Fuzzy Portfolio Model under Investors’ Different Attitudes with Risk Adaptation Value Parameter Based on Possibility Theory

Xue Deng, Yuying Liu, Huidan Zhuang and Zhanye Lin

Abstract—The security market is an uncertain and extremely complicated system, in which few scholars focus on the investor’s behavioral analysis by risk adaptation value parameter in the fuzzy portfolio model. In terms of this issue, this paper attempts to combine the three different investors’ risk attitudes and the security fuzzy uncertainty in the portfolio research. Firstly, the fuzzy return rate is defined as a trapezoidal fuzzy number with risk adaptation value parameter k. Different k values correspond to different risk attitudes which contain risk-averse, risk-neutral and risk-seeking. Secondly, in theory we derive the concrete numerical expressions of the possibilistic mean, variance and covariance with risk adaptation value by strictly mathematical proof based on possibility theory. On the basis of this, the fuzzy portfolio models with three different risk attitudes are developed under the constraints of transaction costs, investment proportion upper and lower bounds. Finally, some numerical examples are verified to show that our models are feasible and effective. Then we discuss and compare the results in the following cases: the influence of three different risk attitudes on the portfolio models when transaction costs are considered and not considered; the change of different k values and the impact of transaction costs when the risk attitude is the same.

Index Terms—portfolio selection, risk adaptation value, trapezoidal fuzzy number, risk attitude, possibility theory

I. INTRODUCTION

SECURITY market is one of the most indispensable parts of the modern financial system, which can facilitate the flow of funds mainly by means of securities trading, then promoting the rational use of effective resources in the financial industry. Generally, investors prefer to choose securities investment due to the much higher return than the profit from bank deposits. The emergence of portfolio enables investors to diversify their funds into a variety of securities and to achieve the goal of ensuring returns while reducing risks. Over the past several decades, the return rates of stocks were usually regarded as stochastic variables. At the same time, numerous portfolio models from different aspects have been proposed on the basis of the probability theory under the assumption above.

In 1952, Markowitz [1] systematically introduced the measurement method and principle of stock returns and risk and built the classic mean-variance model, which laid a solid foundation for subsequent portfolio booms. Since then, on the basis of Markowitz’s theory, many scholars have continuously studied and expanded the theoretical methods of portfolio selection from different perspectives such as risk measurement, model algorithm and constraint conditions, etc., making the portfolio grow into a mature scientific field. For instance, Roy [2] proposed the safety-first portfolio model and obtained some achievements in minimizing the probability of the downside risk. Telser [3] established a model that emphasizes both the fixed level of survival s and the probability of bankruptcy, in which the portfolio is safe if the probability of bankruptcy is no greater than \( \alpha \). Based on a complete diffusion model, Nketi et al. [4] discussed the optimal method of portfolio selecting and expected wealth with random cash flows under inflation protection for an investment company. Nazir [5] presented an efficient financial portfolio selection and optimization implementation of Anticor’s algorithm by the OpenCL framework. Sharpe [6] pointed out the advantages of practical applications of the Markowitz portfolio selection method by means of the relationships among securities. Under the safety first principle, Kataoka [7] put forward a stochastic programming model for portfolio selection problems.

However, more and more researchers have realized that it’s not sufficient to describe the uncertainty of securities market by considering the stock returns as stochastic variables. In 1965, Zadeh [8] put forward the fuzzy set theory and took advantage of membership functions to describe the degree to which elements belong to sets. In 1978, Zadeh [9] introduced the possibility theory in order to explain the difference between randomness and fuzziness. On this basis, Dubois and Prade [10] further developed fuzzy set theory in 1988. Carlsson and Fuller [11], Zhang and Nie [12] introduced possibilistic mean, variances and covariances for fuzzy numbers, [13]-[16] have turned their attention to

Manuscript received August 19, 2020; revised February 7, 2021. This research was supported by the “Humanities and Social Sciences Research and Planning Fund of the Ministry of Education of China, No. 18YJAZH014-x2lx91Yb09090”, “Natural Science Foundation of Guangdong Province, No. 2019A1515011038”, “Guangdong Province Characteristic Innovation Project of Colleges and Universities, No. 2019GKTSCX023”, “Soft Science of Guangdong Province, No. 2018A070120006, 2019A101002118”. The authors are highly grateful to the referees and editor-in-chief for their very helpful comments.

Xue Deng is a Full Professor of the School of Mathematics, South China University of Technology, Guangzhou 510640, China, (e-mail: dxue@scut.edu.cn).

Yuying Liu is a Postgraduate of the School of Mathematics, South China University of Technology, Guangzhou 510640, China, (corresponding author to provide e-mail: 1013277735@qq.com).

Huidan Zhuang is a Teacher of the Primary School of Shenzhen Nanshan Taoyuan, Shenzhen 518000, China, (e-mail: 719732417@qq.com).

Zhanye Lin is a Research Assistant of the School of Mathematics, South China University of Technology, Guangzhou 510640, China, (e-mail: 1134416200@qq.com).
portfolio selection under the environment of fuzzy uncertainty. Chen and Huang [17] used triangular fuzzy numbers to denote the rate of return and risks and attempted to obtain the optimal portfolio through fuzzy optimization. Tsaur [18] studied portfolio selection whose investment proportions were symmetric triangular fuzzy numbers and solved the model with fuzzy constraints.

Different investors take different views of the occurrence for the same event. In general, aggressive investors tend to be more active in security market and stronger in the spirit of taking risks. While conservative decision makers pay more attention to the stability and security of investment returns, and often regard the security of principal as the most important thing. Although quantities of scholars have made efforts to research fuzzy portfolio models, very few of the previous studies attempted to analyze the investors’ risk behaviors with risk adaptation value parameter under a fuzzy environment. Tsaur [19] developed a fuzzy portfolio model when considering three different risk attitudes and discussed the change of the return rate under the aforementioned three different environments. Feng et al. [20] proposed the mean value with pessimism coefficient based on measure theory, and discussed the influence of investors’ pessimism on portfolio. By dividing investors into three groups, Sun [21] established a novel fuzzy random model with respect to portfolio selection to provide them with different investment strategies. Zhou [22] et al. aimed to deal with the problem when investors took conservative, neutral and aggressive attitudes in portfolio selection. As for constant relative risk adverse pension plan members, Nkeki [23] constructed a M-V portfolio model when considering random salary, strategic consumption planning and the time factor.

Therefore, in this study, we mainly aim to construct the portfolio model with three different risk attitudes of investors in fuzzy environment by adding the risk adaption value k to the improved mean-variance model and regarding the stock return rate as a trapezoidal fuzzy number. Based on probability theory, we derive the possibilistic mean, variance and covariance expressions with risk adaption value parameter. Finally, numerical examples are presented to show how the corresponding results vary when the investors’ attitudes change, when the parameter k is assigned to different values under the same risk attitude and analyze the impact of transaction costs on the portfolio.

This paper can be divided into five sections altogether. In Section 2, some conceptions of fuzzy theory and possibilistic mean, variance, and covariance are introduced respectively firstly. Fuzzy portfolio selection models are constructed on the basis of different investor risk attitudes with risk adaption value in Section 3. In Section 4 we design four numerical examples to verify the feasibility of our models. Finally, some helpful conclusions are put forward in Section 5.

II. PRELIMINARIES

A. The Fuzzy Set Theory

In this section, firstly we will review some basic conceptions about fuzzy set theory put forward by Zadeh [8]. Some relevant definitions are as follows:

**Definition 1.** Assume that $\mathcal{N}=\{a,b,\alpha,\beta\}$ is a fuzzy variable and its membership function is normal, convex, and continuous, then the $\gamma$-level set of $\mathcal{N}$ can be represented as:

$$\tilde{N}_\gamma = \{a_\gamma, a_\gamma, \gamma\}, \forall \gamma \in [0,1]$$  \hspace{1cm} (1)

**Definition 2.** Assume that $\mathcal{N}$ is a fuzzy variable, $L_\mathcal{N}$ and $R_\mathcal{N}$ are both monotone nonincreasing continuous functions from $[0,1]$ to $[0,1]$, $L_\mathcal{N}(0) = R_\mathcal{N}(0) = 1$, $L_\mathcal{N}(1) = R_\mathcal{N}(1) = 0$, and the membership function of $\mathcal{N}$ is:

$$\mu_{\mathcal{N}}(x) = \begin{cases} 
L_\mathcal{N}(\frac{a-x}{\alpha}) & a - \alpha \leq x < a, \\
1 & a \leq x < b, \\
R_\mathcal{N}(\frac{x-b}{\beta}) & b \leq x < b + \beta, \\
0 & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (2)

When $L_\mathcal{N}, R_\mathcal{N}$ are degenerated to linear functions, then $\mathcal{N}=\{a,b,\alpha,\beta\}$ is referred to as a trapezoidal fuzzy variable and its membership function is:

$$\mu_{\mathcal{N}}(x) = \begin{cases} 
1 - \frac{a-x}{\alpha} & a - \alpha \leq x < a, \\
1 & a \leq x < b, \\
1 - \frac{x-b}{\beta} & b \leq x < b + \beta, \\
0 & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (3)

When $L_\mathcal{N}, R_\mathcal{N}$ are degenerated to linear functions and $a = b$, then $\mathcal{N}=\{a,\alpha,\beta\}$ is referred to as a triangular fuzzy variable and the membership function of $\mathcal{N}=\{a,\alpha,\beta\}$ is:

$$\mu_{\mathcal{N}}(x) = \begin{cases} 
1 - \frac{a-x}{\alpha} & a - \alpha \leq x < a, \\
1 & a \leq x < b, \\
0 & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (4)

B. Possibilistic Mean and Variance and Covariance of Fuzzy Variables

Assume $\mathcal{N}$ is a fuzzy variable, the upper and lower possibilistic mean, the probabilistic variance and covariance defined in [18] and [19] are demonstrated as follows:

**Definition 3.** The upper possibilistic mean of $\mathcal{N}$ is calculated as follows:

$$M_\mu^+ (\tilde{\mathcal{N}}) = \frac{\int_{0}^{1} \mu_{\tilde{\mathcal{N}}}(\gamma) \mu_{\mathcal{N}}(\gamma) d\gamma}{\int_{0}^{1} \mu_{\mu}(\gamma) d\gamma} = 2\int_{0}^{1} \mu_{\mathcal{N}}(\gamma) d\gamma.$$  \hspace{1cm} (5)

**Definition 4.** The lower possibilistic mean of $\mathcal{N}$ is calculated as follows:

$$M_\mu^- (\tilde{\mathcal{N}}) = \frac{\int_{0}^{1} \mu_{\tilde{\mathcal{N}}}(\gamma) \mu_{\mathcal{N}}(\gamma) d\gamma}{\int_{0}^{1} \mu_{\mu}(\gamma) d\gamma} = 2\int_{0}^{1} \mu_{\mathcal{N}}(\gamma) d\gamma.$$  \hspace{1cm} (6)

Where $\mu$ represents the possibility, and there are:

$$\text{Pos}(\mathcal{N} \in a_\gamma) = [a_\gamma, \gamma] = \sup_{\mu \geq a_\gamma} \mu(\mu) = \gamma,$$  \hspace{1cm} (7)
\[ \text{Pos}[\tilde{A} \leq a_i(\gamma)] = \prod_{i \in [\alpha, \beta]} a_i(\gamma) = \sup_{\mu_{a_i}(\gamma)} \tilde{A}(\mu) = \gamma. \] (8)

**Definition 5.** The possibilistic mean of \( \tilde{A} = (a, b, \alpha, \beta) \) is calculated as follows:
\[ M(\tilde{A}) = \frac{1}{2} (\tilde{A} + M(\tilde{A})). \] (9)

**Theorem 1.** Suppose \( \lambda \) is a real number, we have:
\[ M(\lambda \tilde{A}) = \lambda M(\tilde{A}). \] (10)

**Theorem 2.** Assume that \( \tilde{A} \) and \( \tilde{B} \) are both fuzzy variables, then:
\[ M(\tilde{A} + \tilde{B}) = M(\tilde{A}) + M(\tilde{B}). \] (11)

**Definition 6.** The possibilistic variance of \( \tilde{A} \) is calculated as:
\[ \text{Var} (\tilde{A}) = \frac{1}{2} \int \gamma (a_i(\gamma) - a_i(\gamma))^2 d\gamma. \] (12)

**Definition 7.** Let \( \tilde{A} \) and \( \tilde{B} \) be fuzzy variables, then the possibilistic covariance of \( \tilde{A} \) and \( \tilde{B} \) is calculated as:
\[ \text{Cov}(\tilde{A}, \tilde{B}) = \frac{1}{2} \int \gamma (a_i(\gamma) - a_i(\gamma))(b_i(\gamma) - b_i(\gamma)) d\gamma. \] (13)

**Theorem 3.** Let \( \tilde{A} \) and \( \tilde{B} \) be fuzzy variables, and \( \lambda_i, \lambda_2 \) are real numbers, we have:
\[ \text{Var}(\lambda_i \tilde{A} + \lambda_2 \tilde{B}) = \lambda_i^2 \text{Var}(\tilde{A}) + \lambda_2^2 \text{Var}(\tilde{B}) + 2 |\lambda_i \lambda_2| \text{Cov}(\tilde{A}, \tilde{B}) \] (14)

**III. FUZZY PORTFOLIO MODEL UNDER INVESTORS’ DIFFERENT ATTITUDES**

A. Portfolio Model with the Risk Adaptation Value Parameter

Let \( \tilde{r} = (a_i, b_i; \alpha_i, \beta_i) \), \( i = 1, 2, \ldots, n \) denote the return rate of asset \( i \) and it is regarded as a trapezoidal fuzzy variable in the meanwhile, \( [a_i, b_i] \) denotes the central interval of \( \tilde{r} \), where \( \alpha_i \) and \( \beta_i \) denote the left and right spread values respectively and \( \alpha_i > 0, \beta_i > 0 \). In this paper, we take different investor attitudes towards investment risk into account, hence we introduce the risk adaptation value \( k (k > 0) \) to the membership function of \( \tilde{r} \), then:
\[ \mu_{\tilde{r}}(x) = \begin{cases} 
1 - \frac{(a_i - x)}{\alpha_i} & a_i - \alpha_i \leq x < a_i, \\
1 & a_i \leq x < b_i, \\
1 - \frac{(x - b_i)}{\beta_i} & b_i \leq x < b_i + \beta_i, \\
0 & \text{otherwise}.
\end{cases} \] (15)

It is not difficult to calculate the second derivative of \( \mu_{\tilde{r}}(x) \) at \( i = 1, 2, \ldots, n \):
\[ \mu''_{\tilde{r}}(x) = \begin{cases} 
-k(k-1) \frac{1}{\alpha_i^2} (a_i - x)^{k-2} & a_i - \alpha_i \leq x < a_i, \\
-k(k-1) \frac{1}{\beta_i^2} (x - b_i)^{k-2} & b_i \leq x < b_i + \beta_i, \\
0 & \text{otherwise}.
\end{cases} \] (16)

According to (16), the membership function with risk-averse which refers to investor who is sensitive to risk is concave when \( k < 1 \); when \( k = 1 \) the membership function with risk-neutral which refers to investor holds neutral attitude to risk is linear; and the membership function with risk-seeking which refers to the investor prefers a challenging investment is convex when \( k > 1 \).

According to Definition 1, it’s easy to derive the \( \gamma \)-level set of the return rate \( \tilde{r} \):
\[ \{ \tilde{r} \} = \{ r_i \}, r_i = [a_i - (1 - \gamma)^{\frac{1}{k+1}} \alpha_i, b_i + (1 - \gamma)^{\frac{1}{k+1}} \beta_i] \] (17)

According to (5), the upper possibilistic mean of \( \tilde{r} \) is expressed as:
\[ M^+(\tilde{r}) = 2 \int_0^1 \gamma r_i(\gamma) d\gamma = 2 \int_0^{1}\gamma [b_i + (1 - \gamma)^{\frac{1}{k+1}} \beta_i] d\gamma \]
\[ = b_i + \frac{2k^2}{(1+k)(1+2k)} \beta_i, \forall k > 0. \] (18)

According to (6), the lower possibilistic mean of \( \tilde{r} \) is expressed as:
\[ M^- (\tilde{r}) = 2 \int_0^1 \gamma r_i(\gamma) d\gamma = 2 \int_0^{1}\gamma [a_i - (1 - \gamma)^{\frac{1}{k+1}} \alpha_i] d\gamma \]
\[ = a_i - \frac{2k^2}{(1+k)(1+2k)} \alpha_i, \forall k > 0. \] (19)

According to (9), the possibilistic mean of \( \tilde{r} \) is expressed as:
\[ M (\tilde{r}) = \frac{M^+(\tilde{r}) + M^- (\tilde{r})}{2} \]
\[ = \frac{a_i + b_i}{2} + \frac{k^2}{(1+k)(1+2k)} (\beta_i - \alpha_i), \forall k > 0. \] (20)

According to (12), the possibilistic variance of \( \tilde{r} \) is denoted as:
\[ \text{Var} (\tilde{r}) = \frac{1}{2} \int_0^1 \gamma (r_i(\gamma) - r_i(\gamma))^2 d\gamma \]
\[ = \frac{1}{2} \int_0^1 \gamma (b_i - a_i + (1 - \gamma)^{\frac{1}{k+1}} (\beta_i - \alpha_i))^2 d\gamma \]
\[ = \frac{(b_i - a_i)^2}{4} + \frac{k^2}{4(1+k)(2k)} (\beta_i - \alpha_i)^2 \]
\[ + \frac{k^2}{(1+k)(1+2k)} (b_i - a_i)(\beta_i - \alpha_i). \] (21)

According to (13), the possibilistic covariance between fuzzy return rate \( \tilde{r} \) and \( \tilde{r}(\forall i \neq j) \) can be derived as follows:
kinds of assets, where $x_k$ when the decision maker holds risk-neutral attitude; when $x_k > 0$, decision makers hold risk-averse attitude in Model (27). Similarly we obtain the possibilistic mean–variance portfolio model with fuzzy return rate when making a portfolio selection under the certain risk environment. Therefore, the possibilistic mean–variance portfolio model based on risk-neutral attitude can be denoted as:

$$\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \left[ a_i x_i + \frac{1}{12} (\beta_i - \alpha_i) x_i \right], \\
\text{s.t.} & \quad \sum_{i=1}^{n} \left[ (b_i - a_i)^2 x_i^2 + \frac{1}{60} (\beta_i + \alpha_i)^2 x_i^2 \right] + \frac{1}{12} (b_i - a_i)(\beta_i + \alpha_i) x_i^2 \\
& \quad + \frac{1}{24} \left[ (b_i - a_i)(\beta_i + \alpha_i) + (b_i - a_i)(\beta_i + \alpha_i) \right] \\
& \quad + \frac{1}{24} \left[ (b_i - a_i)(\beta_i + \alpha_i) + (b_i - a_i)(\beta_i + \alpha_i) \right] \leq \sigma^2, \\
& \quad \sum_{i=1}^{n} x_i \leq 1, \\
& \quad l_i \leq x_i \leq u_i, \forall i = 1, 2, ..., n.
\end{align*}$$

B. Portfolio Model with Three Different Investor Risk Attitudes and without Transaction Costs

According to the concept of the mean-variance put forward by Markowitz, investors are eager to obtain the highest return rate when making a portfolio selection under the certain risk environment. Therefore, the possibilistic mean–variance portfolio model with fuzzy return rate is derived as:

$$\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \left[ a_i + \frac{1}{2} b_2 \right] x_i + \frac{k^2}{(1+k)(1+2k)} (\beta_i - \alpha_i) x_i, \\
\text{s.t.} & \quad \text{Var} \left( \sum_{i=1}^{n} x_i d_i \right) \leq \sigma^2, \\
& \quad \sum_{i=1}^{n} x_i \leq 1, \\
& \quad l_i \leq x_i \leq u_i, \forall i = 1, 2, K, n.
\end{align*}$$

Where the objective function represents the goal of maximum returns. $\sigma^2$ represents the maximum risk level value which can be accepted by investors. $u_i$ and $l_i$ represent the upper and lower bounds of the investment proportion $x_i$, respectively, and $\sum_{i=1}^{n} x_i \leq 1$ means the sum of the investment proportions should not exceed 1.

Case 1: ($k = 0.5$) The fuzzy portfolio model based on risk-averse attitude can be denoted as:

$$\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \left[ a_i + \frac{1}{2} b_2 \right] x_i + \frac{k^2}{(1+k)(1+2k)} (\beta_i - \alpha_i) x_i, \\
\text{s.t.} & \quad \text{Var} \left( \sum_{i=1}^{n} x_i d_i \right) \leq \sigma^2, \\
& \quad \sum_{i=1}^{n} x_i \leq 1, \\
& \quad l_i \leq x_i \leq u_i, \forall i = 1, 2, ..., n.
\end{align*}$$
\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \left[ \frac{a_i + b_i}{2} x_i + \frac{1}{6} (\beta_i - \alpha_i) x_i^3 \right], \\
\text{s.t.} & \quad \sum_{i=1}^{n} \left[ \frac{(b_i-a_i)^2}{4} x_i^2 + \frac{1}{24} (\beta_i + \alpha_i)^2 x_i^2 + \frac{1}{6} (b_i-a_i)(\beta_i + \alpha_i) x_i^2 \right] \\
& + \sum_{i,j=1}^{n} 2x_i x_j \left( \frac{(b_i-a_j)(b_j-a_i)}{4} + \frac{1}{12} [(b_i-a_j)(\beta_i + \alpha_j) + (b_j-a_i)(\beta_i + \alpha_j)] \right) \\
& + \frac{1}{24} (\beta_i + \alpha_i)(\beta_j + \alpha_j) \leq \sigma^2, \\
& \sum_{i=1}^{n} x_i \leq 1, \\
& l_i \leq x_i \leq u_i, \forall i = 1,2,K,n.
\end{align*}
\]

(28)

Case 3 \((k = 2)\): The fuzzy portfolio model based on risk-seeking attitude can be denoted as:

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \left[ \frac{a_i + b_i}{2} x_i + \frac{4}{15} (\beta_i - \alpha_i) x_i^2 \right], \\
\text{s.t.} & \quad \sum_{i=1}^{n} \left[ \frac{(b_i-a_i)^2}{4} x_i^2 + \frac{1}{12} (\beta_i + \alpha_i)^2 x_i^2 + \frac{4}{15} (b_i-a_i)(\beta_i + \alpha_i) x_i^2 \right] \\
& + \sum_{i,j=1}^{n} 2x_i x_j \left( \frac{(b_i-a_j)(b_j-a_i)}{4} + \frac{2}{15} [(b_i-a_j)(\beta_i + \alpha_j) + (b_j-a_i)(\beta_i + \alpha_j)] \right) \\
& + (b_j - a_i)(\beta_i + \alpha_j) + \frac{1}{12} (\beta_i + \alpha_i)(\beta_j + \alpha_j) \leq \sigma^2, \\
& \sum_{i=1}^{n} x_i \leq 1, \\
& l_i \leq x_i \leq u_i, \forall i = 1,2,K,n.
\end{align*}
\]

(29)

C. Portfolio Model with Three Different Risk Attitudes and Transaction Costs

In fact, it’s necessary for investors to focus on not only the return and risk of a portfolio, but also the transaction costs. And it’s possible to result in a useless portfolio due to the neglect of transaction costs. Therefore, it’s reasonable to take transaction costs into account. In this section, the transaction costs are regarded as a \( V - \) function. Then, the corresponding transaction costs of asset \( a \) are \( t_a \), \( t_b \), \( t_c \), \( \ldots \), \( t_n \). where \( t_i \geq 0 \) represents the transaction cost per unit in assets, \( x_0 = (x_{0, i}, x_{0, j}, \ldots, x_{0, n}) \) is a given portfolio distribution and \( x = (x_{1, i}, x_{1, j}, \ldots, x_{1, n}) \) is a new portfolio distribution, then the transaction costs are \( \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} t_i |x_i - x_{0, i}| \) in total. On basis of this, Model (26) can be changed into Model (30):

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \left[ \frac{a_i + b_i}{2} x_i + \frac{k^2}{(1+k)(1+2k)} (\beta_i - \alpha_i) x_i \right] - \sum_{i=1}^{n} t_i |x_i|, \\
\text{s.t.} & \quad \sum_{i=1}^{n} \left[ \frac{(b_i-a_i)^2}{4} x_i^2 + \frac{1}{60} (\beta_i + \alpha_i)^2 x_i^2 + \frac{1}{12} (b_i-a_i)(\beta_i + \alpha_i) x_i^2 \right] \\
& + \sum_{i,j=1}^{n} 2x_i x_j \left( \frac{(b_i-a_j)(b_j-a_i)}{4} + \frac{1}{24} [(b_i-a_j)(\beta_i + \alpha_j) + (b_j-a_i)(\beta_i + \alpha_j)] \right) \\
& + (b_j - a_i)(\beta_i + \alpha_j) + \frac{1}{60} (\beta_i + \alpha_i)(\beta_j + \alpha_j) \leq \sigma^2, \\
& \sum_{i=1}^{n} x_i \leq 1, \\
& l_i \leq x_i \leq u_i, \forall i = 1,2,\ldots, n.
\end{align*}
\]

(30)

Models (27)-(29) can be changed into Models (31)-(33) as follows when we set \( t_i = 0.005 \).

Case 1: \((k = 0.5)\) The fuzzy portfolio model with risk-averse attitude and transaction costs is denoted as:

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \left[ \frac{a_i + b_i}{2} x_i + \frac{k^2}{(1+k)(1+2k)} (\beta_i - \alpha_i) x_i \right] - \sum_{i=1}^{n} t_i |x_i|, \\
\text{s.t.} & \quad \sum_{i=1}^{n} \left[ \frac{(b_i-a_i)^2}{4} x_i^2 + \frac{1}{60} (\beta_i + \alpha_i)^2 x_i^2 + \frac{1}{12} (b_i-a_i)(\beta_i + \alpha_i) x_i^2 \right] \\
& + \sum_{i,j=1}^{n} 2x_i x_j \left( \frac{(b_i-a_j)(b_j-a_i)}{4} + \frac{1}{24} [(b_i-a_j)(\beta_i + \alpha_j) + (b_j-a_i)(\beta_i + \alpha_j)] \right) \\
& + (b_j - a_i)(\beta_i + \alpha_j) + \frac{1}{60} (\beta_i + \alpha_i)(\beta_j + \alpha_j) \leq \sigma^2, \\
& \sum_{i=1}^{n} x_i \leq 1, \\
& l_i \leq x_i \leq u_i, \forall i = 1,2,\ldots, n.
\end{align*}
\]

(31)
Case 2 ($k = 1$): The fuzzy portfolio model with risk-neutral attitude and transaction costs is denoted as:

$$\begin{align*}
\max & \quad \sum_{i=1}^{n} \left[ a_i + b_i \cdot x_i + \frac{1}{6} (\beta_i - \alpha_i) x_i \right] - \sum_{i=1}^{n} \frac{5}{10} x_i, \\
\text{s.t.} & \quad \sum_{i=1}^{n} \left( \frac{(b_i - a_i)^2}{4} x_i^2 + \frac{1}{24} (\beta_i + \alpha_i)^2 x_i^2 + \frac{1}{6} (b_i - a_i)(\beta_i + \alpha_i) x_i^2 \right) \\
& \quad + \sum_{i,j=1}^{n} 2x_i x_j (b_i - a_i)(b_j - a_j) + \frac{1}{12} (b_i - a_i)(\beta_i + \alpha_i) \\
& \quad + (b_j - a_j)(\beta_i + \alpha_i) + \frac{1}{24} (\beta_i + \alpha_i)(\beta_j + \alpha_j) \leq \sigma^2, \\
& \quad \sum_{i=1}^{n} x_i \leq 1, \\
& \quad l_i \leq x_i \leq u_i, \quad \forall i = 1, 2, K, n.
\end{align*}$$

(32)

Case 3 ($k = 2$): The fuzzy portfolio model with risk-seeking attitude and transaction costs is denoted as:

$$\begin{align*}
\max & \quad \sum_{i=1}^{n} \left[ a_i + b_i \cdot x_i + \frac{4}{15} (\beta_i - \alpha_i) x_i \right] - \sum_{i=1}^{n} \frac{5}{10} x_i, \\
\text{s.t.} & \quad \sum_{i=1}^{n} \left( \frac{(b_i - a_i)^2}{4} x_i^2 + \frac{1}{12} (\beta_i + \alpha_i)^2 x_i^2 + \frac{4}{15} (b_i - a_i)(\beta_i + \alpha_i) x_i^2 \right) \\
& \quad + \sum_{i,j=1}^{n} 2x_i x_j (b_i - a_i)(b_j - a_j) + \frac{2}{15} (b_i - a_i)(\beta_i + \alpha_i) \\
& \quad + (b_j - a_j)(\beta_i + \alpha_i) + \frac{1}{12} (\beta_i + \alpha_i)(\beta_j + \alpha_j) \leq \sigma^2, \\
& \quad \sum_{i=1}^{n} x_i \leq 1, \\
& \quad l_i \leq x_i \leq u_i, \quad \forall i = 1, 2, K, n.
\end{align*}$$

(33)

IV. Numerical Example

For verifying the feasibility of the models we proposed, and expert’s advice in relevant fields, the possibility distributions of these 5 stocks are shown in Table 1:

A. Discuss Fuzzy Portfolio Model with Three Different Risk Attitudes and without Transaction Costs

Firstly, five stocks are selected for the next empirical analysis. With the help of historical data, market information

<table>
<thead>
<tr>
<th>Asset 1</th>
<th>$a_1$</th>
<th>$b_1$</th>
<th>$\alpha_1$</th>
<th>$\beta_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>1.200</td>
<td>2.700</td>
<td>1.500</td>
<td>0.400</td>
</tr>
<tr>
<td>Asset 2</td>
<td>1.100</td>
<td>1.900</td>
<td>1.400</td>
<td>0.700</td>
</tr>
<tr>
<td>Asset 3</td>
<td>2.100</td>
<td>3.000</td>
<td>2.300</td>
<td>1.000</td>
</tr>
<tr>
<td>Asset 4</td>
<td>1.100</td>
<td>2.000</td>
<td>1.600</td>
<td>0.900</td>
</tr>
<tr>
<td>Asset 5</td>
<td>1.200</td>
<td>2.200</td>
<td>1.800</td>
<td>1.100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Asset 6</th>
<th>$a_6$</th>
<th>$b_6$</th>
<th>$\alpha_6$</th>
<th>$\beta_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 6</td>
<td>1.200</td>
<td>2.700</td>
<td>1.500</td>
<td>0.400</td>
</tr>
</tbody>
</table>

Table I: Possibility Distribution of the Fuzzy Returns on the Five Assets

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>0.9432</td>
<td>1.2513</td>
<td>1.5591</td>
<td>1.8093</td>
<td>1.9872</td>
<td>1.9917</td>
<td>1.9917</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.2892</td>
<td>0.3000</td>
<td>0.3000</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.2313</td>
<td>0.1108</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0.1228</td>
<td>0.2490</td>
<td>0.3751</td>
<td>0.4000</td>
<td>0.4000</td>
<td>0.4000</td>
<td>0.4000</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
</tbody>
</table>

| $\sum_{i=1}^{n} x_i \leq 1$ | 0.5228 | 0.6490 | 0.7751 | 0.9313 | 1.0000 | 1.0000 | 1.0000 |

Table II: Fuzzy Portfolio Model (27) with Risk-Averse Attitude and without Transaction Costs
Let \((u_1, u_2, u_3, u_4, u_5)^T = (0.3, 0.3, 0.4, 0.5, 0.6)^T\) and \((l_1, l_2, l_3, l_4, l_5)^T = (0.1, 0.1, 0.1, 0.1, 0.1)^T\) be the upper and lower bounds of \(x_i (i = 1, 2, ..., 5)\) respectively. After the analysis and application of Models (26), (27) and (28), the efficient portfolios are obtained respectively and shown in Tables 2, 3 and 4 by the MATLAB software.

As demonstrated in Table 2, we can find that: for a risk-averse decision maker, when \(\sum_{i=1}^{5} x_i \leq 1\) and the risk value \(\sigma \leq 0.7\), then the return \(M \leq 1.8093\); when the risk value \(\sigma\) increases from 0.4 to 0.7, we can increase return by increasing the investment proportion of assets 2 and 3; when \(\sigma \geq 0.8\), we can increase the return through increasing the value of \(x_1\) and reducing the value of \(x_2\); when \(\sigma \geq 0.9\), the investment proportion remains unchanged, which indicates that 0.9 is the largest risk value that the risk-averse decision makers can accept.

As demonstrated in Table 3: for a risk-neutral decision maker, when \(\sum_{i=1}^{5} x_i \leq 1\) and the risk value \(\sigma \leq 0.9\), then the return \(M \leq 1.7048\); when the risk value \(\sigma\) increases from 0.5 to 0.9, we can increase return by increasing the values of \(x_2\) and \(x_1\); when \(\sigma \geq 1\), the investment proportion of assets 3, 4 and 5 has not changed and we can increase the return through increasing the value of \(x_1\) and reducing the value of \(x_2\); when \(\sigma \geq 1\), the investment proportion remains unchanged, which indicates that 1.1 is the maximum acceptable risk value for a risk-neutral investor.

As shown in Table 4, it can be easily found that: for a risk-seeking decision maker, when \(\sum_{i=1}^{5} x_i \leq 1\) and the risk value \(\sigma \leq 1.2\), then the return \(M \leq 1.7005\); when the risk value \(\sigma\) increases from 0.8 to 1.2, the investment proportion of assets 1, 4 and 5 has not changed, and the return is increased mainly by increasing the investment proportion of assets 2 and 3; when \(\sigma \geq 1.3\), the investment proportion remains unchanged, which indicates that 1.3 is the maximum acceptable risk for a risk-seeking investor.

When we compare Tables 2-4, it can be seen that: Firstly, the larger the risk investors pursue the larger the risk they could accept in portfolio selection. Secondly, the higher the level of risk, the more the expected return. Finally, when the risk value exceeds some certain range, the efficient portfolio maintains the same values in full investment.

In order to more clearly illustrate the different investment choices corresponding to different risk attitudes, the results in Table 2, 3 and 4 are presented as shown in Fig. 1:
Fig. 1. The efficient frontier for Models (27), (28) and (29)

Fig. 2. The effective frontier for different $k$ values ($k = 0.2$, $0.5$ and $0.8$) in Model (26)

Fig. 3. The effective frontier for different $k$ values ($k = 2$, $5$ and $8$) in Model (26)
From Fig. 1, we can conclude that: Firstly, the efficient frontier of the portfolio with risk-neutral attitude lies between that of the portfolio model with risk-averse attitude and risk-seeking. Secondly, when the risk value is constant, the risk-averse person gains the maximum return, followed by the risk-neutral person, and finally the risk-seeking person. Last but not least, with the increase (decrease) of the return, the variance also increases (decreases) correspondingly. Moreover, the shape of the effective frontier in Models (27), (28) and (29) is a good parabola, which is completely consistent with the shape and trend of the classical Markowitz model. This indicates that the model this paper proposed is reasonable.

B. Discuss Different K Values in the Fuzzy Portfolio Model under the Same Investor Risk Attitude

As mentioned above, different decision makers who hold different risk attitudes need different portfolio selections. In this section, we consider taking different \( k \) respectively for comparison for risk-averse attitude and risk-seeking attitude. Firstly we set \( k = 0.2, 0.5 \) and \( 0.8 \) in the fuzzy portfolio model with risk-averse attitude, and the effective frontier for the optimal portfolio is demonstrated in Fig. 2. Similarly, we set \( k = 2, 5 \) and \( 8 \) respectively based on risk-seeking attitude, and the effective frontier of the optimal portfolio is demonstrated in Fig. 3.

In Fig. 2, it’s obvious that with the increase of \( k \) values, the effective frontier moves to the right-upper corner, and the expected return decreases accordingly in the fuzzy portfolio model on the basis of risk-averse attitude.

In Fig. 3, it’s also obvious that with the increase of \( k \) values, the effective frontier moves to the right-upper corner, and the return decreases accordingly in the portfolio model with risk-seeking attitude.

C. Discuss Fuzzy Portfolio Model with Three Different Investor Risk Attitudes and Transaction Costs

In this section, we will analyze the impact of transaction costs on a portfolio with different attitudes. MATLAB is used to optimize the Models (31), (32) and (33), and the results obtained are shown in Tables 5, 6 and 7.

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( 0.4 )</th>
<th>( 0.5 )</th>
<th>( 0.6 )</th>
<th>( 0.7 )</th>
<th>( 0.8 )</th>
<th>( 0.9 )</th>
<th>( 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>0.6818</td>
<td>0.9268</td>
<td>1.1716</td>
<td>1.3436</td>
<td>1.4872</td>
<td>1.4917</td>
<td>1.4917</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.2892</td>
<td>0.3000</td>
<td>0.3000</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.2313</td>
<td>0.1108</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0.1000</td>
<td>0.2490</td>
<td>0.3751</td>
<td>0.4000</td>
<td>0.4000</td>
<td>0.4000</td>
<td>0.4000</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( \sum_{i=1}^{5} x_i \leq 1 )</td>
<td>0.5228</td>
<td>0.6490</td>
<td>0.7751</td>
<td>0.9313</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( 0.5 )</th>
<th>( 0.6 )</th>
<th>( 0.7 )</th>
<th>( 0.8 )</th>
<th>( 0.9 )</th>
<th>( 1.0 )</th>
<th>( 1.1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>0.6170</td>
<td>0.7881</td>
<td>0.9591</td>
<td>1.1301</td>
<td>1.2535</td>
<td>1.3704</td>
<td>1.4033</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1817</td>
<td>0.2740</td>
<td>0.3000</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0.1000</td>
<td>0.2026</td>
<td>0.2959</td>
<td>0.3892</td>
<td>0.4000</td>
<td>0.4000</td>
<td>0.4000</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( \sum_{i=1}^{5} x_i \leq 1 )</td>
<td>0.5093</td>
<td>0.6026</td>
<td>0.6959</td>
<td>0.7892</td>
<td>0.8817</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( 0.8 )</th>
<th>( 0.9 )</th>
<th>( 1.0 )</th>
<th>( 1.1 )</th>
<th>( 1.2 )</th>
<th>( 1.3 )</th>
<th>( 1.4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>0.7897</td>
<td>0.9133</td>
<td>1.0368</td>
<td>1.1355</td>
<td>1.2266</td>
<td>1.2973</td>
<td>1.2973</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0.2378</td>
<td>0.3103</td>
<td>0.3829</td>
<td>0.4000</td>
<td>0.4000</td>
<td>0.4000</td>
<td>0.4000</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>( \sum_{i=1}^{5} x_i \leq 1 )</td>
<td>0.6378</td>
<td>0.7103</td>
<td>0.7829</td>
<td>0.8601</td>
<td>0.9388</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
As shown in Table 5, it can be found that clearly: when \( \sum_{i=1}^{5} x_i \leq 1 \) and the risk value \( \sigma \leq 0.7 \), then the return \( M \leq 1.3436 \). When the risk value \( \sigma \leq 0.7 \), the optimal investment returns with and without transaction costs are 1.3436 and 1.8093 respectively by comparing Table 2, which means the transaction costs play a significant role in the portfolio selection. And a similar conclusion can be obtained from Tables 6 and 7.

Similarly, in order to more clearly demonstrate the optimal portfolio with transaction costs and different risk attitudes, the results in Tables 5, 6 and 7 are shown in Fig. 4.

![Fig. 4. The effective frontier for Models (31), (32) and (33)](image)

![Fig. 5. The efficient frontier for Models (27) and (31)](image)

![Fig. 6. The efficient frontier for Models (28) and (32)](image)
From Fig. 4, it can be concluded that the effective frontiers are increasing functions for the fuzzy portfolio model with transaction costs. It moves to the right-upper corner in the above figure under different risk attitudes in the fuzzy portfolio model when investors consider transaction costs.

D. Discuss the Impact of Transaction Costs on the Portfolio Selection

In this section, we will focus on how the investors make a decision for the efficient portfolio when we consider and don’t consider transaction costs under the same risk attitude.

According to above discussion, the results in Tables 2 and 5, Tables 3 and 6, Tables 4 and 7 based on some different risk attitudes can be clearly shown in Fig. 5-7.

By comparing Fig. 5 to Fig. 7, we can see clearly that the effective frontier for the model with no transaction costs constraint is higher than that of the model with transaction costs constraint. If investors obtain the same expected returns, the risk of the portfolio with transaction costs is higher than that of the portfolio without transaction costs, which indicates that the portfolio selection will be affected by transaction costs to some extent.

V. CONCLUSION

In this paper, not only fuzzy theory, but also three different risk attitudes on the portfolio of investors are considered to get the optimal portfolio. To fully take the factor of investors’ own risk attitudes into accounts, we regard the return rate as a trapezoidal fuzzy variable and introduce the risk adaptation parameter $k$ to the membership function, while different $k$ correspond to different risk attitudes of investors. Then the mean-variance model with risk adaptation value parameter under the constraints of transaction costs, investment proportion upper and lower bounds is constructed to maximize the return within some certain range of risk by taking advantage of the probability theory. To reveal the effectiveness of the approaches we put forward for a portfolio selection problem, firstly, this paper made a study on the optimal portfolio based on three different risk attitudes of investors (risk-averse when $k = 0.5$, risk-neutral when $k = 1$ and risk-seeking when $k = 2$). Secondly, under the conditions of risk-averse behavior ($k = 0.2$, 0.5 and 0.8) and risk-seeking behavior for investors ($k = 2$, 5 and 8), the fuzzy portfolio selections were discussed. Thirdly, we take three different risk attitudes and transaction costs into account at the same time in the fuzzy portfolio model. Finally, the influence of transaction costs under the same risk attitude on the fuzzy portfolio model is presented.

The results show that: Firstly, the higher the returns, the higher the risk in the fuzzy portfolio model when investors take different risk attitudes, which corresponds to the nature that risks and returns increase and decrease at the same time. Secondly, risk-seeking decision makers are able to withstand the higher return and the larger risk than the risk-neutral and risk-averse decision makers when making portfolio selection. Thirdly, the investment proportion will maintain the same when the risk exceeds some certain range. Finally, as the risk adaptation $k$ value increases, the effective frontier for the portfolio moves to the right-upper corner in the corresponding figure. Last but not least, it’s necessary for investors to consider transaction costs which can affect the return when they make portfolio selection.

REFERENCES


