On Some Semigroups Characterized in Terms of Bipolar Fuzzy Weakly Interior Ideals

Thiti Gaketem, Pannawit Khamrot

Abstract—**In this article, we shall give the concepts of bipolar fuzzy weakly interior ideals of semigroups, and we provide some interesting properties of bipolar fuzzy weakly interior ideals of semigroups. The relationship between bipolar fuzzy weakly interior ideals and bipolar fuzzy left (right) ideals, and the relationship between bipolar fuzzy weakly interior ideals and bipolar fuzzy interior ideals are also discussed. In the results, we proceed to characterize some semigroups by using bipolar fuzzy weakly interior ideals. Finally, we discuss the image and pre-image of bipolar fuzzy weakly interior ideals of semigroups.**

Index Terms—**Bipolar fuzzy set, bipolar fuzzy weakly interior ideal, regular semigroup, weakly regular semigroup**

I. Introduction

IN REAL life, the bipolar is theory distinguishes between positive and negative information responses. Positive information representations are compiled to be possible, while negative information representations are impossible [1]. The bipolar information of evaluation can help to evaluate decisions. Sometimes, decisions are not only influenced by the positive decision criterion but also by the negative decision criterion, for example, environmental and social impact assessments. Evaluated alternative considerations should weigh the negative effects to select the optimal choice. Therefore, bipolar information affects the effectiveness and efficiency of decision making. It is used in decision-making problems, organization problems, economic problems, and evaluation, risk management, environmental and social impact assessments. Thus, the concept of bipolar fuzzy sets is more relevant in mathematics [2], [3], [4], [5], [6]. The classical notion of fuzzy sets was innovated by Zadeh in 1965 [7], and many scientists used fuzzy sets in different fields of science. The use of fuzzy sets in algebraic structures was carried out in 1971 by Rosenfeld [8]. He defined fuzzy subgroups and discussed their important properties. The idea of fuzzy ideals and fuzzy bi-ideals in semigroups was given in 1979 by Kuroki [9]. In 1994 Zhang [10] introduced the notion of bipolar fuzzy sets with the extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$, and used them for modeling and decision analysis. In 2000, Lee [11] used the term bipolar valued fuzzy sets and applied it to algebraic structures. In 2017, Kavikumar et al. [12] discussed acknowledgments of bipolar fuzzy finite switchboard state machines. Kuanyun Zhu et al. [13], [14] purposed lattices of (generalized) fuzzy ideals in residuated lattices and a study on Z-Soft fuzzy rough sets in BCI-Algebras.

In this article, we give the concepts of bipolar fuzzy weakly interior ideals of semigroups, and we provide some interesting properties of bipolar fuzzy weakly interior ideals of semigroups. On top of that, we study relationships between bipolar fuzzy weakly interior ideals and bipolar fuzzy left (right) ideals and relationships between bipolar fuzzy weakly interior ideals and bipolar fuzzy interior ideals. In the results, we proceed to characterize some semigroups by using bipolar fuzzy weakly interior ideals. Finally, we discuss the image and pre-image of bipolar fuzzy weakly interior ideals of semigroups.

II. Preliminaries

In this topic, we give some basic definitions which will be helpful in next topic.

By a **subsemigroup** of a semigroup S we mean a non-empty subset I of S such that $I^2 \subseteq I$. A non-empty subset I of a semigroup S is called a **left (right) ideal** of S if $SI \subseteq I$ ($IS \subseteq I$). By an **ideal** of a semigroup S we mean a left ideal and a right ideal of S. A **generalized bi-ideal** of a semigroup S is a non-empty subset I of S if $ISI \subseteq I$. A subsemigroup I of a semigroup S is called a **bi-ideal** of S if $ISI \subseteq I$. A subsemigroup I of a semigroup S is called an **interior ideal** of S if $SIS \subseteq I$. A semigroup I of a semigroup S is called a **weakly interior ideal** of S if $SIS \subseteq I$.

A semigroup S is said to be **regular** if for each $k \in S$, there exists $x \in S$ such that $k = kxk$. A semigroup S is called **left (right) regular** if for each $k \in S$, there exists $a \in S$ such that $k = ak^2$ ($k = k^2a$). A semigroup S is said to be **intra-regular** if for each $k \in S$, there exist $a, b \in S$ such that $k = ak^2b$. A semigroup S is called **semisimple** if every ideal of S is an idempotent. It is evident that S is semisimple if and only if $k \in (SkS)(SkS)$ for every $k \in S$, that is there exist $w, y, z \in S$ such that $k = wkykz$. A semigroup S is called **weakly regular** if for every $k \in S, k \in (kS)^2$, that is there exist $x, y \in S$ such that $k = kxky$. A semigroup S called **quasi-regular** if for each $k \in S$, there exist $x, y \in S$ such that $k = xkyk$ ($k = kxky$). We know that every quasi-regular is weakly regular.

For any $m_i \in [0, 1]$, $i \in \mathcal{A}$, define

$$\bigvee_{i\in\mathcal{A}} m_i := \sup_{i\in\mathcal{A}}\{m_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{A}} m_i := \inf_{i\in\mathcal{A}}\{m_i\}.$$

We see that for any $m, n \in [0, 1]$, we have

$$m \vee n = \max\{m, n\} \quad \text{and} \quad m \wedge n = \min\{m, n\}.$$

A **fuzzy set (fuzzy subset)** of a non-empty set S is a function $f : S \to [0, 1]$.

Manuscript received Jun 10, 2020 ; revised Dec 14, 2020.

T. Gaketem is a lecturer at the School of Science, University of Phayao, Phayao, Thailand. (e-mail: Newtonisaac41@yahoo.com).

P. Khamrot is a lecturer at the Department of Mathematics, Faculty of Science and Agricultural Technology, Ralamangala University of Technology Lanna of Phitsanulok, Phitsanulok, Thailand. (corresponding author to provide: pk_g@rmutl.ac.th).

Definition 2.1. *Let S be a non-empty set. A **bipolar fuzzy set** (BF set) f on S is an object having the form*

$$f := \{(u, f^p(u), f^n(u)) \mid u \in S\},$$

where $f^p : S \to [0,1]$ and $f^n : S \to [-1,0]$.

Remark 2.2. *For the sake of simplicity we shall use the symbol $f = (S; f^p, f^n)$ for the BF set*
$f = \{(u, f^p(u), f^n(u)) \mid u \in S\}$.

The following example of a BF set.

Example 2.3. *Let $S = \{21, 22, 23...\}$. Define $f^p : S \to [0,1]$ is a function*

$$f^p(u) = \begin{cases} 0 & \text{if } u \text{ is old number} \\ 1 & \text{if } u \text{ is even number} \end{cases}$$

and $f^n : S \to [-1,0]$ is a function

$$f^n(u) = \begin{cases} -1 & \text{if } u \text{ is old number} \\ 0 & \text{if } u \text{ is even number.} \end{cases}$$

Then $f = (S; f^p, f^n)$ is a BF set.

For $u \in S$, define $F_u = \{(y,z) \in S \times S \mid u = yz\}$.

Define products $f^p \circ g^p$ and $f^n \circ g^n$ as follows: For $u \in S$

$$(f^p \circ g^p)(u) = \begin{cases} \bigvee_{(y,z)\in F_u} \{f^p(y) \wedge g^p(z)\} & \text{if } u = yz \\ 0 & \text{if otherwise.} \end{cases}$$

and

$$(f^n \circ g^n)(u) = \begin{cases} \bigwedge_{(y,z)\in F_u} \{f^n(y) \vee g^n(z)\} & \text{if } u = yz \\ 0 & \text{if otherwise.} \end{cases}$$

Definition 2.4. *Let I be a non-empty set of a semigroup S. A **positive characteristic function and a negative characteristic function** are respectively defined by*

$$\lambda_I^p : S \to [0,1], u \mapsto \lambda_I^p(u) := \begin{cases} 1 & u \in I, \\ 0 & u \notin I, \end{cases}$$

and

$$\lambda_I^n : S \to [0,1], u \mapsto \lambda_I^n(u) := \begin{cases} -1 & u \in I, \\ 0 & u \notin I. \end{cases}$$

Remark 2.5. *For the sake of simplicity we shall use the symbol $\lambda_I = (S; \lambda_I^p, \lambda_I^n)$ for the BF set*
$\lambda_I := \{(u, \lambda_I^p(u), \lambda_I^n(u)) \mid u \in I\}$.

Definition 2.6. *[15] A BF set $f = (S; f^p, f^n)$ on a semigroup S is called a **BF subsemigroup** on S if it satisfies the following conditions:*

(1) $f^p(uv) \geq f^p(u) \wedge f^p(v)$,
(2) $f^n(uv) \leq f^n(u) \vee f^n(v)$

for all $u, v \in S$.

The following example of a BF subsemigroup.

Example 2.7. *Let S be a semigroup defined by the following table:*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
a	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

Define a BF set $f = (S; f^p, f^n)$ on S as follows :

S	a	b	c	d	e
f^p	0.9	0.8	0.5	0.3	0.3
f^n	-0.8	-0.8	-0.6	-0.5	-0.3

Then $f = (S; f^p, f^n)$ is a BF subsemigroup.

Definition 2.8. *[15] A BF set $f = (S; f_p, f_n)$ on a semigroup S is called a **BF left (right) ideal** on S if it satisfies the following conditions:*

(1) $f^p(uv) \geq f^p(v)$ ($f^p(uv) \geq f^p(u)$),
(2) $f^n(uv) \leq f^n(v)$ ($f^n(uv) \leq f^n(u)$)

for all $u, v \in S$.

Definition 2.9. *[15] A BF set $f = (S; f_p, f_n)$ on a semigroup S is called a **BF generalized bi-ideal** on S if it satisfies the following conditions:*

(1) $f^p(uvw) \geq f^p(u) \wedge f^p(w)$,
(2) $f^n(uvw) \leq f^n(u) \vee f^n(w)$

for all $u, v, w \in S$.

Definition 2.10. *[15] A BF subsemigroup $f = (S; f^p, f^n)$ on a semigroup S is called a **BF bi-ideal** on S if it satisfies the following conditions:*

(1) $f^p(uvw) \geq f^p(u) \wedge f^p(w)$,
(2) $f^n(uvw) \leq f^n(u) \vee f^n(w)$

for all $u, v, w \in S$.

Definition 2.11. *[15] A BF subsemigroup $f = (S; f^p, f^n)$ on a semigroup S is called a **BF interior ideal** on S if it satisfies the following conditions:*

(1) $f^p(uav) \geq f^p(v)$,
(2) $f^n(uav) \leq f^n(v)$

for all $u, v, a \in S$.

Remark 2.12. (1) *Every BF left (right) ideal of a semigroup S is a BF bi-ideal of S.*
(2) *Every BF left (right) ideal of a semigroup S is a BF generalized bi-ideal of S.*
(3) *Every BF left (right) ideal of a semigroup S is a BF interior ideal of S.*
(4) *Every BF bi-ideal of a semigroup S is a BF generalized bi-ideal of S.*

III. Bipolar fuzzy weakly interior ideals on semigroups

In this topic, we shall give concepts of bipolar fuzzy weakly interior ideals and discuses important properties it.

Definition 3.1. *A BF subset $f = (S; f^p, f^n)$ of a semigroup S is said to be a **BF weakly interior ideal** of S if*

(1) $f^p(uav) \geq f^p(a)$,
(2) $f^n(uav) \leq f^n(a)$

for all $u, v, a \in S$.

Remark 3.2. *Every BF interior ideal of a semigroup S is a BF weakly interior ideal of S. But the converse is not true in Example 3.3*

Example 3.3. *Let S be a semigroup defined by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	b	a
d	a	a	b	b

Define a BF set $f = (S; f^p, f^n)$ on S as follows : Then

S	a	b	c	d
f^p	0.5	0.7	0.2	0.9
f^n	-0.6	-0.4	-0.3	-0.3

$f = (S; f^p, f^n)$ is a BF weakly interior ideal on S. But $f = (S; f^p, f^n)$ is not a BF interior ideal on S, since $f^p(dd) = f^p(b) = 0.7 \not\geq 0.9 = f^p(d) \wedge f^p(d)$.

The following theorem we will present that BF weakly interior ideals and BF interior ideals coincide for some types of semigroups.

Theorem 3.4. *In left (right) regular, regular, intra-regular, semisimple or weakly regular semigroup S, the BF weakly interior ideals and BF interior ideals coincide.*

Proof: Suppose that $f = (S; f^p, f^n)$ is a BF weakly interior ideal of S and let $u, v \in S$. Since S is left regular, there exists $x \in S$ such that $u = xu^2$. Thus,
$f^p(uv) = f^p((xu)uv) \geq f^p(u) \geq f^p(u) \wedge f^p(v)$ and $f^n(uv) = f^n((ux)uv) \leq f^n(u) \leq f^n(u) \vee f^n(v)$.
Hence $f = (S; f^p, f^n)$ is a BF subsemigroup of S.
By Definition 2.11 we have $f = (S; f^p, f^n)$ is a BF interior ideal of S. Similarly, we can prove the other cases also. ■

The following theorem we will study relationship of BF ideal and BF weakly interior ideals, which readers can easily prove.

Theorem 3.5. *Every BF left (right) ideal of a semigroup S is a BF weakly interior ideal of S.*

The following theorem show that the BF weakly interior ideals and BF ideals coincide for some types of semigroups.

Theorem 3.6. *In regular, left (right) regular, intra-regular, semisimple or weakly regular, quasi-regular semigroup, the BF weakly interior ideals and the BF ideals coincide.*

Proof: Suppose that $f = (S; f^p, f^n)$ is a BF weakly interior ideal of a regular semigroup S and let $u, v \in S$.
Since S is regular, there exists $x \in S$ such that $u = uxu$.
Thus, $f^p(uv) = f^p((uxu)v) = f^p((ux)uv) \geq f^p(u)$ and $f^n(uv) = f^n((uxu)v) = f^n((ux)uv) \leq f^n(u)$.
Hence f is a BF right ideal of S. Similarly, we can prove that $f = (S; f^p, f^n)$ is a BF left ideal of S.
Thus $f = (S; f^p, f^n)$ is a BF ideal of S. Similarly, we can prove the other cases also. ■

The following theorem show that the intersection of two BF weakly interior ideals.

Theorem 3.7. *The intersection of two BF weakly interior ideals of a semigorup S is a BF weakly interior ideal of S.*

Proof: Let f and g be BF weakly interior ideals of a semigorup S and let $u, a, v \in S$. Then

$$\begin{aligned} (f^p \cap g^p)(uav) &= f^p(uav) \wedge g^p(uav) \\ &\geq f^p(a) \wedge g^p(a) \\ &= (f^p \cap g^p)(a). \end{aligned}$$

Thus $(f^p \cap g^p)(uav) \geq (f^p \cap g^p)(a)$. Similarly, we can show that $(f^n \cap g^n)(uav) \leq (f^n \cap g^n)(a)$. Hence $f \cap g$ is a BF weakly interior ideal of S. ■

Corollary 3.8. *The intersection of $\{B_i \mid i \in \mathcal{A}\}$ BF weakly interior ideals of a semigroup S is a BF weakly interior ideal of S.*

The following theorem is an important property for an equivalent of a BF interior ideal of a semigroup.

Theorem 3.9. *A BF subset $f = (S; f^p, f^n)$ is a BF weakly interior ideal of a semigroup S if and only if $\lambda^p_S \circ f^p \circ \lambda^p_S \leq f^p$ and $\lambda^n_S \circ f^n \circ \lambda^n_S \geq f^n$.*

Proof: ($\Rightarrow$) Assume that $f = (S; f^p, f^n)$ is a BF weakly interior ideal of S and let $u \in S$.
If $F_u = \emptyset$, then it is easy to verify that,

$$(\lambda^p_S \circ f^p \circ \lambda^p)(u) \leq f^p(u).$$

If $F_u \neq \emptyset$ then

$$\begin{aligned} &(\lambda^p_S \circ f^p \circ \lambda^p_S)(u) = \bigvee_{(i,y)\in F_u} \{(\lambda^p_S \circ f^p)(i) \wedge \lambda^p_S(y)\} \\ &= \bigvee_{(i,y)\in F_u} \{(\bigvee_{(x,a)\in F_i} (\{\lambda^p_S(x) \wedge f^p(a)\})) \wedge \lambda^p_S(y)\} \\ &= \bigvee_{(i,y)\in F_u} \{(\bigvee_{(x,a)\in F_i} (\{1 \wedge f^p(a)\})) \wedge 1\} \\ &= \bigvee_{(i,y)\in F_u} \{(\bigvee_{(x,a)\in F_i} \{f^p(a)\})\} \\ &\leq \bigvee_{(i,y)\in F_u} \{(\bigvee_{(x,a)\in F_i} \{f^p(xay)\})\} \\ &= f^p(u). \end{aligned}$$

Thus, $(\lambda^p_S \circ f^p \circ \lambda^p_S)(u) \leq f^p(u)$. Hence, $\lambda^p_S \circ f^p \circ \lambda^p_S \leq f^p$. Similarly, we can show that $\lambda^n_S \circ f^n \circ \lambda^n_S \geq f^n$.

($\Leftarrow$) Conversely, let $a, u, v \in S$. Since $\lambda^p_S \circ f^p \circ \lambda^p_S \leq f^p$ we have $(\lambda^p_S \circ f^p \circ \lambda^p_S)(uav) \leq f^p(uav)$. Thus

$$\begin{aligned} &f^p(uav) \geq (\lambda^p_S \circ f^p \circ \lambda^p_S)(uav) \\ &= \bigvee_{(i,j)\in F_{uav}} \{(\lambda^p_S \circ f^p)(i) \wedge \lambda^p_S(j)\} \\ &= \bigvee_{(i,j)\in F_{uav}} \{(\bigvee_{(p,q)\in F_i} (\{\lambda^p_S(p) \wedge f^p(q)\}) \wedge \lambda^p_S(j))\} \\ &= \bigvee_{(i,j)\in F_{uav}} \{(\bigvee_{(p,q)\in F_i} (\{1 \wedge f^p(q)\}) \wedge 1)\} \\ &= \bigvee_{(i,j)\in F_{uav}} \{(\bigvee_{(p,q)\in F_i} f^p(q)\})\} \geq f^p(a). \end{aligned}$$

Hence, $f^p(uav) \geq f^p(a)$. Similarly, we can show that $f^n(uav) \leq f^n(a)$. Therefore f is a BF weakly interior ideal of S. ■

The following theorem is an important property for an equivalent of a BF weakly interior ideal of a regular, left (right) regular, intra-regular semigroup.

Theorem 3.10. *For a regular or a left (right) regular or an intra-regular semigroup S, f is a BF weakly interior ideal if and only if $\lambda^p_S \circ f^p \circ \lambda^p_S = f^p$ and $\lambda^n_S \circ f^n \circ \lambda^n_S = f^n$*

Proof: Assume that f is a BF weakly interior ideal of a regular semigroup S and let $u \in S$. Then there exists $x \in S$ such that $u = uxu = (uxu)xu = u(xux)u$. Thus

$\lambda_S^p \circ f^p \circ \lambda_S^p(u) = \bigvee_{(i,j)\in F_u} \{(\lambda_S^p \circ f^p)(i) \wedge \lambda_S^p(j)\}$
$= \bigvee_{(i,j)\in F_{u(xux)u}} \{(\lambda_S^p \circ f^p)(i) \wedge \lambda_S^p(j)\}$
$\geq (\lambda_S^p \circ f^p)(u(xux))\lambda_S^p(u) = (\lambda_S^p \circ f^p)(u(xux)) \wedge 1$
$= (\lambda_S^p \circ f^p)(u(xux)) = \bigvee_{(i,j)\in F_{u(xux)}} \{\lambda_S^p(i) \wedge f^p(j)\}$
$\geq (\lambda_S^p(u) \wedge f^p(xux))$
$= (1 \wedge f^p(xux))$
$= f^p(xux) \geq f^p(u)$.

Hence, $\lambda_S^p \circ f^p \circ \lambda_S^p(u) \geq f^p(u)$.
Therefore, $f^p \leq \lambda_S^p \circ f^p \circ \lambda_S^p$.
By Theorem 3.9 we have $\lambda_S^p \circ f^p \circ \lambda_S^p \leq f^p$.
Thus, $\lambda_S^p \circ f^p \circ \lambda_S^p = f^p$. Similarly, we can prove that $\lambda_S^n \circ f^n \circ \lambda_S^n = f^n$.

For the converse, it follows from Theorem 3.9.

Similarly we can prove the other cases also. ■

Theorem 3.11. *Let S be a semigroup. Then for every $k \in S$, $\lambda_{k\cup SkS} = (S; \lambda^p_{k\cup SkS}, \lambda^n_{k\cup SkS})$ is a BF weakly interior ideals of S.*

Proof: Let $u, a, v \in S$. If $a \in k\cup SkS$, then $uav \in SkS$. Thus $\lambda^p_{k\cup SkS}(uav) = 1 \geq \lambda^p_{k\cup SkS}(a)$ and $\lambda^n_{k\cup SkS}(uav) = -1 \leq \lambda^n_{k\cup SkS}(a)$.

On the other hand, if $a \notin (k \cup SkS)$, then we have $\lambda^n_{(k\cup SkS)}(uav) \geq \lambda^n_{(k\cup SkS)}(a)$ and $\lambda^n_{(k\cup SkS)}(uav) \leq \lambda^n_{(a\cup SaS)}$.
Hence $\lambda_{k\cup SkS}$ is a BF weakly interior ideals of S. ■

Theorem 3.12. *Let S be a semigroup. Then for every $k \in S$, $\lambda_{k\cup kSk} = (S; \lambda^p_{k\cup kSk}, \lambda^n_{k\cup kSk})$ is a BF generalized bi-ideals of S.*

Proof: Let $u, v, w \in S$. If $u, w \in (k \cup kSk)$, then $uvw \in kSk$. Thus
$\lambda^p_{(k\cup kSk)}(uvw) = 1 \geq \lambda^p_{(k\cup kSk)}(u) \wedge \lambda^n_{(k\cup kSk)}(w)$ and
$\lambda^n_{(k\cup kSk)}(uvw) = -1 \leq \lambda^n_{(k\cup kSk)}(u) \wedge \lambda^n_{(k\cup kSk)}(w)$.

On the other hand, if $u \notin (k \cup SkS)$ or $w \notin (k \cup SkS)$, then we have $\lambda^n_{(k\cup kSk)}(uvw) \geq \lambda^n_{(k\cup kSk)}(u) \wedge \lambda^n_{(k\cup kSk)}(w)$ and $\lambda^n_{(k\cup kSk)}(uvw) \leq \lambda^n_{(k\cup kSk)}(u) \wedge \lambda^n_{(k\cup kSk)}(w)$.
Hence $\lambda_{(k\cup kSk)}$ is a BF generalized bi-ideals of S. ■

In the following theorem, we give a relationship between a weakly interior ideal and the bipolar characteristic function.

Theorem 3.13. *Let I be a non-empty subset of a semigroup S. Then I is a weakly interior ideal of S if and only if $\lambda_I = (S; \lambda_I^p, \lambda_I^n)$ is a BF weak interior ideal of S.*

Proof: Let $a, u, v \in S$ and I is a weakly interior ideal of S. By Theorem 3.11(1) we have $\lambda_I = (S; \lambda_I^p, \lambda_I^n)$ is a BF weakly interior ideal of S.

Conversely let $a, u, v \in S$ and $a \in I$.
Since $\lambda_I = (S; \lambda_K^p, \lambda_I^n)$ is a BF weakly interior ideal of S we have $\lambda_I^p(uav) \geq \lambda_I^p(a) = 1$ and $\lambda_I^n(uav) \leq \lambda_I^n(a) = -1$.
Thus $uav \in I$. Hence I is a weakly interior ideal of S. ■

This following definition is of an (s,t)-level subset of a bipolar fuzzy set.

Definition 3.14. *Let $f = (S; f^p, f^n)$ be a BF set and $(s,t) \in [-1,0] \times [0,1]$. Define the set $X_f^{(t,s)} = \{u \in X \mid f^p(x) \geq t, f^n(x) \leq s\}$ is called an (s,t)-**level subset** of a bipolar fuzzy set of f.*

In the following theorem, we give a relationship between a weakly interior ideal and the (s,t)-level subset of a BF set.

Theorem 3.15. *A BF set f is a weakly interior ideal of a semigroup if and only if the level set $X_f^{(t,s)}$ is a weakly interior ideal of S for all $(s,t) \in [-1,0] \times [0,1]$.*

Proof: Let f be a BF weakly interior ideal of S and let $u, a, v \in S, (s,t) \in [-1,0] \times [0,1], u, a, v \in X_f^{(t,s)}$. Then $f^p(u) \geq t, f^p(a) \geq t$ and $f^p(v) \geq t$, also
$f^n(u) \leq s, f^n(a) \leq s$ and $f^n(v) \leq s$. Thus $f^n(uav) \geq t$ and $f^p(uav) \leq s$ implies $uav \in X_f^{(t,s)}$.
Hence $X_f^{(t,s)}$ is a weakly interior ideal of S.

Conversely, suppose that $X_f^{(t,s)}$ is a weakly interior ideal of S. If f is not a BF weakly interior ideal of S, then there exists $u, a, v \in S$ such that $f^p(uav) < f^p(a)$ or $f^n(uav) < f^n(a)$. By assumption, we have $uav \in X_f^{(t,s)}$. Thus $f^p(uav) \geq f^p(a)$ and $f^n(uav) \leq f^n(a)$. It is a contradiction. Hence f is a BF weakly interior ideal of S. ■

This following definition is of a bipolar (α,β)-cut set.

Definition 3.16. *[15] For a BF set $f = (S; f^p, f^n)$ and $(\alpha,\beta) \in [-1,0) \times (0,1]$ we define*

$$P(f;\beta) := x \in S \mid f^p(u) \geq \beta,$$

$$N(f;\alpha) := x \in S \mid f^n(u) \leq \alpha$$

*which are called the **positive** $\beta - cut$ of $f = (S; f^p, f^n)$ and the **negative** α**-cut** of $f = (S; f^p, f^n)$ respectively. The set $C(f;(\alpha,\beta)) := P(f;\beta) \cap N(f;\alpha)$ is called the **bipolar** (α,β)**-cut set** of $f = (S; f^p, f^n)$.*

In the following theorem, we give a relationship between a weakly interior ideal and the bipolar (α,β)-cut set.

Theorem 3.17. *For a BF set $f = (S; f^p, f^n)$ on semigroup S and $(\alpha,\beta) \in [-1,0) \times (0,1]$, $f = (S; f^p, f^n)$ is a BF weakly interior ideal of S if and only if every bipolar (α,β)-cut set of $f = (S; f^p, f^n)$ is a weakly interior ideal of S when it is non-empty set.*

Proof: Suppose that $f = (S; f^p, f^n)$ is a BF weakly interior ideal of S and $(\alpha,\beta) \in [-1,0) \times (0,1]$ such that $C(f;(\alpha,\beta)) \neq \emptyset$. Let $u, v \in C(f;(\alpha,\beta))$ and $a \in S$. Then $f^p(u) \geq \beta$, $f^p(v) \geq \beta$, $f^n(u) \leq \alpha$ and $f^n(v) \leq \alpha$. Thus

$$f^p(uav) \geq f^p(a) \geq \beta$$

and

$$f^n(uav) \leq f^n(a) \leq \alpha.$$

Hence $uav \in C(f;(\alpha,\beta))$. Therefore $C(f;(\alpha,\beta))$ is a weakly interior ideal of S.

Conversely, suppose that the bipolar (α,β)-cut set of $f = (S; f^p, f^n)$ is a weakly interior ideal of S.
If $f = (S; f^p, f^n)$ is not a BF weakly interior ideal of S, then there exist $a, u, v \in S$ such that $f^p(uav) > f^p(a)$ or $f^n(uav) < f^n(a)$. Let $\beta = \frac{f^p(uav)+f^p(a)}{2}$. Then $f^p(uav) < \beta < f^p(a)$. Thus $a \in C(f; f^p(a), \beta)$ but $uav \notin C(f; f^p(a), \beta)$. Similarly we take $\alpha = \frac{f^n(uav)+f^n(a)}{2}$ we have $a \in C(f; f^n(a), \alpha)$ but $uav \notin C(f; f^n(a), \alpha)$.
Thus $C(f;(\alpha,\beta))$ is not a weakly interior ideal of S.
It is a contradiction. Hence $f = (S; f^p, f^n)$ is a BF weakly interior ideal of S. ■

IV. Characterizing some semigroups in terms of BF weakly interior ideals.

In this topic, we will characterize some semigroups in terms of BF weakly interior ideals.

The following lemma is basic properties of a positive characteristic function and a negative characteristic function.

Lemma 4.1. *Let I and L be non-empty subsets of a semigroup S. Then the following statements hold.*
(1) $\lambda_I^p \wedge \lambda_L^p = (\lambda_{I\cap L})^p$.
(2) $\lambda_I^n \vee \lambda_L^n = (\lambda_{I\cup L})^n$.
(3) $\lambda_I^p \circ \lambda_L^p = (\lambda_{IL})^p$.
(4) $\lambda_I^n \circ \lambda_L^n = (\lambda_{IL})^n$.

Proof: It is straightforward. ■

Theorem 4.2. *Let S be a semigroup. Then the following are equivalents:*
(1) *S is regular.*
(2) *$f^p \wedge g^p \wedge h^p \leq f^p \circ g^p \circ h^p$ and $f^n \vee g^n \vee h^n \geq f^n \circ g^n \circ h^n$, for every BF right ideal f, for every BF weakly interior ideal g and for every BF left ideal h of S.*
(3) *$f^p \wedge g^p \wedge h^p \leq f^p \circ g^p \circ h^p$ and $f^n \vee g^n \vee h^n \geq f^n \circ g^n \circ h^n$, for every BF right ideal f, for every BF weakly interior ideal g and for every BF generalized bi-ideal h of S.*
(4) *$f^p \wedge g^p \wedge h^p \leq f^p \circ g^p \circ h^p$ and $f^n \vee g^n \vee h^n \geq f^n \circ g^n \circ h^n$, for every BF weakly interior ideal g and for every BF generalized bi-ideal f and h of S.*
(5) *$\lambda^p_{(k)_r} \wedge \lambda^p_{k\cup SkS} \wedge \lambda^p_{(k)_l} \leq \lambda^p_{(k)_r} \circ \lambda^p_{k\cup SkS} \circ \lambda^p_{(k)_l}$ and $\lambda^n_{(k)_r} \vee \lambda^n_{k\cup SkS} \vee \lambda^n_{(k)_l} \geq \lambda^n_{(k)_r} \circ \lambda^n_{k\cup SkS} \circ \lambda^n_{(k)_l}$, for all $k \in S$, where $(k)_r$ is the right ideal generated by k and $(k)_l$ is the left ideal generated by k.*

Proof: (1) $\Rightarrow$ (4) Suppose that f, h are BF generalized bi-ideals and g is a BF weakly interior ideal of S and let $k \in S$. Since S is regular, there exists $x \in S$ such that $k = kxk = (kxk)xk = kxkxk = (kxk)(xkx)(kxk)$. Thus

$$\begin{aligned}
(f^p \circ g^p \circ h^p)(k) &= (\bigvee_{(i,j)\in F_k} \{f^p(i) \wedge (g^p \circ h^p)(j)\}\\
&= (\bigvee_{(i,j)\in F_{(kxk)(xkx)(kxk)}} \{f^p(i) \wedge (g^p \circ h^p)(j)\}\\
&\geq f^p(kxk) \wedge (g^p \circ h^p)(xkx)(kxk)\\
&= f^p(uxu) \wedge (\bigvee_{(a,b)\in F_{(xkx)(kxk)}} \{g^p(a) \wedge h^p(b)\})\\
&\geq f^p(kxk) \wedge (g^p((xkx)) \wedge h^p(kxk))\\
&\geq f^p(k) \wedge f^p(k) \wedge (g^p(k) \wedge h^p(k) \wedge h^p(k))\\
&= f^p(k) \wedge (g^p(k) \wedge h^p(k))\\
&= f^p(k) \wedge (g^p \wedge h^p)(k)\\
&= (f^p \wedge g^p \wedge h^p)(k).
\end{aligned}$$

Hence, $(f^p \circ g^p \circ h^p)(k) \geq (f^p \wedge g^p \wedge h^p)(k)$.
Therefore $f^p \wedge g^p \wedge h^p \leq f^p \circ g^p \circ h^p$.
Similarly, we can show that $f^n \vee g^n \vee h^n \geq f^n \circ g^n \circ h^n$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) This is obvious because every BF right ideal is a BF bi-ideal of S and every BF right ideal is a BF generalized bi-ideal of S.

(2) $\Rightarrow$ (5) By Theorem 3.11 we have $\lambda_{k\cup SkS} = (S; \lambda^p_{k\cup SkS}, \lambda^n_{k\cup SkS})$ is a BF weakly interior ideal of S. Then for all $k \in S$,

$$(\lambda^p_{(k)_r} \wedge \lambda^p_{k\cup SkS} \wedge \lambda^p_{(k)_l})(k) \leq (\lambda^p_{(k)_r} \circ \lambda^p_{k\cup SkS} \circ \lambda^p_{(k)_l})(k)$$

and

$$(\lambda^n_{(k)_r} \wedge \lambda^n_{k\cup SkS} \wedge \lambda^n_{(k)_l})(k) \geq (\lambda^n_{(k)_r} \circ \lambda^n_{k\cup SkS} \circ \lambda^n_{(k)_l})(k).$$

(5) $\Rightarrow$ (1) Let $k \in S$ and let $(k)_r$ is a right ideal generated by k and $(k)_l$ is a left ideal generated by k. Then by Theorem 3.11 $\lambda_{(k)_r} = (S; \lambda^p_{(k)_r}, \lambda^n_{(k)_r})$ is a BF right ideals generated by k and $\lambda_{(k)_l} = (S; \lambda^p_{(k)_l}, \lambda^n_{(k)_l})$ is a BF left ideals generated by k.
By assumption,

$$\begin{aligned}
1 &= (\lambda^p_{(k)_r} \wedge \lambda^p_{k\cup SkS} \wedge \lambda^p_{(k)_l})(k)\\
&\leq (\lambda^p_{(k)_r} \circ \lambda^p_{k\cup SkS} \circ \lambda^p_{(k)_l})(k) = 1.
\end{aligned}$$

and

$$\begin{aligned}
-1 &= (\lambda^n_{(k)_r} \vee \lambda^n_{k\cup SkS} \vee \lambda^n_{(k)_l})(k)\\
&\geq (\lambda^n_{(k)_r} \circ \lambda^n_{k\cup SkS} \circ \lambda^n_{(k)_l})(k) = -1.
\end{aligned}$$

Thus $k \in ((k)_r)(k \cup SkS)((k)_l)$ implies
$k \in (k \cup kS)(k \cup SkS)(k \cup Sk)$ so $k \in k^3$ or $k \in kSk$.
Hence $k = k^3$ or $k = kxk$. Therefore S is regular. ■

Theorem 4.3. *A semigroup S is semisimple if and only if $f^p \wedge g^p \leq f^p \circ g^p$ and $f^n \vee g^n \geq f^n \circ g^n$, for each BF weakly interior ideals f and g of S.*

Proof: ($\Rightarrow$) Let f and g be BF weakly interior ideals of S and let $k \in S$. Since S is semisimple, there exist $x, y, z \in S$ such that $k = xkykz = (xky)(xkykzz) = (xky)(xk(ykz^2))$. Thus

$$\begin{aligned}
(f^p \circ g^p)(k) &= \bigvee_{(i,j)\in F_k} \{f^p(i) \wedge g^p(j)\}\\
&= (\bigvee_{(i,j)\in F_{(xky)(xk(ykz^2))}} \{f^p(i) \wedge g^p(j)\}\\
&\geq f^p(xky) \wedge g^p(xk(ykz^2))\\
&= (f^p(k) \wedge g^p(k)) = (f^p \wedge g^p)(k).
\end{aligned}$$

Hence, $(f^p \wedge g^p)(k) \leq (f^p \circ g^p)(k)$.
Therefore $f^p \wedge g^p \leq f^p \circ g^p$. Similarly, we can show that $f^n \vee g^n \geq f^n \circ g^n$.

($\Leftarrow$) Let $k \in S$. Then by Theorem 3.11 $\lambda_{k\cup SkS} = (S; \lambda^p_{k\cup SkS}, \lambda^n_{k\cup SkS})$ is a BF weakly interior ideals of S. By assumption,

$$\begin{aligned}
1 &= (\lambda^p_{k\cup SkS} \wedge \lambda^p_{k\cup SkS})(k)\\
&\leq (\lambda^p_{k\cup SkS} \circ \lambda^p_{k\cup SkS})(k) = 1
\end{aligned}$$

and

$$\begin{aligned}
-1 &= (\lambda^n_{k\cup SkS} \wedge \lambda^n_{k\cup SkS})(k)\\
&\geq (\lambda^n_{k\cup SkS} \circ \lambda^n_{k\cup SkS})(k) = -1.
\end{aligned}$$

Thus $k \in (k \cup SkS)(k \cup SkS)$ so $k \in k^2$ or $k \in kSkS$ or $k \in SkSk$ Hence $k = kk = kkkk = kkkkk$ or $k = kxky = kxkxkyy = (kx)kxk(y^2)$ or $k = xkyk = xkyxkyk = xk(yk)k(yk)$ or $k = xkykz$.
Therefore S is semisimple. ■

Theorem 4.4. *A semigroup S is semisimple if and only if $f^p \wedge g^p \wedge h^p \leq f^p \circ g^p \circ h^p$ and $f^n \vee g^n \vee h^n \geq f^n \circ g^n \circ h^n$, for every right ideal g and for every BF weakly interior ideals f and h of S.*

Proof: Suppose that f, h are BF weakly interior ideals and g is a BF right ideal of S and let $k \in S$. Since S is semisimple, there exist $x, y, z \in S$ such that
$k = xkykz = (x^2ky)(kz)(ykz)$. Thus

$(f^p \circ g^p \circ h^p)(k) = (\bigvee_{(i,j)\in F_k} \{f^p(i) \wedge (g^p \circ h^p)(j)\}$
$= \bigvee_{(i,j)\in F_{(x^2ky)(kz)(ykz)}} \{f^p(i) \wedge (g^p \circ h^p)(j)\}$
$\geq f^p(x^2ky) \wedge (g^p \circ h^p)(kz)(ykz)$
$= f^p(x^2ky) \wedge (\bigvee_{(a,b)\in F_{(kz)(ykz)}} \{g^p(a) \wedge h^p(b)\})$
$\geq f^p(k) \wedge (g^p(kz) \wedge h^p(ykz))$
$\geq f^p(k) \wedge (g^p(k) \wedge h^p(k))$
$= f^p(k) \wedge (g^p \wedge h^p)(k)$
$= (f^p \wedge g^p \wedge h^p)(k)$.

Hence, $(f^p \circ g^p \circ h^p)(k) \geq (f^p \wedge g^p \wedge h^p)(k)$.
Therefore $f^p \wedge g^p \wedge h^p \leq f^p \circ g^p \circ h^p$.
Similarly, we can show that $f^n \vee g^n \vee h^n \geq f^n \circ g^n \circ h^n$.

Conversely, let $k \in S$ and let $(k)_r$ be a right ideal generated by k. Then by Theorem 3.11, $\lambda_{(k)_r} = (S; \lambda^p_{(k)_r}, \lambda^n_{(k)_r})$ is a BF right ideals generated by k. By assumption,

$$\begin{aligned} 1 &= (\lambda^p_{k\cup SkS} \wedge \lambda^p_{(k)_r} \wedge \lambda^p_{k\cup SkS})(k) \\ &\leq (\lambda^p_{k\cup SkS} \circ \lambda^p_{(k)_r} \circ \lambda^p_{k\cup SkS}(k) = 1 \end{aligned}$$

and

$$\begin{aligned} -1 &= (\lambda^n_{k\cup SkS} \vee \lambda^n_{(k)_r} \vee \lambda^n_{k\cup SkS})(u) \\ &\leq (\lambda^n_{k\cup SkS} \circ \lambda^n_{(k)_r} \circ \lambda^n_{k\cup SkS})(k) = -1 \end{aligned}$$

Thus $k \in (k \cup SkS)((k)_r)(k \cup SkS)$ implies
$k \in (k \cup SkS)(k \cup kS)(k \cup SkS)$ so $k \in k^3$ or $k \in k^2Sk$ or $k \in k^2SkS$ or $k \in SkSkS$.
Hence $k = kkk = kkkkk$ or $k = k^2xky = kkxky$ or
$k = k^2xk = kkxk = kkxkkxk = kk(xk)k(xk)$ or
$k = xkykz$. Therefore S is semisimple. ∎

Theorem 4.5. *A semigroup S is weakly regular if and only if $f^p \wedge g^p \leq f^p \circ g^p$ and $f^n \vee g^n \geq f^n \circ g^n$, for every BF generalized bi-ideal f and for every BF weakly interior ideal g of S.*

Proof: ($\Rightarrow$) Let f and g be a BF generalized bi-ideal and BF weakly interior ideal of S respectively let $k \in S$. Since S is weakly regular, there exist $x, y \in S$ such that $k = kxky = (kxk)(xky^2)$. Thus

$(f^p \circ g^p)(k) = \bigvee_{(i,j)\in F_k} \{f^p(i) \wedge g^p(j)\}$
$= \bigvee_{(i,j)\in F_{(kxk)(xky^2)}} \{f^p(i) \wedge g^p(j)\}$
$\geq f^p(kxk) \wedge g^p(xky^2) \geq (f^p(k) \wedge f^p(k) \wedge g^p(k))$
$= (f^p(k) \wedge g^p(k)) = (f^p \wedge g^p)(k)$.

Hence, $(f^p \wedge g^p)(k) \leq (f^p \circ g^p)(k)$.
Therefore $f^p \wedge g^p \leq f^p \circ g^p$. Similarly, we can show that $f^n \vee g^n \geq f^n \circ g^n$.

($\Leftarrow$) Let $k \in S$. Then by Theorem 3.11
$\lambda_{k\cup SkS} = (S; \lambda^p_{k\cup SkS}, \lambda^n_{k\cup SkS})$ is a BF weakly interior ideals and $\lambda_{k\cup kSk} = (S; \lambda^p_{k\cup kSk}, \lambda^n_{k\cup kSk})$ is a BF generalized bi-ideals of S.
By assumption,

$$\begin{aligned} 1 &= (\lambda^p_{k\cup SkS} \wedge \lambda^p_{k\cup kSk})(k) \\ &\leq (\lambda^p_{k\cup SkS} \circ \lambda^p_{k\cup kSk})(k) = 1 \end{aligned}$$

and

$$\begin{aligned} -1 &= (\lambda^n_{k\cup SkS} \wedge \lambda^n_{k\cup SkS})(k) \\ &\geq (\lambda^n_{k\cup SkS} \circ \lambda^n_{k\cup SkS})(k) = -1. \end{aligned}$$

Thus $k \in (k \cup SkS)(k \cup kSk)$ so $k \in k^2$ or $k \in kSkS$.
Hence $k = kk = kkkk$ or $k = kxky$.
Therefore S is weakly regular. ∎

Theorem 4.6. *A semigroup S is left quasi-regular if and only if $f^p \wedge g^p \leq f^p \circ g^p$ and $f^n \vee g^n \geq f^n \circ g^n$, for every BF weakly interior ideal f and every BF generalized bi-ideal g of S.*

Proof: ($\Rightarrow$) Let f and g be a BF weakly interior ideal and a BF generalized bi-ideal of S respectively and let $u \in S$. Then there exist $x, y \in S$ such that $k = xkyk$. Thus,

$(f^p \circ g^p)(k) = (\bigvee_{(i,j)\in F_u} \{f^p(i) \wedge g^p(j)\}$
$= (\bigvee_{(i,j)\in F_{xkyk}} \{f^p(i) \wedge g^p(j)\} \geq f^p(xky) \wedge g^p(k)$
$\geq f^p(k) \wedge g^p(k) = (f^p \wedge g^p)(k)$.

Hence, $(f^p \wedge g^p)(k) \leq (f^p \circ g^p)(k)$.
Therefore $f^p \wedge g^p \leq f^p \circ g^p$. Similarly, we can show that $f^n \vee g^n \geq f^n \circ g^n$.

($\Leftarrow$) Let $k \in S$. Then by Theorem 3.11 $\lambda_{k\cup SkS} = (S; \lambda^p_{k\cup SkS}, \lambda^n_{k\cup SkS})$ is a BF weakly interior ideals and $\lambda_{k\cup kSk} = (S; \lambda^p_{k\cup kSk}, \lambda^n_{k\cup kSk})$ is a BF generalized bi-ideals of S. By assumption,

$$\begin{aligned} 1 &= (\lambda^p_{k\cup SkS} \wedge \lambda^p_{k\cup kSk})(k) \\ &\leq (\lambda^p_{k\cup SkS} \circ \lambda^p_{k\cup kSk})(k) = 1 \end{aligned}$$

and

$$\begin{aligned} -1 &= (\lambda^n_{k\cup SkS} \wedge \lambda^n_{k\cup kSk})(k) \\ &\geq (\lambda^n_{k\cup SkS} \circ \lambda^n_{k\cup kSk})(k) = -1. \end{aligned}$$

Thus $k \in (k \cup SkS)(k \cup kSk)$ so $k \in k^2$ or $k \in SkSk$.
Hence $k = kk = kkkk$ or $k = xkyk$.
Therefore S is left quasi-regular. ∎

Theorem 4.7. *Let S be a semigroup. Then the following are equivalent:*

(1) *S is right quasi-regular,*
(2) *$f^p \wedge g^p \leq f^p \circ g^p$ and $f^n \vee g^n \geq f^n \circ g^n$, for every BF right ideal f and every BF weakly interior ideal g of S.*

Proof: (1) $\Rightarrow$ (2) Let f and g be a BF right ideal and a BF weakly interior ideal of S respectively and let $k \in S$. Then there exist $x, y \in S$ such that $k = kxky$. Thus,

$(f^p \circ g^p)(k) = \bigvee_{(i,j)\in F_k} \{f^p(i) \wedge g^p(j)\}$
$= \bigvee_{(i,j)\in F_{kxky}} \{f^p(i) \wedge g^p(j)\}$
$\geq f^p(k) \wedge g^p(xky) \geq f^p(k) \wedge g^p(k) = (f^p \wedge g^p)(k)$.

Hence, $(f^p \wedge g^p)(k) \leq (f^p \circ g^p)(k)$.
Therefore $f^p \wedge g^p \leq f^p \circ g^p$. Similarly, we can show that $f^n \vee g^n \geq f^n \circ g^n$.

(2) $\Rightarrow$ (1) Let $k \in S$ and let $(k)_r$ be a right ideal generated by k. Then by Theorem 3.11, $\lambda_{(k)_r} = (S; \lambda^p_{(k)_r}, \lambda^n_{(k)_r})$ is a BF right ideals generated by k. By assumption,

$$\begin{aligned} 1 &= (\lambda^p_{k\cup SkS} \wedge \lambda^p_{(k)_r} \wedge \lambda^p_{k\cup SkS})(k) \\ &\leq (\lambda^p_{k\cup SkS} \circ \lambda^p_{(k)_r} \circ \lambda^p_{k\cup SkS}(k) = 1 \end{aligned}$$

and

$$\begin{aligned} -1 &= (\lambda^n_{k\cup SkS} \vee \lambda^n_{(k)_r} \vee \lambda^n_{k\cup SkS})(u) \\ &\leq (\lambda^n_{k\cup SkS} \circ \lambda^n_{(k)_r} \circ \lambda^n_{k\cup SkS})(k) = -1 \end{aligned}$$

Thus $k \in (k \cup SkS)((k)_r)(k \cup SkS)$ implies
$k \in (k \cup SkS)(k \cup kS)(k \cup SkS)$ so $k \in k^2$ or $k \in kSkS$.
Hence$k = kk = kkkk$ or $k = kxky$.
Therefore S is right quasi-regular. ■

Theorem 4.8. *Let S be a semigroup. Then the following are equivalent:*

(1) *S is right quasi-regular,*
(2) *$f^p \wedge g^p \leq f^p \circ g^p$ and $f^n \vee g^n \geq f^n \circ g^n$, for every BF generalized bi-ideal f and every BF weakly interior ideal g of S.*

Proof: It follows form Theorem 4.5. ■

V. The image and pre-image of BF weakly interior ideals.

In this section, we introduce the notion of image and pre-image of the BF subsemigroups and discuss some of its properties.

Definition 5.1. *[16] A mapping ϕ from a semigroup S_1 to a semigroup S_2 is said to be* ***homomorphism*** *if $\phi(uv) = \phi(u)\phi(v)$ for all $u, v \in S_1$.*

Definition 5.2. *[17] Let ϕ be a mapping a semigroup S_1 to a semigroup S_2 and let $f = (f^p, f^n)$ and $g = (g^p, g^n)$ are bipolar fuzzy subsets in S_1 and S_2 respectively. Then the image $\phi(f)$ of f is the bipolar fuzzy subset $\phi(f) = (\phi(f)^p, \phi(f)^n)$ of S_2 defined by for $u \in S_2$,*

$$\phi(f)^p(u) = \begin{cases} \bigvee\limits_{x \in \phi^{-1}(u)} f^p(x) & \text{if } \phi^{-1}(u) \neq \emptyset \\ 0 & \text{if otherwise.} \end{cases}$$

and

$$\phi(f)^n(u) = \begin{cases} \bigvee\limits_{x \in \phi^{-1}(u)} f^n(x) & \text{if } \phi^{-1}(u) \neq \emptyset \\ 0 & \text{if otherwise.} \end{cases}$$

and the pre-image $\phi^{-1}(f)$ of f under f is the bipolar fuzzy subset of S_1 defined bu for $x \in S_1$, $(\phi^{-1}(f)^p)(x) = f^p(\phi(x)), (\phi^{-1}(f))^p(x) = f^p(\phi(x))$.

Theorem 5.3. *Let U and V be two semigroups and let $\phi : U \rightarrow V$ be an epimomorphism. If $\overline{f}$ and $\overline{g}$ are BF weakly interior ideal of U and V respectively, then*

(1) *$\phi(\overline{f})$ is a BF weakly interior ideal of V,*
(2) *$\phi^{-1}(\overline{g})$ is a BF weakly interior ideal of U.*

Proof: Suppose that $\overline{f}$ and $\overline{g}$ are BF weakly interior ideal of U and V respectively.

(1) Let $v_1, v_2, v_3 \in V$. Since ϕ is a surjective, there exist $u_1, u_2, u_3 \in U$ such that $\phi(u_1) = v_1, \phi(u_2) = v_2$ and $\phi(u_3) = v_3$. Thus

$$\begin{aligned} \phi(f)^p(v_1 v_2, v_3) &= \bigvee_{z \in \phi^{-1}(v_1 v_2 v_3)} f^p(z) \\ &= \bigvee_{\phi(u_1)=v_1, \phi(u_2)=v_2, \phi(u_3)=v_3} f^p(u_1 u_2 u_3) \\ &\geq \bigvee_{\phi(u_1)=v_1, \phi(u_2)=v_2,, \phi(u_3)=v_3} f^p(u_2) \\ &= \bigvee_{z=\phi^{-1}(v_2)} f^p(z) = \phi(f)^p(v_2). \end{aligned}$$

Hence, $\phi(f)^p(v_1 v_2, v_3) \geq \phi(f^p)(v_2)$.
Similarly $\phi(\phi(f^n)(v_1 v_2, v_3) \leq \phi(f^n)(v_2)$.
Therefore $\phi(\overline{f})$ is a BF weakly interior ideal of V.

(2) Let $a, b, c \in U$. Then

$$\begin{aligned} \phi^{-1}(g^p)(abc) &= g^p(\phi(abc)) = g^p(\phi(a)\phi(b))\phi(c) \\ &\geq (g^p)(\phi(b)) = \phi^{-1}(g^p)(b). \end{aligned}$$

Thus, $\phi^{-1}(g^p)(abc) \geq \phi^{-1}(g^p)(\phi(b))$.
Similarly $\phi^{-1}(g^n)(abc) \leq \phi^{-1}(g^p)(\phi(b))$.
Hence $\phi^{-1}(g^p)$ is a BF weakly interior ideal of U.
■

References

[1] Isabelle Bloch, "Bipolar fuzzy mathematical morphology for spatial reasoning," International Journal of Approximate Reasoning, vol. 53, no. 7, pp1031-1070, 2012
[2] Rukchart Prasertpong, and Manoj Siripitukdet, "Rough set models induced by serial fuzzy relations approach in semigroups," Engineering Letters, vol. 27, no. 1, pp216-225, 2019
[3] John T Cacioppo, Wendi L Gardner, and Gary Berntson, "Beyondbipolar conceptualizations and measures: the case of attitudes and evaluative space," Personality & Social Psychology Review, vol. 1, no. 1, pp3-25, 1997
[4] Paul Slovic, Melissa Finucane, Ellen Peters, and Donald G Macgregor, "Rational actors or rational fools: Implications of the affect heuristic for behavioral economics," The Journal of Socio-Economics, vol. 31, no. 4, pp329-342, 2002
[5] Gianluca Campanella, and Rita A. Ribeiro, "A framework for dynamic multiple-criteria decision making," Decision Support Systems, vol. 52, no. 1, pp52-60, 2011
[6] Hongbin Liu, Le Jiang, and Luis Martnez, "A dynamic multi-criteria decision making model with bipolar linguistic term sets," Expert Systems With Applications, vol. 95, pp104-112, 2018
[7] Lotfi A. Zadeh "Fuzzy sets," Information and Control, vol. 8, pp338-353, 1965
[8] Azriel Rosenfeld, "Fuzzy group," Journal of Mathematical Analysis and Application, pp338-353, 1971
[9] Nobuaki Kuroki, "Fuzzy bi-ideals in semigroup," Commentarii Mathematici Universitatis Sancti Pauli, vol. 5, pp128-132, 1979
[10] Wen-Ran Zhang, "Bipolar fuzzy sets and relations: A computational framework forcognitive modeling and multiagent decision analysis," *In proceedings of IEEE conference*, 1994, pp305-309
[11] Keon Myung Lee, "Bipolar-valued fuzzy sets and their operations," *In proceeding International Conference on Intelligent Technologies Bangkok, Thailand*, 2000, pp307-312
[12] Kavikumar Jacob, Khamis Bin and Rozainin Roslan, "Bipolar-valued fuzzy finite switchboard state machines," Proceedings of the World Congress on Engineering and Computer Science, pp571-576, 2012
[13] Kuanyun Zhu , Jingru Wamg and Yongwei Yang, "Lattices of (generalized) fuzzy ideals in residuated lattices," IAENG International Journal of Applied Mathematics, vol. 50, no. 3, pp505-511, 2020
[14] Kuanyun Zhu , Jingru Wamg and Yongwei Yang, "A study on Z-Soft fuzzy rough sets in BCI-Algebras," IAENG International Journal of Applied Mathematics, vol. 50, no. 3, pp577-583, 2020
[15] Chang Su Kim, Jeong Gi Kang, and Jung Mi Kang "Ideal theory of semigroups based on the bipolar valued fuzzy set theory," Annals of Fuzzy Mathematics and Informatics, vol. 2, no. 2, pp193-206, 2012
[16] John N. Mordeson, Davender S. Malik, and Nobuaki Kuroki, "Fuzzy semigroup," Springer Science and Business Media, 2003
[17] Starmuthu R Muthuraj and Mohan Sridharan, "Homomorphism and Anti homomorphism on bipolar fuzzy sub HX groups," General Mathenatics Notes, vol. 17, no. 2, pp53-65, 2013