comparing integer and fractional order PIDs by using a unitary step reference signal. In addition, the closed-loop response of the real system is evaluated under two types of reference signals: square signal and sinusoidal signal. It can be seen that the proposed implementation results in a more efficient response by using the same parameters in terms of steady-state error and settling-time. This paper is organized as follows: in section II is presented the mathematical foundation of the fractional-order PID control and the proposed discrete implementation of the fractional operator. In section III the evaluation of the fractional operator over a simulated squared signal is presented, as well as the performance of the proposed approach over a real system with constant and time-varying reference signals. And finally, in section IV the conclusions and final remarks are presented.

II. Theoretical framework

A. Fractional calculus

Fractional calculus is an extension of calculus to integrate non-integer operators in derivation and integration. This was born with the calculation itself, but it was not widely developed, until the 24th century along, with the advances of control theory [18].

To give a definition of fractional integrals we can start with the definition of the first integral of a function as shown in (1):

$$\nabla^{-1} f(t) = \int_0^t f(x)dx \tag{1}$$

Applying the integral operator again, the second integral is obtained, in the form:

$$\nabla^{-2} f(t) = \int_0^t \int_0^x f(y)dydx \tag{2}$$

By reversing the order of integration and making the respective limit changes, we obtain:

$$\nabla^{-2} f(t) = \int_0^t \int_y^t f(y)dxdy \tag{3}$$

Manuscript received September 9, 2020; revised November 14, 2020. This work was carried out under the funding of the Universidad Tecnológica de Pereira, Vicerrectoría de Investigación, Innovación y Extensión. Research project: 6-20-7 "Estimación Dinámica de estados en sistemas multivariables acoplados a gran escala".

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Since $f(y)$ is constant with respect to x, the second integral would be of the form:

$$\nabla^{-2} f(t) = \int_0^t (t-y) f(y) dy \quad (4)$$

In a similar way, the third integral can be obtained as (5):

$$\nabla^{-3} f(t) = \frac{1}{2} \int_0^t (t-y)^2 f(y) dy \quad (5)$$

In general, for an operator of order n, following the previous procedure, we obtain:

$$\nabla^{-n} f(t) = \frac{1}{(n-1)!} \int_0^t (t-y)^{n-1} f(y) dy \quad (6)$$

Now, making use of the Γ function shown in (7) and (8) applied in (6), the Riemann-Liouville equation presented in (9) for fractional integrals is obtained as follows:

$$\int_0^\infty e^{-t} t^{z-1} dt = \Gamma(z) \quad (7)$$

$$\Gamma(z) = (n-1)!, \quad z \in \mathbb{R}^+ \quad (8)$$

The Γ function allows us to evaluate the factorial operator not only on positive integers, but also on all positive real numbers:

$$\nabla^{-n} f(t) = \frac{1}{\Gamma(n)} \int_0^t f(y)(t-y)^{n-1} dy, \quad n \in \mathbb{R}^+ \quad (9)$$

In the beginning of (6) it is possible to also obtain the Riemann-Liouville definition for derivatives of non-integer order, as shown in (10), in that case it is necessary to introduce a new variable m.

$$\nabla^\alpha f(t) = \nabla I^{1-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt}^m \int_\alpha^t \frac{f(y)}{(t-y)^{\alpha-m+1}} dy \quad (10)$$

where $m-1 < \alpha < m, \ m \in \mathbb{N}$.

Another way of realizing non-integer derivatives is proposed in [1], which does not require the initial conditions of fractional order of the function.

$$\nabla^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha-m+1}} dt \quad (11)$$

B. Fractional-order PID

There are several controllers that use the fractional calculation, where one of the first developed is the fractional-order $PI^\lambda D^\mu$ control proposed by [2] for commensurable order systems using an integral action of order λ and a derivative action of order μ. For this, we start from the fact that a differential equation of fractional order, linear and invariant with time can be defined as:

$$\sum_{k=0}^{m} a_k D^{\alpha_k} y(t) = \sum_{k=o}^{l} b_k D^{\beta_k} \quad (12)$$

The previous equation would be of a commensurable order if it is true that all the orders of derivation are integer multiples of a base order, therefore:

$$\alpha_k, \beta_k = n\alpha, \ \alpha \in \mathbb{R}, \ n \in \mathbb{Z} \quad (13)$$

In this way, (12) would be as follows:

$$\sum_{k=0}^{m} a_k D^{n\alpha} y(t) = \sum_{k=o}^{l} b_k D^{n\alpha} \quad (14)$$

If that is also true $\alpha = \frac{1}{q}, q \in \mathbb{Z}^+$, then the system is said to be of a rational order.
The transfer function of the PID controller can be defined in terms of the error as follows:

$$C(s) = K_p + \frac{K_i}{s} + K_d s \quad (15)$$

On the other hand, the transfer function of a $PI^\lambda D^\mu$ can be expressed as:

$$C(s) = K_p + \frac{K_i}{s^\lambda} + K_d s^\mu \quad (16)$$

The main advantage of the fractional control is the possibility of giving more degrees of freedom the order of the integral (λ) and derivative (μ) actions.

A discrete implementation of the fractional-order PID is proposed in [17]. In this work, by using a backwards operator, the following equivalence is used:

$$s = \frac{1-z^{-1}}{T} \quad (17)$$

being T the sample time. Therefore, the application of (17) on (15) results in the discrete difference equation of the PID, as follows:

$$\begin{aligned} e_i[k] &= Te[k] + e_i[k-1] \\ u[k] &= K_p e[k] + K_d \frac{e[k]-e[k-1]}{T} + K_i e_i[k] \end{aligned} \quad (18)$$

with e_i the integral error and $e_i[0] = 0$.

In [19] a discrete fractional operator is defined as

$$s^\mu = \left(\frac{1-z^{-1}}{T} \right)^\mu \quad (19)$$

By applying a binomial expansion of (19), the discrete time fractional order operator can be obtained

$$D^\mu = T^{-\mu} \sum_{j=0}^{\infty} b_j z^{-j} \quad (20)$$

being b_j defined as

$$b_j = \left(1 - \frac{1+\mu}{j} \right) b_{j-1} \quad (21)$$

with $j = 1, 2, \ldots$ and $b_0 = 1$.

By considering (21) and (16) the following fractional-order $PI^\lambda D^\mu$ is proposed:

$$\begin{aligned} u[k] &= K_p e[k] \\ &+ K_d T^{-\mu} \sum_{j=0}^{L} b_j e[k-j] + K_i T^\lambda \sum_{j=0}^{L} c_j e[k-j] \end{aligned} \quad (22)$$

being L the number of samples of the window and being b_j and c_j defined as

$$b_j = \left(1 - \frac{1+\mu}{j} \right) b_{j-1} \quad (23)$$

$$c_j = \left(1 - \frac{1-\lambda}{j} \right) c_{j-1} \quad (24)$$

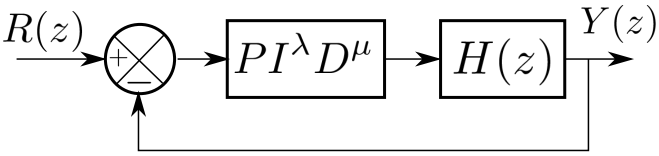

Fig. 1. Closed-loop system with a fractional-order PID

with $b_0 = 1$ and $c_0 = 1$. The resulting closed-loop system with a fractional-order $PI^{\lambda}D^{\mu}$ is shown in Fig. 1, being $H(z)$ the discrete transfer function of the system to be controlled.

III. Results

In order to evaluate the performance of the discrete fractional-order PID, a comparison analysis is performed. This evaluation is achieved using simulated and real systems where the real system is implemented using operational amplifiers and a USB Data AcQuisition (USB-DAQ) card. Initially, the fractional operator's evaluation is performed over a simulated square signal with unitary amplitude, $\mu = 0.2$, $\lambda = 0.6$, and sample time $T = 50$ milliseconds. In Fig. 2 is presented the fractional operator (integral and derivative) with a window length of $L = 200$.

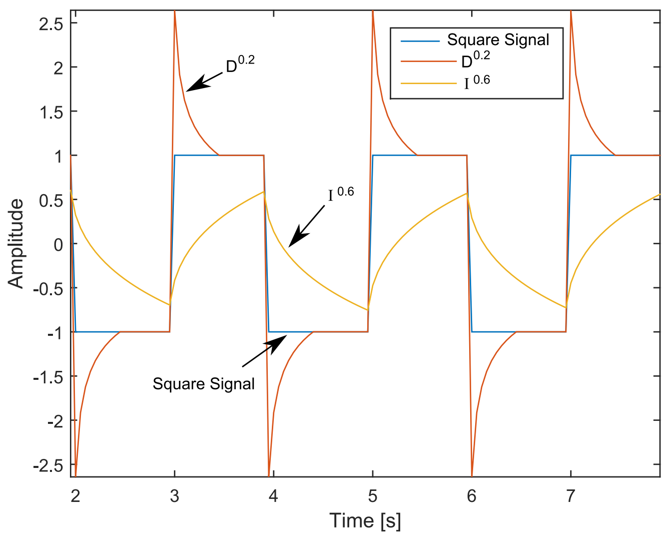

Fig. 2. Evaluation of fractional order operator (integrative and derivative) with a simulated square signal

An additional evaluation is performed over a closed-loop control system by using a simulated discrete transfer function with a sample time of 0.1 seconds defined by

$$H(z) = \frac{0.004837z + 004679}{z^2 - 1.905z + 0.9048} \quad (25)$$

By using (25) a a comparison analysis by using integer and fractional-order controllers is performed by considering a unitary step reference signal. In Fig. 3 is presented a comparison between a integer-order PID with parameters $K_p = 1$, $K_d = 1$ and $K_i = 1$, and a fractional-order PID with the same Kp, K_i and K_d parameters and $\lambda = 0.5$ and $\mu = 0.5$.

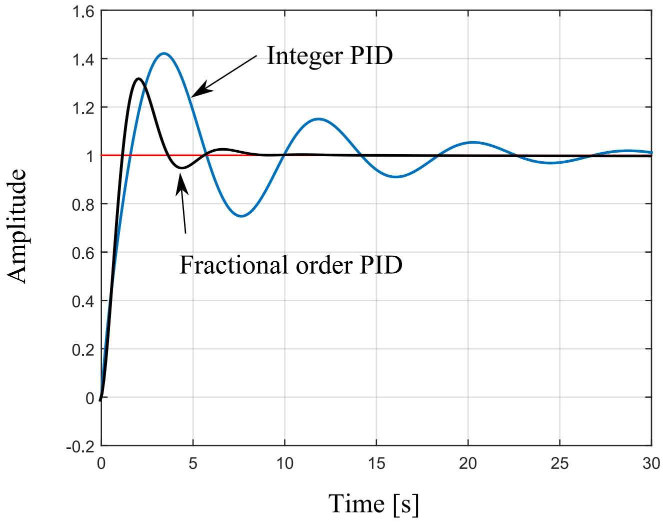

Fig. 3. Comparison analysis by using an integer and fractional order PID with $\lambda = 0.5$ and $\mu = 0.5$

It can be seen from Fig. 3 that the fractional-order PID outperform the integer-order PID by using the same parameters. An additional comparison is performed by modifying exclusively the λ and μ parameters. In Fig. 4 is presented a comparison between a integer-order PID with parameters $K_p = 1$, $K_d = 1$ and $K_i = 1$, and a fractional-order PID with the same Kp, K_i and K_d parameters and $\lambda = 0.3$ and $\mu = 0.7$.

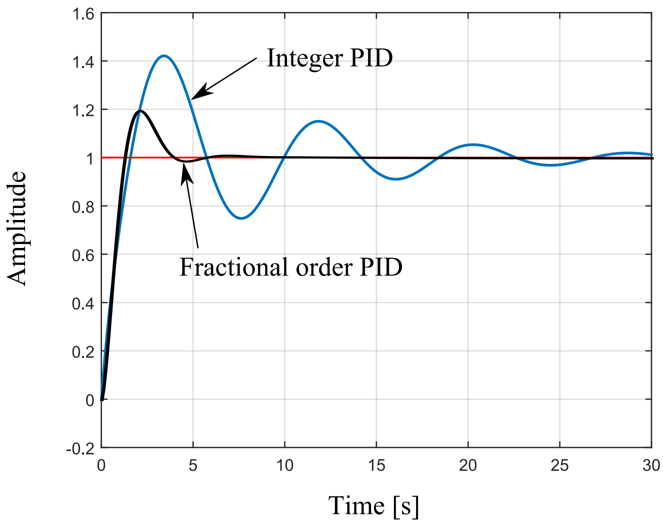

Fig. 4. Comparison analysis by using an integer and fractional-order PID with $\lambda = 0.3$ and $\mu = 0.7$

From Fig. 4, it can be seen that the closed-loop response of the integer-order PID is also outperformed by using the fractional-order PID in terms of settling-time and maximum peak. In addition, by comparing the closed-loop responses of the fractional-order PIDs presented in Fig. 3 and Fig. 4, it can be seen that the fractional-order PID of Fig. 4 shows a lower maximum peak as well as a lower settling-time.

The performance of the proposed $PI^{\lambda}D^{\mu}$ method is compared for an integer-order PID and evaluated over a real third-order single input single output system [20], for constant and time-varying references. It is worth noting that the third-order real system is implemented with operational amplifiers. In order to obtain the comparison analysis, the proposed $PI^{\lambda}D^{\mu}$ method is implemented according to (22) with $L = 1000$. The discrete-time control system is implemented over $LabVIEW^{TM}$ 2013 with a NI-DAQ USB 6009 and a sample time $T = 50$ milliseconds. The control signal is saturated in $0 \leq u[k] \leq 5$ volts range.

In Fig. 5 is presented the closed loop response of the $PI^{\lambda}D^{\mu}$ with a square signal.

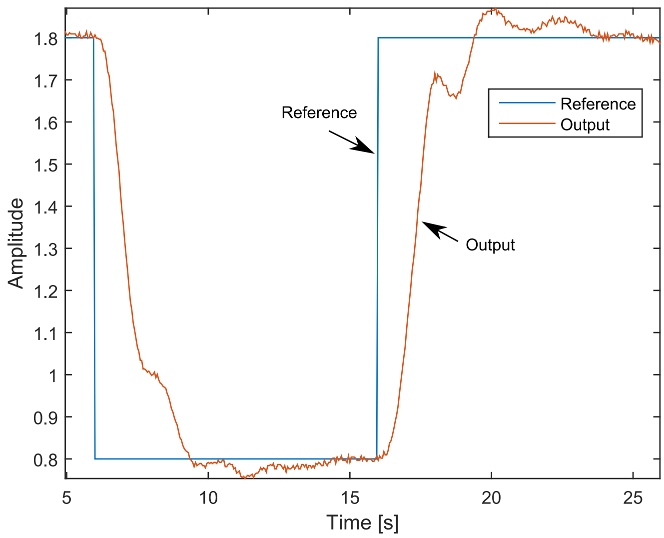

Fig. 5. $PI^{\lambda}D^{\mu}$ with $\mu = 0.8$ and $\lambda = 0.9$, $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

In Fig. 6 is presented the corresponding control signal of the closed loop of Fig. 5.

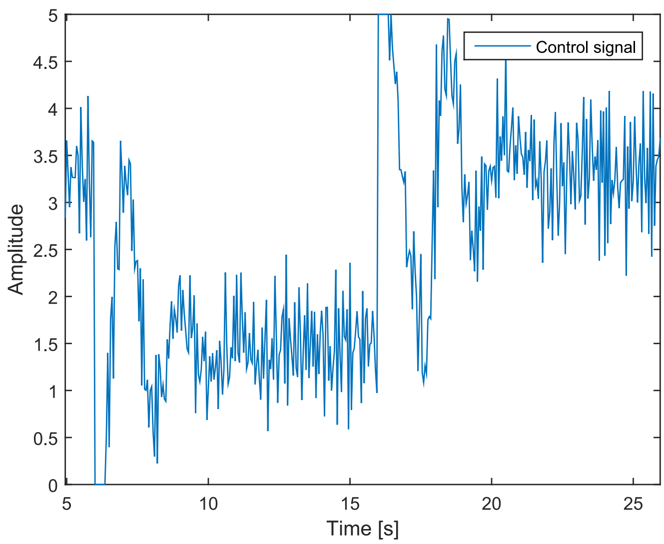

Fig. 6. Control signal of the $PI^{\lambda}D^{\mu}$ with $\mu = 0.8$ and $\lambda = 0.9$, $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

In Fig. 7 is presented the closed-loop response of the PID with a square signal.

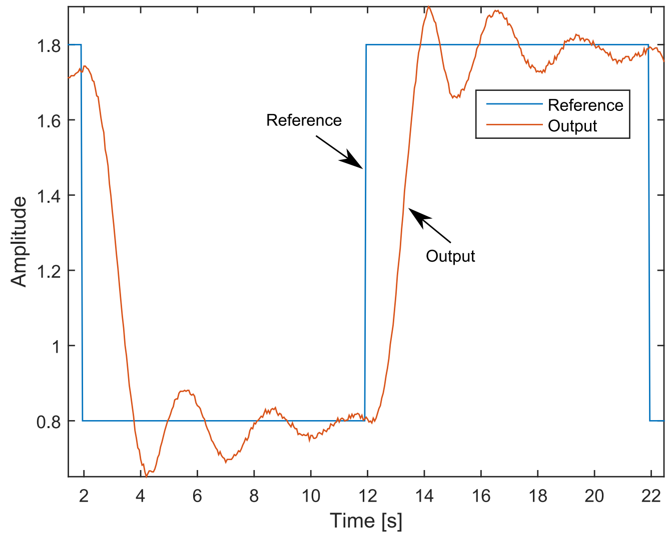

Fig. 7. $PI^{\lambda}D^{\mu}$ with $\mu = 0.7$ and $\lambda = 0.9$, $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

In Fig. 8 is presented the corresponding control signal of the closed-loop of Fig. 7.

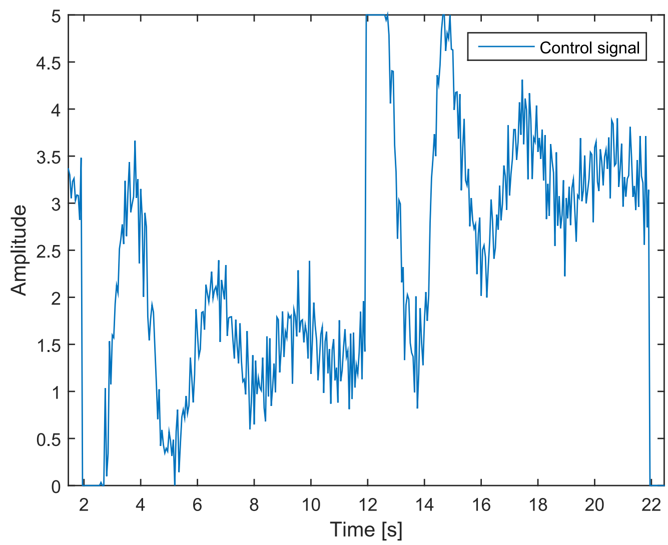

Fig. 8. Control signal of the $PI^{\lambda}D^{\mu}$ with $\mu = 0.7$ and $\lambda = 0.9$, $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

In Fig. 9 is presented the closed-loop response of the PID with a square signal. It can be seen that the closed-loop response has a higher settling-time for the same parameters.

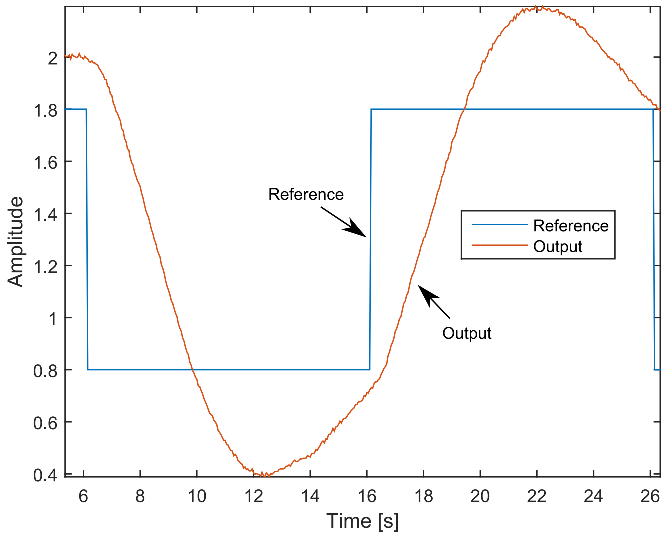

Fig. 9. PID with $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

In Fig. 10 is presented the corresponding control signal of the closed-loop of Fig. 9.

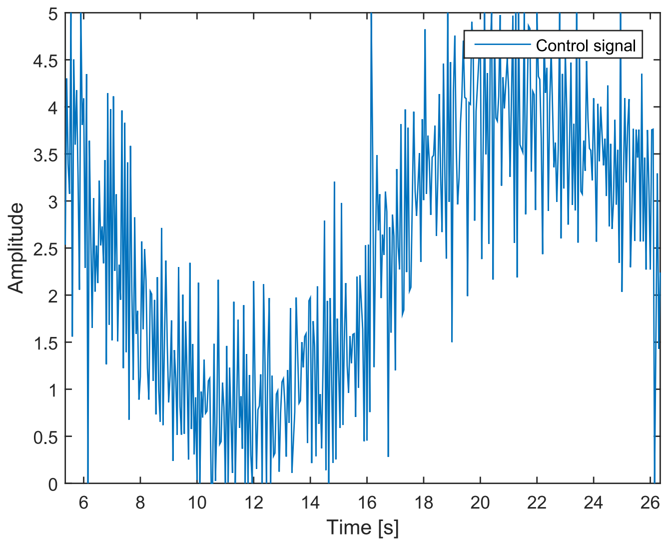

Fig. 10. Control signal of the PID with $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

By increasing the time before a reference change, it can be seen that the closed loop response of Fig. 9 achieves steady state, as presented in Fig. 11.

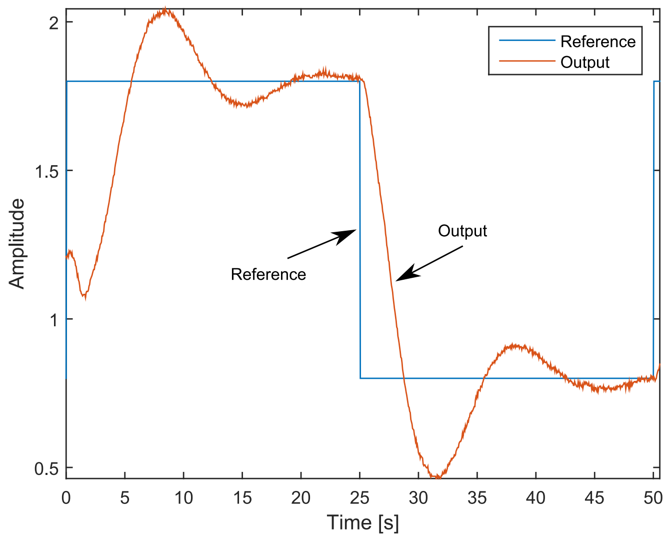

Fig. 11. PID with $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

In Fig. 12 is presented the corresponding control signal of the closed-loop of Fig. 11.

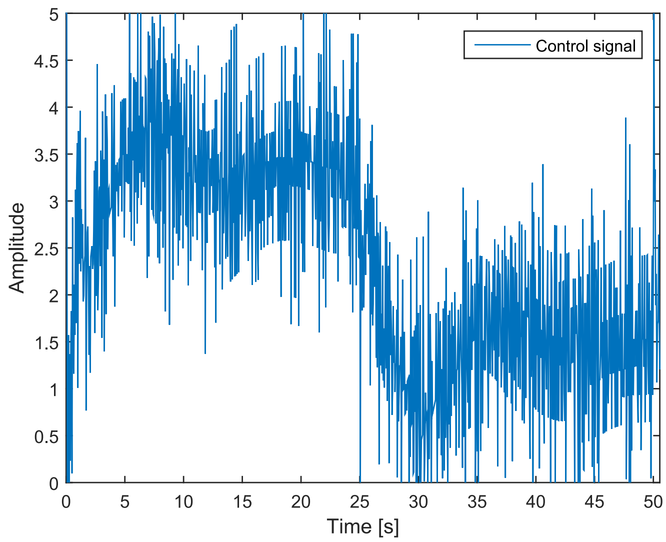

Fig. 12. Control signal of the PID with $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

The real system is evaluated for a time varying reference signal by using a sinusoidal signal. In Fig. 13 is presented the closed-loop response of the $PI^{\lambda}D^{\mu}$ with a sinusoidal reference signal.

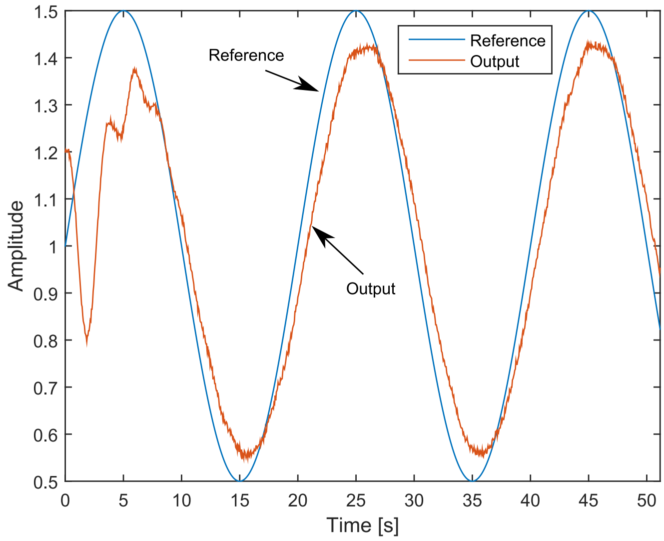

Fig. 13. $PI^{\lambda}D^{\mu}$ with $\mu = 0.7$ and $\lambda = 0.9$, $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

In Fig. 14 is presented the corresponding control signal of the closed-loop of Fig. 13.

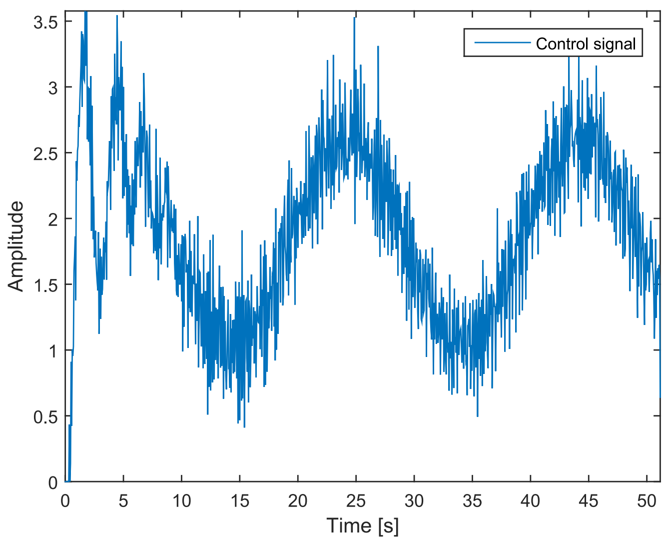

Fig. 14. Control signal of the $PI^{\lambda}D^{\mu}$ with $\mu = 0.7$ and $\lambda = 0.9$, $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

In Fig. 15 is presented the closed-loop response of the PID with a sinusoidal reference signal.

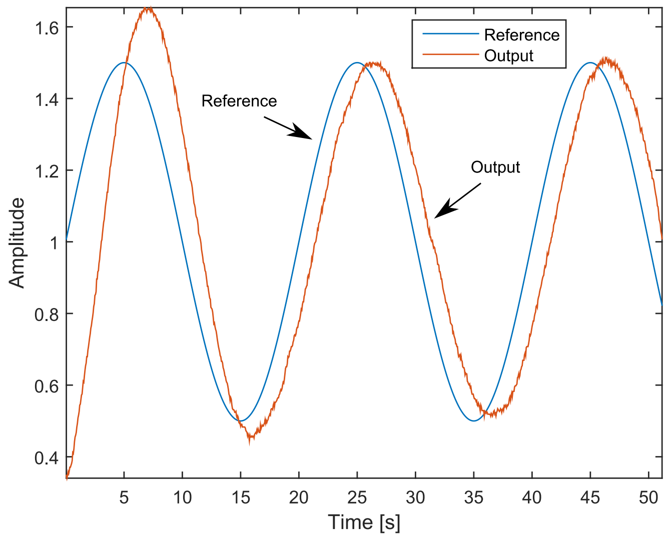

Fig. 15. PID with $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

In Fig. 16 is presented the corresponding control signal of the closed-loop of Fig. 15.

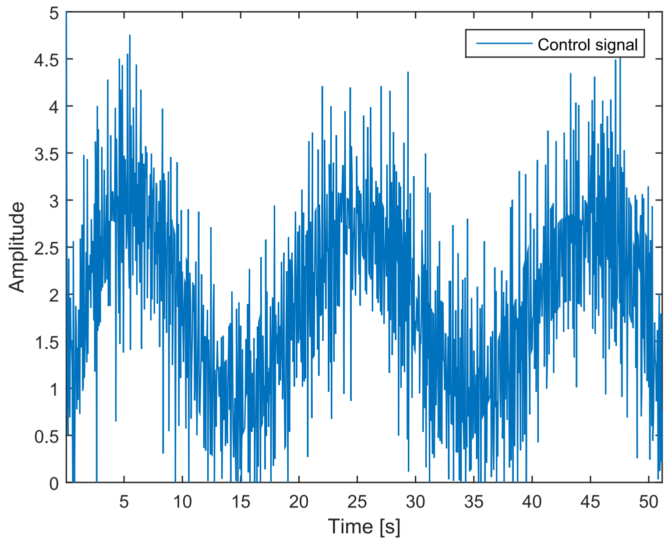

Fig. 16. Control signal of the PID with $K_p = 0.8$, $K_d = 5.9$, $K_i = 1.8$

IV. Conclusions

In this work, a novel implementation in discrete-time of a fractional-order PID control is presented. The system is evaluated over a real third-order single input single output system where the proposed implementation of the fractional-order PID is compared with a classical integer-order PID. It can be seen that the closed-loop response is more efficient by using the same parameters in terms of steady-state error and settling-time for constant reference signals and for time-varying reference signals. Also, it can be seen that a reduction in high-frequency noise is diminished by decreasing the μ value of the differential operator. As future work, a multivariable adaptive fractional-order $PI^{\lambda}D^{\mu}$ will be developed where the automatic tuning of parameters can be obtained.

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