Some Results of Derivations on FI-lattices

Mei Wang, Ting Qian, Jun Tao Wang*

***Abstract*—In this paper, we further study implicative derivations on FI-lattices and obtain some new results of them. In particular, we prove that the set of all implicative derivations on FI-lattice forms a monoid. Also, we generalize the implicative derivations by homomorphism, which is called generalized implicative derivation, on FI-lattices and obtain some important results of them. The results of this paper generalize the corresponding results of derivations on the logical algebras of t-norm based fuzzy logic.**

***Index Terms*—Logical algebra, FI-lattice, (generalized) implicative derivation.**

I. Introduction

FUZZY logic takes the advantage of the classical logic to handle uncertain information and fuzzy information. In recent decades, various logical algebras have been proposed as the semantical systems of fuzzy logic systems, for example, MV-algebras [1], BL-algebras [2], MTL-algebras [3], residuated lattices [4], hoops[5] and FI-algebras [16]. In the past decades, these fuzzy logical algebras and their axiomatization have become important topics in theoretical research and in the applications of fuzzy logic. It is well known that the fuzzy implication connective plays a crucial role in fuzzy logic and reasoning. Various authors studied fuzzy implications from different perspectives. Seeking the common properties of some important fuzzy implications used in fuzzy logic is meaningful. Consequentially, Wu[16] introduced a class of fuzzy logical algebras called fuzzy implication algebras, FI-algebras for short, in 1990. After then, various interesting results of FI-algebras, regular FI-algebras[14], commutative FI-algebras[17] were reported. In 2011, Pei[15] introduced a new type of FI-algebras, called fuzzy implication lattices, FI-lattices for short, and studied some results.

The notion of derivations, introduced from the analytic theory, is helpful for studying algebraic structures and properties in algebraic systems. In 1957, Posner [12] introduced the notion of derivations in a prime ring $(R, +, \cdot)$, which is a mapping $d : R \longrightarrow R$ satisfying the following conditions:

$$\text{(i) } d(x + y) = d(x) + d(y), \text{ (ii)}$$
$$d(x \cdot y) = (d(x) \cdot y) + (x \cdot d(y)),$$

for any $x, y \in R$, and serval authors also gave some characterizations of commutative prime ring in terms of derivations [31], [32]. Subsequently, Jun et. al [11] applied the notion of derivations to BCI-algebras $(L, *)$, which is a mapping $d : L \longrightarrow L$ satisfying the following conditions:

$$d(x * y) = (d(x) * y) \wedge (x * d(y)),$$

and gave characterizations of p-semisimple BCI-algebra by regular derivations. Inspired by this, several authors have studied generalized derivations in BCI-algebras [8], [24]. In the past few years, Xin [19], [20] introduced the concept of derivations in a lattice, where operations $+$ and $\cdot$ are interpreted as lattice operations $\vee$ and $\wedge$, respectively, which is a mapping $d : L \longrightarrow L$ satisfying the following conditions:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)),$$

and characterized modular lattices and distributive lattices by isotone derivations; Alshehri et. al [7], [9], [33] introduced the notion of (additive) derivations for an MV-algebra, where operations $+$ and $\cdot$ are interpreted as $\oplus$ and $\odot$, which is a mapping $d : L \longrightarrow L$ satisfying the following conditions:

$$d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y)),$$

and discussed some related properties; Xin et. al [34], [35] also introduced and studied derivation on BL-algebras and give some characterization of Gödel algebras in terms of derivations; Sang and Yong [13], [22] investigate derivation and generalized derivation in lattice implication algebra and characterized the fixed set by these derivations; He [10] investigated derivations in residuated lattices and characterize Heyting algebras in terms of derivations; Zhu [23] introduced some derivations in linguistic truth-valued lattice implication algebras and discussed the relationship between them; Wang [21] investigated derivations in commutative multiplicative semilattices and characterize quantales in terms of derivations; Liang [25] introduced the notion of derivations of EQ-algebra and gave some characterizations of them. In 2019, Wang [5] introduced the notion of implicative derivations on FI-lattices and studied some of their basic properties. 2020, Zhu et. al introduced and studied generalized derivations of residuated lattices and obtain some related results [26], [27], [28], [29].

The paper is organized as follows: In Section 2, we review some basic definitions and results about FI-lattices. In Section 3, we review the notion of implicative derivations on FI-lattices and give some characterizations of them. In Section 4, we introduce the notion of generalized implicative derivations and study some of their related properties of FI-lattcies.

II. Preliminaries

In this section, we summarize some definitions and results about FI-lattices, which will be used in the following sections.

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Mei Wang is a doctoral student at the School of Arts and Sciences, Shaanxi University of Science & Technology, Xi'an 710021, China, e-mail: wm311@126.com.

Ting Qian is a Lecturer at the School of Science, Xi'an Shiyou University, Xi'an 710065, China, e-mail: qiant2000@126.com,

Juntao Wang is a Lecturer at the School of Science, Xi'an Shiyou University, Xi'an 710065, China, wjt@xsyu.edu.cn.

Definition *2.1 ([15]):* An algebra $(L, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a *FI-lattice* if it satisfies the following conditions:

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(2) $p \rightarrow (q \rightarrow w) = q \rightarrow (p \rightarrow w)$,
(3) $(p \rightarrow q) \rightarrow ((q \rightarrow w) \rightarrow (p \rightarrow w)) = 1$,
(4) $p \rightarrow p = 1$,
(5) if $p \rightarrow p = 1$ and $q \rightarrow p = 1$, then $p = q$,
(6) $0 \rightarrow p = 1$,
(7) $p \rightarrow (p \vee w) = (p \rightarrow q) \vee (p \rightarrow w)$,

for any $p, q, w \in L$.

In what follows, by L we denote the universe of a FI-lattice $(L, \wedge, \vee, \rightarrow, 0, 1)$. On each L, we define

$$p \leq q \text{ iff } p \rightarrow q = 1.$$

Then $\leq$ is a partial order relation on L and for all $p \in L, 0 \leq p \leq 1$.

Proposition *2.2 ([14], [15], [16], [17], [27], [30]):* In any FI-lattice, the following properties hold: for all $p, q, w \in L$,

(1) if $p \leq q$, then $q \rightarrow w \leq p \rightarrow w$, $w \rightarrow p \leq w \rightarrow q$,
(2) $p \rightarrow (q \rightarrow p) = 1$,
(3) $1 \rightarrow p = p$,
(4) $((p \rightarrow q) \rightarrow q) \rightarrow q = p \rightarrow q$,
(5) $(p \vee q) \rightarrow w = (p \rightarrow w) \wedge (q \rightarrow w)$.

Definition *2.3:* Let L_1 and L_2 be two FI-lattices, $f : L_1 \rightarrow L_2$ be a homomorphism of FI-lattices if for all $p, q \in L$,

(1) $f(p \wedge q) = f(p) \wedge f(q)$,
(2) $f(p \vee q) = f(p) \vee f(q)$,
(3) $f(p \odot q) = f(p) \odot f(q)$.

III. Implicative derivations on FI-lattices

In this section, we review implicative derivations in FI-lattices and give some characterizations of them. Then we further study some algebraic properties of implicative derivations on FI-lattices.

Definition *3.1:* [5] Let L be a FI-lattice. A map $I : L \longrightarrow L$ is called an *implicative derivation* on L if it satisfies the following condition:

$$I(p \rightarrow q) = (I(p) \rightarrow q) \vee (p \rightarrow I(q)),$$

for any $p, q \in L$.

We will denote by $I(L)$ be the set of all implicative derivations of L.

Now, we present some examples for implicative derivations on FI-lattices.

Example *3.2:* (1) Define a map on L by $1_I : L \rightarrow L$ $1_I(p) = 1$ for all $p \in L$. Then 1_I is an implicative derivation on L, which is called the one implicative derivation. Moreover, we define a map $I_1 : L \rightarrow L$ by $I_1(p) = p$ for all $p \in L$. Then I_1 is also an implicative derivation on L, which is called the identity implicative derivation.

(2) Let L be a FI-lattice. Then $I_a(p) = a \rightarrow p$ for any $a, p \in L$ is an implicative derivation on L.

(3) Let $L = [0, 1]$ be the unit interval. Defining $\odot$ and $\rightarrow$ on L as follows:

$$p \vee q = \max\{p, q\},\ p \wedge q = \min\{p, q\}.$$

$$p \rightarrow q = \begin{cases} 1, & p \leq q, \\ q, & p \geq q \end{cases}$$

Then $(L, \wedge, \vee, \rightarrow, 0, 1)$ is a FI-lattice. Now, we define $I : L$ as follows:

$$I(p) = \begin{cases} p, & p < 0.5, \\ 1, & p \geq 0.5. \end{cases}$$

Then I is an implicative derivation on L.

(4) Let $L = \{0, p, q, w, 1\}$ be a chain. Defining operation $\rightarrow$ is as follows:

$\rightarrow$	0	p	q	w	1
0	1	1	1	1	1
p	q	1	1	1	1
q	p	p	1	1	1
w	0	p	q	1	1
1	0	p	q	w	1

Then $(L, \wedge, \vee, \rightarrow, 0, 1)$ is a FI-lattice. Now, we define $I : L \rightarrow L$ as follow:

$$I(x) = \begin{cases} 0, & x = 0, \\ p, & x = p, \\ q, & x = q, \\ 1, & x = w, 1 \end{cases},$$

It is verified that I is an implicative derivation on L.

Proposition *3.3:* [5] Let L be a FI-lattice and I be an implicative derivation on L. Then we have: for any $p, q \in L$,

(1) $I(1) = 1$,
(2) $p \leq I(p)$,
(3) $I(p) \wedge I(q) \leq I(p \rightarrow q)$,
(4) $I(p) \rightarrow q \leq p \rightarrow I(q)$,
(5) $I(p \rightarrow q) = p \rightarrow I(q)$,
(6) $I(p \rightarrow q) \geq I(p) \rightarrow I(q)$.

Theorem *3.4:* [5] Let L be a FI-lattice and $I : L \rightarrow L$ be a map on L. Then the following statements are equivalent:

(1) I is an implicative derivation on L,
(2) $I(p \rightarrow q) = p \rightarrow I(q)$ for any $p, q \in L$.

Proposition *3.5:* Let L be a FI-lattice and I an implicative derivation on L. Then the following conditions hold, for any $p, q \in L$,

(1) $I(\neg p) = p \rightarrow I(0)$,
(2) $I(0) \leq I(\neg p)$,
(3) if $I(0) = 0$, then $I(\neg p) = \neg p$,
(4) if $p \leq I(0)$, then $I(\neg p) = 1$,
(5) $I^n(p \rightarrow q) = p \rightarrow I^n(q)$, for any $n \geq 1$,
(6) $I(p) \rightarrow q \leq p \rightarrow I(q)$, $I(q) \leq I(p \rightarrow q)$ and so $q \leq I(p \rightarrow q)$,
(7) if $p \leq q$, then $p \leq I(q)$,
(8) if F is an upper set of L, then $I(F) \subseteq F$,
(9) if $p \in B(L)$, then $I(p) \vee \neg p = p \vee I(\neg p) = I(p) \vee I(\neg p) = 1$.

Proof: (1) It is verified that

$$I(\neg p) = I(p \rightarrow 0).$$

Moreover, by Theorem 3.4, we have

$$I(p \rightarrow 0) = p \rightarrow I(0),$$

which shows that $I(\neg p) = p \rightarrow I(0)$ for any $p \in L$.

(2) It follows from (1) that

$$I(\neg p) = p \rightarrow I(0) \geq I(0),$$

which shows that $I(0) \leq I(\neg p)$ for any $p \in L$.

(3) If $I(0) = 0$, then by (2), we have

$$I(\neg p) = p \to I(0) = p \to 0 = \neg p,$$

for any $x \in L$.

(4) If $x \leq I(0)$, then $p \to I(0) = 1$. Moreover, by Theorem 3.4, we have

$$I(\neg p) = I(p \to 0) = p \to I(0),$$

which shows that $I(\neg p) = 1$.

(5) For $n = 1$, by Theorem 3.4, we have

$$I(p \to q) = p \to I(q)$$

and for $n = 2$,

$$\begin{aligned} I^2(p \to q) &= I(I(p \to q)) \\ &= I(p \to I(q)) \\ &= p \to I(I(q)) \\ &= p \to I^2(q). \end{aligned}$$

(6) By Proposition 3.3(2), we have $p \leq I(p)$. So

$$I(p) \to q \leq p \to q$$

and since $q \leq I(q)$, we also have

$$p \to q \leq p \to I(q),$$

which shows that $I(p) \to q \leq p \to I(q)$, for any $p, q \in L$. Moreover, we have

$$I(q) \leq p \to I(q) = I(p \to q).$$

So $I(q) \leq I(p \to q)$. Now, we prove $q \leq I(p \to q)$, by Proposition 3.3(2), we have $q \leq I(q)$, and hence

$$p \to q \leq p \to I(q).$$

Since $q \leq p \to q$, by Theorem 3.4, we have

$$q \leq p \to I(q) = I(p \to q)$$

for any $p, q \in L$.

(7) If $p \leq q$, then $p \to q = 1$. So by Proposition 3.3(1) and Theorem 3.4, we have

$$1 = I(1) = I(p \to q) = p \to I(q),$$

which shows that $p \leq I(p)$.

(8) If F is an upper set of L and $x \in I(F)$, then there exists an element $f \in F$ such that $x = I(f)$. By Proposition 3.3 (2), we have $f \leq I(f)$. Since $f \in F$, then $I(f) \in F$.

(9) By Proposition 3.3(2), we have

$$p \vee \neg p \leq I(p) \vee \neg p.$$

Since $x \in B(L)$, we have

$$1 = p \vee \neg p \leq I(p) \vee \neg p,$$

which shows that

$$I(p) \vee \neg p = 1.$$

Similarity, we can obtain

$$1 = p \vee \neg p \leq p \vee I(\neg p),$$

which shows that

$$p \vee I(\neg p) = 1.$$

■

Proposition *3.6:* If I_1,I_2 are two implicative derivation on a FI-lattice L, then $I_1 \circ I_2$ is an implicative derivation on L.

Proof: For any implicative derivations I_1, I_2, we have

$$\begin{aligned} (I_i \circ I_j)(p \to q) &= I_1(I_2(p \to q)) \\ &= I_1(p \to I_2(y)) \\ &= p \to I_1(I_2(y)) \\ &= p \to (I_i \circ I_j)(q), \end{aligned}$$

which shows that $I_1 \circ I_2$ is an implicative derivation on L. ■

Corollary *3.7:* $(I(L), \circ, I_1)$ is a monoid.

Proof: It follows from Proposition 3.6. ■

In the following proposition, we observe a result about the implicative derivation of the direct product of FI-lattices. In fact, let L_1 and L_2 be two FI-lattices. Then $L_1 \times L_2$ is also a FI-lattice with respect to the point-wise operators given by:

$$\begin{gathered} (p_1, p_2) \wedge (q_1, q_2) = (p_1 \wedge q_1, p_2 \wedge q_2), \\ (p_1, p_2) \vee (q_1, q_2) = (p_1 \vee q_1, p_2 \vee q_2), \\ (p_1, p_2) \to (q_1, q_2) = (p_1 \to q_1, p_2 \to q_2). \end{gathered}$$

Proposition *3.8:* Let L_1 and L_2 be two FI-lattices. Define a map $I : L_1 \times L_2 \to L_1 \times L_2$ by

$$I(p, q) = I(p, 1)$$

for all $(p, q) \in L_1 \times L_2$. Then I is an implicative derivation of $L_1 \times L_2$ with respect to point-wise operations.

Proof: Let $(p_1, p_2), (q_1, q_2) \in L_1 \times L_2$. Then we have

$$\begin{aligned} I((p_1, p_2) \to (q_1, q_2)) &= I(p_1 \to q_1, p_2 \to q_2) \\ &= (p_1 \to q_1, 1) \\ &= (p_1 \to q_1, p_2 \to 1) \\ &= (p_1, p_2) \to (q_1, 1) \\ &= (p_1, p_2) \to I(q_1, q_2), \end{aligned}$$

which shows that I is an implicative derivation on the direct product $L_1 \times L_2$. ■

The following theorem is an extension of the above proposition.

Theorem *3.9:* Let $\{L_i\}_{i \in \triangle}$ be a family of FI-lattices and $\triangle_0$ a finite subset of $\triangle$. Let

$$I_{\triangle_0} : \prod_{i \in \triangle} L_i \to \prod_{i \in \triangle} L_i$$

defined by

$$I((p_i)_{i \in \triangle}) = (q_i)_{i \in \triangle}$$

where

$$q_i = \begin{cases} p_i, & i \in \triangle_0, \\ 1, & i \notin \triangle_0 \end{cases}.$$

Then $I_{\triangle_0}$ is an implicative derivation of the FI-lattice $\prod_{i \in \triangle} L_i$.

Proof: It follows from Proposition 3.8. ■

IV. Generalized implicative derivations on FI-lattices

In this section, we introduce the notion of generalized implicative derivations on FI-lattices and study some of their algebraic properties.

***Definition** 4.1:* Let L be a FI-lattice. A map $GI : L \longrightarrow L$ is called a *generalized implicative derivation* on L for the given two homomorphisms $f, g : L \to L$, if

$$GI(p \to q) = (GI(p) \to f(q)) \vee (g(p) \to GI(q)),$$

for any $p, q \in L$.

***Remark** 4.2:* (1) If we replace the homomorphisms f and g by the identity map I_1, then we have $GI = I$. In this case, one can see that every implicative derivation is a generalized implicative derivation on FI-lattice.

(2) A generalized implicative derivation is called a f-implicative derivation if $f = g$ in Definition 4.1. It is noted that every implicative derivation is a f-implicative derivation on FI-lattice.

***Example** 4.3:* Let $L = \{0, p, q, w, 1\}$ such that $0 \le p, q \le w \le 1$. . Defining operation $\to$ is as follows:

$\to$	0	p	q	w	1
0	1	1	1	1	1
p	q	1	q	1	1
q	p	p	1	1	1
w	0	p	q	1	1
1	0	p	q	w	1

Then $(L, \wedge, \vee, \to, 0, 1)$ is a FI-lattice. Now, we define $I : L \to L$ as follow:

$$I(x) = \begin{cases} 0, & x = 0, \\ p, & x = q, \\ 1, & x = p, w, 1, \end{cases} \quad , \qquad f(x) = \begin{cases} 0, & x = 0 \\ q, & x = p \\ p, & x = q \\ w, & x = w \\ 1, & x = 1 \end{cases}$$

$$g(x) = \begin{cases} 0, & x = 0 \\ q, & x = p \\ p, & x = q \\ 1, & x = w, 1 \end{cases}$$

It is verified that GI is a generalized implicative derivation on L.

***Proposition** 4.4:* Let L be a FI-lattice and I a generalized implicative derivation on L. Then the following statements are satisfied, for any $p, q \in L$,

(1) $GI(1) = 1$,
(2) $GI(p) \le f(p)$ or $g(p) \le GI(p)$,
(3) If $GI(p) = 1$ and $g(p) \to GI(p) \ne 1$, then $f(p) = 1$,
(4) If $g(p) \to GI(\neg\neg p) \ne 1$, then $GI(p) \le f(\neg\neg p)$,
(5) If $g(p) \to GI(p) \ne 1$ and $g(p) \to GI(\neg\neg p) \ne 1$, then $GI(\neg p) \le \neg GI(p)$,
(6) If $g(\neg p) \to GI(\neg p) \ne 1$, then $GI(\neg p) \le f(\neg x)$,
(7) $f(p) \le GI(p)$,
(8) $GI(p) \vee GI(q) \le GI(p \to q)$,

Proof: (1) By Definition 4.1, we have

$$\begin{aligned} GI(1) &= GI(1 \to 1) \\ &= (G1(1) \to f(1)) \vee (g(1) \to GI(1)) \\ &= (GI(1) \to 1) \vee (1 \to GI(1)) \\ &= 1 \vee (1 \to G1(1)) \\ &= 1. \end{aligned}$$

(2) By (1), we have

$$(GI(p) \to f(p)) \vee (g(p) \to GI(P))1,$$

and hence

$$GI(p) \to f(p) = 1 \text{ or } g(p) \to GI(p) = 1,$$

that is, $GI(p) \le f(p)$ or $g(p) \le GI(p)$.

(3) Since $g(p) \to GI(p) \ne 1$ and $GI(p) = 1$, by (2), we have $GI(p) \le f(p)$ and $1 \le f(p)$, which show that $f(p) = 1$.

(4) Since $x \le \neg\neg x$, we have $x \to \neg\neg x = 1$, and hence

$$\begin{aligned} GI(p \to \neg\neg p) &= (GI(p) \to f(\neg\neg p)) \vee (g(p) \to G1(\neg\neg p)) \\ &= 1, \end{aligned}$$

and so $GI(p) \to f(\neg\neg p) = 1$, that is, $GI(p) \le \neg\neg f(p)$.

(5) Since $\neg\neg\neg p = \neg p$, and $GI(p) \le \neg\neg f(p)$, we have

$$GI(\neg x) \le \neg\neg f(\neg p) = f(\neg p),$$

and thus $GI(p) \le f(p)$, that is, $\neg f(p) \le \neg GI(p)$, hence $GI(\neg x) \le \neg GI(x)$.

(6) By Definition 4.1, we have

$$\begin{aligned} GI(\neg p \to \neg p \neg) &= (GI(\neg p) \to f(\neg p)) \vee (g(\neg p) \to GI(\neg p)) \\ &= 1, \end{aligned}$$

and hence $GI(\neg p) \to f(\neg p) = 1$, that is, $GI(\neg x) \le f(\neg x)$.

(7) By (1), we have

$$\begin{aligned} GI(1 \to x) &= (GI(1) \to f(p)) \vee (g(1) \to GI(p)) \\ &= (1 \to f(p)) \vee (1 \to GI(p)) \\ &= (1 \to f(p)) \vee (1 \to GI(p)) \\ &= GI(p) \vee f(p), \end{aligned}$$

which show that $f(p) \le GI(p)$.

(8) By $GI(p) \le f(p) \to GI(p)$ and $GI(q) \le g(p) \to GI(q)$, we have

$$\begin{aligned} GI(p) \vee GI(q) &\le (f(q) \to GI(p)) \vee (g(p) \to GI(q)) \\ &= GI(p \to q). \end{aligned}$$

■

***Theorem** 4.5:* Let f be an endomorphism of a FI-lattice L and GI_a a unary operation defined on L by

$$GI_a(p) = a \to f(p)$$

for all $p \in L$, where a is a given element of L. Then teh following statements are satisfied:

(1) GI_a is a f-implicative derivation on L,
(2) GI_a is monotone,
(3) $GI_a(p) \vee GI_a(q) \le GI_a(p \vee q)$ and $GI_a(p \wedge q) = GI_a(p) \wedge GI_a(q)$ for all $p, q \in L$.

Proof: (1) Let f be an endomorphism of a FI-lattice L. Then we have

$$\begin{aligned} GI_a(p \to q) &= a \to (f(p) \to f(q)) \\ &= ((a \to f(p)) \to f(q)) \vee (a \to ((f(p \to q))) \\ &= (GI_a(p) \to f(q)) \vee (f(p) \to GI_a(q)), \end{aligned}$$

which show that GI_a is a generalized implicative derivation on L.

(2) It follows from the fact that f is a homomorphism.
(3 It follows from Definition 4.1. ■

V. Conclusion

The notion of derivations is helpful for studying structures and properties in algebraic systems. In this paper, we review the implicative derivations on FI-lattices and give some characterizations of them. We also introduce the notion of generalized implicative derivations on FI-lattices and obtain some important results. In our further work, we will introduce the other types of derivations on FI-algebras and discuss the relationship between implicative derivations and them.

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