The Maximum (k,m)-Subsets Problem is in the Class NEXP

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***Abstract*—In this paper we define and solve the following new problem: Given a set of n elements, we are interested in determining the largest number of subsets of size k that have at most m elements in common. We prove that in the worst-case scenario, a brute force solution requires a double exponential time with respect to n. Afterwards, we show that our problem is in the class NEXP by reducing it to the succinct K-coloring problem. We use in our proof ordered binary decision diagrams to determine whether there is an edge between two nodes of a graph that corresponds to an instance of our problem. We also show that it is at least EXP-hard.**

***Index Terms*—K-coloring, binary decision diagrams, complexity classes.**

I. Introduction

GIVEN a cluster of n computers, we wish to separate them in tightly connected sub-clusters of size k each, such that any couple of sub-clusters have at most m computers in common. More precisely, we are interested in maximizing the number of sub-clusters that satisfy the abovementioned condition.

Using set theory, we can formulate this problem simply as follows: Let S be a set of n elements and $U_{k,n}$ its subsets of size $k < n$. Compute the largest set $S_k \subseteq U_{k,n}$ such that any couple of S_k share at most m elements. A formal description of our problem is given in Section III. Since two additional parameters (i.e. k and m) are included in the formulation of this problem, we decided to name it the *maximum (k,m)-subsets problem*. To our knowledge, no one has solved it before, although one can find similar problems in the literature such as the maximum subarray problem [24], and the k maximum sums one [27].

Our main aim in this paper is to determine how much hard it is to solve this problem in general, which is a fundamental step prior to proposing any algorithm that answers our question for specific values of n, k, and m. The seminal paper by Cook [2] where he showed that Boolean satisfiability is NP-complete, followed by Karp [6] who extended Cook's result to include twenty-one new NP-complete problems, paved the way for many researchers to classify a wide variety of problems in the computer science field [26], [30], [21].

We determine an upper bound for the maximum (k,m)-subsets problem by reducing it to the succinct K-coloring problem. We already know that the latter is complete for the class NEXP [15]. For technical reasons, our proof in Section V uses a well-known data structure (namely binary decision diagrams), instead of Boolean circuits. It is worth noting that this structure was first introduced by Lee [1], before being investigated further by many computer scientists [7], [8], [9], [20], [4]. We also show in Section IV that this problem is EXP-hard. Determining the difficulty of solving a problem is of utmost importance [33], [25], [32]. Complexity results and optimization algorithms were suggested in other fields such as in mathematics [22], [23], artificial intelligence [3], theoretical computer science [5], [28], [31], and biology [28].

The remainder of this paper is divided as follows.

Section II gives a brief overview of some previously established results that are relevant to our problem, including Boolean circuits and the succinct K-coloring problem. We formally define the maximum (k,m)-subsets problem in Section III. We propose in Section IV a brute force algorithm for solving it. In the worst case scenario, we show that the leading factor is a double exponential with respect to n (we adopt here the notation given by Cormen [19] to compute the running time of our algorithm). We briefly describe ordered binary decision diagrams in Section V, before showing that our problem is in the class NEXP. Section VI concludes our work.

II. Preliminaries

We recall in this section some previously established results we need for solving our problem. As we already stated in the Introduction, our main proof in Section V consists in reducing the maximum (k,m)-subsets problem to the succinct K-coloring problem. Galperin and Wigderson [13] showed that the former problem is in the class NEXP. Later on, Papadimitriou and Yannakakis [15] proved that it is complete for its class.

Generally speaking, we know from Papadimitriou [17] that all the NP-complete problems become NEXP-complete when the input is exponentially more succinct than the description of the original problem. This result can be extended to include many well known NP-complete problems such as SAT, Hamilton Path, K-Coloring, Knapsack, and Max Cut, where Boolean circuits were chosen to (succinctly) represent graphs.

Shannon [18] was the first to apply Boolean logic [10] to electric circuit design. Since then, his seminal work found applications in a wide variety of different fields such as cryptography, mathematics, and biology [11], [14], [12].

Recall that a Boolean circuit is a directed acyclic graph, where a node falls into one of the following three categories: an input node with no incoming edges, a constant node that is either true or false, and a gate node labeled with one of the following logical connectives $\{\vee, \wedge, \neg\}$. Figure 1 includes two examples of such circuits. The one in (a) computes the exclusive-or function, and the one in (b) represents the function $f(x_1, x_2) = (x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2)$.

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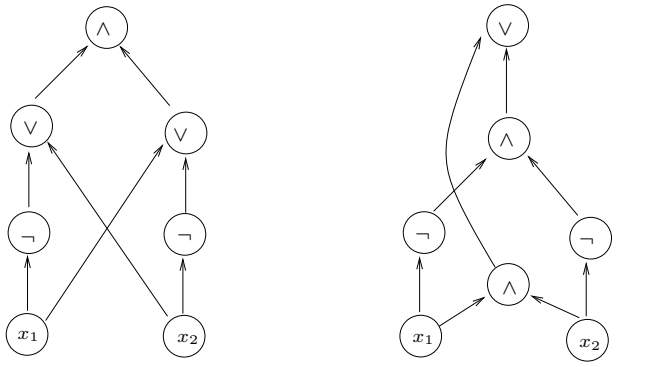

(a) $x_1 \oplus x_2$ (b) $(x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2)$

Fig. 1. Example of Boolean circuits.

Papadimitriou and Yannakakis [15] used a succinct representation of a graph (with exponentially many nodes) in order to prove the NEXP-completeness of the problems they tackled. Formally speaking, given a graph G_C with n nodes (where $n = 2^b$), a succinct representation of G is a Boolean circuit C with $2b$ input nodes such that the following holds: for every couple of nodes $1 \leq i < j \leq n$, there is an edge between i and j if and only if C accepts the binary representation of the b-bit integers i and j.
In our case, we used an ordered binary decision diagram (as described by Bryant [16]) to represent a Boolean function instead of using a Boolean circuit. The reason behind this choice is straightforward. As we shall see in Section V, an ordered binary decision diagram (OBDD) allows us to naturally perform the following task: given a set S of size n and two subsets $L \subseteq S$ and $L' \subseteq S$ of size $k < n$, determine the number of elements the subsets L and L' have in common.
We next formulate our problem.

III. Problem Formulation

Given a set U_n that contains n elements, and given two integers k and m that satisfy $m \leq k < n$, we consider the subsets of U_n that have size k. We know that there are exactly C_n^k such subsets. More precisely, we are interested in solving the problem of determining the maximum number of subsets of size k that have at most m elements in common. Formally, the problem can be defined as follows:

Definition 1: **Maximum (k,m)-subsets problem**
Let n, k, m, where $U_n = \{a_1, \ldots, a_n\}$, and $0 \leq m \leq k < n$. We denote by $U_{k,n} = \{L \subseteq U_n : Card(L) = k\}$ the set of subsets of U_n that have exactly k elements.
We say that $F_{k,n}^m \in 2^{U_{k,n}}$ is a maximum (k,m)-subset of U_n if it satisfies the following conditions:

1) $\forall L, L' \in F_{k,n}^m$, $Card(L \cap L') \leq m$.
2) $\forall F' \in 2^{U_{k,n}}$ that satisfies the above condition, we have $Card(F') \leq Card(F_{k,n}^m)$.

In this case we say that this problem has a solution of size $K = Card(F_{k,n}^m) \leq C_n^k$.

Example 1: Let $U_4 = \{a_1, a_2, a_3, a_4\}$, $k = 2$, and $m = 1$. One can easily check that a solution to this problem is simply given by:
$F_{2,4}^1 = \{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_3, a_4\}\}$, where $F_{2,4}^1$ happens to be equal to the set of all subsets of U_4 of size two (i.e. $U_{2,4}$).
Indeed, each element of $F_{2,4}^1$ has a cardinal equal to two, and the intersection of any couple of elements in $F_{2,4}^1$ is less than or equal to one. To conclude, it is easy to see that the above set is the largest one that satisfies the conditions stated in Definition 1 since $Card(F_{2,4}^1) = C_4^2 = 6$.

Example 2: Let $U_5 = \{a_1, a_2, a_3, a_4, a_5\}$, $k = 3$, and $m = 1$.
It is straightforward to see that a possible solution for this problem is given by $F_{3,5}^1 = \{\{a_1, a_2, a_3\}, \{a_1, a_4, a_5\}\}$.
Note that this solution is not unique since $F' = \{\{a_1, a_2, a_3\}, \{a_3, a_4, a_5\}\}$ is another possible one.
Now any attempt to add a set of size $k = 3$ to $F_{3,5}^1$ results in violating the first constraint given in Definition 1 .

The trivial case where $m = 0$ can be solved in linear time. Indeed, given U_n and a positive integer k, we consider subsets that contain k successive elements from U_n. By construction, the intersection of any two subsets in $F_{k,n}^0$ is empty. Moreover, $card(F_{k,n}^0) = n/k$.

Example 3: Let $U_7 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$, $k = 2$, and $m = 0$.
A solution to our problem has size three:

$$F_{2,7}^0 = \{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}\}$$

and $card(F_{2,7}^0) = 7/2 = 3$.

It is worth noting that if we modify condition (1) in Definition 1 so as to reason with subsets of $U_{k,n}$ that have exactly m elements in common, then solving our problem becomes trivial as shown in Proposition 1.

Proposition 1: Using the same notation given in Definition 1, let the first condition be: $\forall L, L' \in F_{k,n}^m$, $Card(L \cap L') = m$.
We certify that $Card(F_{k,n}^m) = C_{n-m}^{k-m}$.
Without loss of generality, consider any subsets of $U_{k,n}$ that have m elements in common (e.g. $\{a_1, a_2, \ldots, a_m\}$). Once we fix m elements in a subset of size $k \geq m$, we choose the remaining $k-m$ elements out of $n-m$ elements. Therefore, the total number of subsets of size k we can build and that have m elements in common is equal to C_{n-m}^{k-m}.

Example 4: Let U_{100}, $k = 20$, and $m = 10$.
There are at most $C_{100-10}^{20-10} = C_{90}^{10}$ subsets of U_{100} of size 20 that have exactly 10 elements in common.

In the next section we show how to solve our problem using a brute-force approach.

IV. Brute force

Given a set U_n of size n, an integer $k < n$ and a target $0 \leq m < k$, a straightforward algorithm for solving our problem consists in enumerating all the subsets of U_n of size k. Then we process the set of all subsets of $U_{k,n}$ as follows: for each subset we check whether the intersection of any couple of its elements is less than or equal to m. If this is the case, and if the selected subset is the largest one we considered so far, then we update $F_{k,n}^m$ accordingly.

Proposition 2: The running time of Algorithm 1 is

$$T(n) = O(n^2 \times (C_n^k)^2 \times 2^{C_n^k})$$

proof Obviously, lines 1 and 2 require $\Theta(C_n^k + 2^{C_n^k}) = \Theta(2^{C_n^k})$ steps. Next we determine the running time of the

Algorithm 1 Maximum (k,m)-subsets problems

Require: $n, k, m \in \mathbb{N}$, where $k < n$ and $0 \leq m < k$
1: Compute $U_{k,n}$
2: Compute $2^{U_{k,n}}$
3: $F^m_{k,n} = \emptyset$ {Initialization to the empty set}
4: **for all** $T_k \in 2^{U_{k,n}}$ **do**
5: **for all** $L, L' \in S_k$, (where $L \neq L'$) **do**
6: **if** $Card(L \cap L') > m$ **then**
7: break
8: **end if**
9: **end for**
10: **if** $Card(T_k) > Card(F^m_{k,n})$ **then**
11: $F^m_{k,n} = T_k$
12: **end if**
13: **end for**
14: **return** $F^m_{k,n}$

nested two for loops.
Let $T_k \in 2^{U_{k,n}}$. For each couple of elements in T_k, we test whether they have more than m elements in common. If this is the case we discard this set. Otherwise (i.e. we have $\forall L, L' \in T_k;\ |L \cap L'| \leq m$), we compute the cardinal of T_k and, if necessary, update $F^m_{k,n}$ (lines 10 and 11).
Note that computing the intersection of two sets of size k requires $\Theta(k^2)$. Therefore, the test at line 6 requires at most $O(k^2)$ operations. Moreover, we have to repeat this operation for any couple of elements in T_k (i.e. inner for loop at line 5), for a total of $O(k^2) \times \Theta(l(l+1)/2) = O((kl)^2)$, where $Card(T_k) = l$.
Since the for loop at line 4 is repeated $\Theta(2^{C^k_n})$ times, we deduce that the overall running time of the algorithm is $T(n) = O((kl)^2 \times 2^{C^k_n})$.
It is easy to see that $S_{T_k} \in 2^{U_{k,n}}$ may contain up to C^k_n elements, which gives us $l = O(C^k_n)$.
Before concluding, note that k may also depend on n (e.g. $k = n/2$). Therefore the running time of Algorithm 1 is $T(n) = O(n^2 \times (C^k_n)^2 \times 2^{C^k_n})$.

Proposition 3: For $k = \Theta(n)$, we need $O(2^{n-1})$ operations to compute C^k_n.

proof Let $k = \frac{n}{2}$. We have:

$$\begin{aligned} C^k_n &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{\frac{n}{2}!(n-\frac{n}{2})!} \\ &= \frac{n!}{\frac{n}{2}! \times \frac{n}{2}!} \end{aligned}$$

Recalling Stirling's approximation for the factorial function (i.e. $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$):

$$\begin{aligned} C^{\frac{n}{2}}_n &\approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\left(\sqrt{2\pi\frac{n}{2}}\left(\frac{\frac{n}{2}}{e}\right)^{\frac{n}{2}}\right)^2} \\ &\approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{2\pi\frac{n}{2}\left(\frac{n}{e}\right)^n \times \frac{1}{2^n}} \\ &\approx \frac{\sqrt{2\pi n}}{2\pi n} \times 2^{n-1} \\ &= \Theta\left(\frac{1}{\sqrt{n}} \times 2^{n-1}\right) \\ &= O(2^{n-1}) \end{aligned}$$

Corollary 1: A set U_n that contains n elements has $O(2^n)$ subsets of size $k = \Theta(n)$ each.

Corollary 2: In the worst case scenario, Algorithm 1 runs in

$$T(n) = O(n^2 \times 2^{2n} \times 2^{2^{n-1}})$$

where the leading factor is a double exponential.

Proposition 4: The maximum (k,m)-subsets problem is EXP-hard.

proof Let $k = \Theta(n)$. The number of subsets of size k is obviously exponential with respect to n.
Let $m = k - 1$. Our solution $F^m_{k,n}$ contains all the subsets of U_n of size k, and enumerating them requires exponential time.

V. NEXP-completness result

We show in this section that our problem belongs to the class NEXP by reducing it to the succinct K-coloring problem.

Definition 2: We say that an ordered binary decision diagram OBDD represents the set U_n for an integer $m < n$ if it is built as described below, where we assume that the levels of the OBDD are numbered from 1 (i.e. the root) to $n+1$. For all i where $1 \leq i \leq n$, level i contains 2^{i-1} internal nodes a_i.
The last level consists of terminal nodes with values 0 or 1. Given such a node x, we determine its value as follows: We consider the path from the root to x. If the number of times we go to the left is less than or equal to m, then we assign to x the value 1; otherwise we assign it the value 0 (see Figure 2 (a) for an example).

Definition 3: Let an OBDD that represents a set U_n for an integer m. We say that the OBDD accepts two subsets $L \in U_n$ and $L' \in U_n$ if we reach a terminal node with value 1 after following these steps: Starting from the root of the OBDD, and for all $1 \leq i \leq n$ we check to see if the processed node a_i belongs to both L and L'. If this is the case then we go to the left, otherwise we go to the right.

In other words, an OBDD accepts two subsets L and L' if and only if $card(L \cap L') \leq m$.
Examples 5 and 6 illustrate our idea.

Example 5: Let $U_4 = \{a_1, a_2, a_3, a_4\}$ and $m = 1$.
The OBDD in Figure 2 (a) (or its reduced form in part (b)) allows us to reason about the intersection of subsets of U_4.

Given $L = \{a_1, a_3\}$ and $L' = \{a_3, a_4\}$ for example, we check that: a_1, a_2 and a_4 do not belong to $L \cap L'$, and that

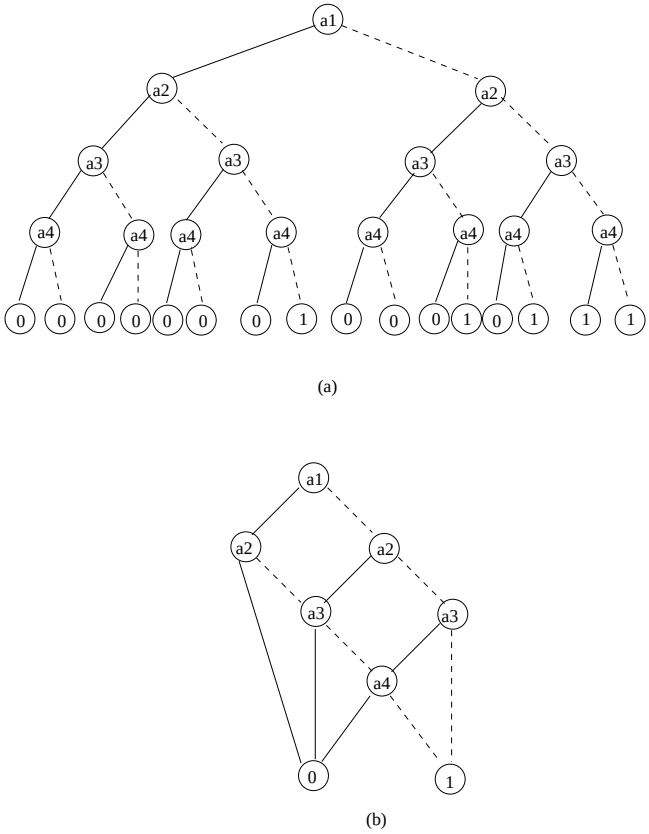

Fig. 2. OBDD (and its reduced form) that represents the set U_4 for $m = 1$.

$a_3 \in L \cap L'$. So starting from the root, we go right-right-left-right.
In this case the OBDD accepts the subsets L and L'.

Example 6: Let $U_5 = \{a_1, a_2, a_3, a_4, a_5\}$ and $m = 1$. Given $L = \{a_1, a_3, a_5\}$ and $L' = \{a_1, a_3, a_4\}$, we easily check that the OBDD in Figure 3 does not accept the subsets L and L'.

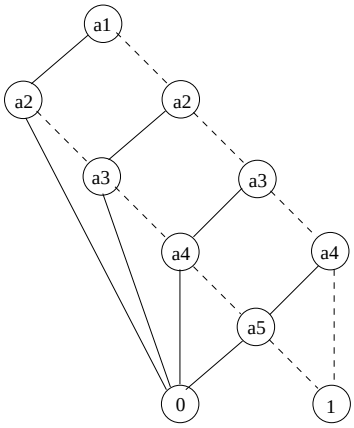

Fig. 3. Reduced OBDD that represents the set U_5 for $m = 1$.

We indicate in Figure 4 the general form of a reduced ordinary binary decision diagram where the initial set has n elements. This efficient representation requires only $O(n)$ space instead of the original $O(2^n)$ space. We notice that we have only two final states (i.e. 0 and 1), and that the graph has two diagonals: one with the elements $a_1, a_2, \ldots, a_{n-1}$, and the other one with the elements $a_2, a_2, \ldots, a_n$.

Below is a pseudo-code that determines whether or not an OBDD accepts two subsets of U_n.

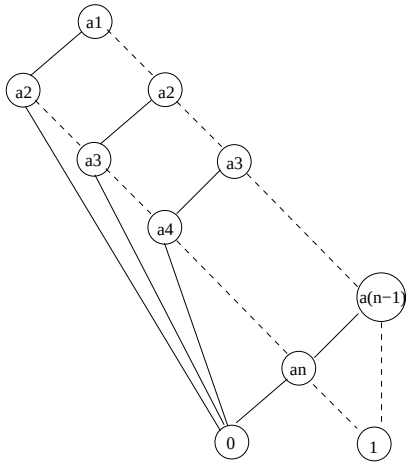

Fig. 4. General form of a reduced OBDD that represents the set U_n for $m = 1$.

Algorithm 2 Return true if $card(L \cap L') \leq m$, and false otherwise.

Require: OBDD representing $U_n = \{a_1, \ldots, a_n\}$ for m
1: Let $L \subseteq U_n$, $L' \subseteq U_n$, where $card(L) = card(L') = k$
2: (Optional) Sort the elements in L and L'
3: $i = 1$, $counter = 0$
4: **while** $(i \leq n)$ **do**
5: **if** $a_i \in L \cap L'$ **then**
6: go to the left
7: $i = i + 1$, $counter = counter + 1$
8: **else**
9: go to the right
10: $i = i + 1$
11: **end if**
12: **end while**
13: **if** $(counter \leq m)$ **then**
14: **return** 1
15: **else**
16: **return** 0
17: **end if**

Proposition 5: Determining if the intersection of two subsets of size k contains less than m elements can be done in $T(n) = O(n \log n)$.

proof The result can be easily derived from Algorithm 2. Indeed, sorting the subsets requires $O(k \log k)$. The test on line 5 can be done in constant time since the subsets are sorted. For that we can simply use two counters j and j' for the subsets L and L', respectively. We compare a_i to $a_j \in L$ and to $a_{j'} \in L'$ in constant time, then increment the counters accordingly. Since the while loop is entered $O(n)$ times, the overall running time of the algorithm is $O(n+klogk)$. In the worst case we have $k = O(n)$, therefore $T(n) = O(n \log n)$. Note that if the subsets are not sorted, then the test on line 5 would require $O(k)$ steps. The leading factor for the while loop (and also for the whole algorithm) becomes $O(kn) = O(n^2)$.

We next prove that our problem is NEXP-complete by reducing to it the succinct K-coloring problem. For this purpose we consider graphs that have C_n^k nodes and where $k = \Theta(n)$. We know from Proposition 3 that they contain an exponential number of nodes with respect to n (and therefore

w.r.t. k).
Intuitively, each node of the graph corresponds to a subset of U_n of size k.

Proposition 6: Let K be some positive integer, and m, k, n such that $0 < m \leq k < n$.
For any instance $P_{m,k,n}$ of the maximum (k,m)-subsets problem, there is a graph $G_{m,k,n}$ such that if $P_{m,k,n}$ has a solution of size K then $G_{m,k,n}$ is K-colorable.
proof Our reduction R is straightforward and can be done in polynomial time with respect to the size of the input.
Intuitively, a node i of the graph $G_{m,k,n}$ is the encoding of a distinct subset of U_n of size k, that we denoted by S_i (recall Definition 1). Let $P_{m,k,n}$ be an instance of the maximum (k,m)-subsets problem. The reduction R associates to this instance the graph $G_{m,k,n}$ with C_n^k vertices, where node i corresponds to the subset S_i. We next consider the OBDD that represents the set U_n for m (as described in Definition 3). For any couple of nodes i and j of $G_{m,k,n}$, we check whether the OBDD accepts the subsets S_i and S_j. If this is the case (i.e. the subsets S_i and S_j have less than m elements in common), we add an edge between i and j; otherwise we don't.
Clearly, if $P_{m,k,n}$ has a solution of size K then $G_{m,k,n}$ is K-colorable.

VI. Conclusion

Given a set S of size n, we tackled in this paper the problem of determining the largest number of subsets of S of size k that have at most m elements in common. We showed that this problem is at least EXP-hard and at most NEXP-complete. Our main proof reduces the maximum (k,m)-subsets problem to the succinct K-coloring problem, and uses very well-known data structures to represent graphs; namely ordered binary decision diagrams.
Besides showing whether this problem is EXP-complete or NEXP-complete, we intend to determine its tractable instances. More precisely, we would like to investigate whether there is a polynomial-time solution to our problem for some specific values of m, k, and n.

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