Non-equidistance Fractional Order Accumulation Grey Model NFGM (1, 1) and Its Application

Xiaogao Yang, Deqiong Ding*, Youxin Luo

Abstract—**In scientific research and engineering applications, there are a large number of research objects with non-equidistant fractional-order characteristics. The existing fractional-order grey theory model was no longer suitable for solving such problems. Based on the integer-order grey theory model, the non-equidistant fractional-order accumulation generation and subtraction generation operators are proposed. The background value of the non-equidistant grey theory analysis model is reconstructed. By taking fractional-order as a design variable and minimum mean relative error and minimum mean square as an objective function, two kinds of optimization models are established. In the application of strength, it is found that the accuracy of the background values reconstructed by the two models is much higher than that of the reference analysis results. The proposed model in the present paper improves the accuracy of the grey theory analysis model, expands the scope of application of the model, and provides references for scientific research and practical application of engineering.**

Index Terms—**Non-equidistance, fractional-order accumulation grey model, background value, average generation**

I. INTRODUCTION

THE Numerical simulation and analysis play an important role in engineering application and scientific research [1],[2],[3]. A large number of effective numerical simulation analyses [4],[5],[6],[7] have been carried out by domestic and foreign scientists and professional technicians. The accuracy of the model description of the research object is related to the validity of the analysis conclusion. A large number of effective data is the basis of numerical analysis and simulation. However, there is a large number of information and poor data phenomenon in scientific research and engineering applications. At this time, the accuracy of numerical analysis model description and data mining ability become the key to effective analysis and research. The GM (1, 1) model plays an important role in grey prediction because it has the characteristics of an accurate description model and high precision in dealing with practical engineering problems with small samples and poor information [8],[9]. To this end, a large number of analysis models were derived from the basic grey model GM (1, 1) to solve practical engineering problems [10],[11],[12],[13], which have made the grey theory developed rapidly and widely used in engineering application. Because the grey model GM (1, 1) is an integer-order derivative model, it is not suitable for analyzing systems with fractional order characteristics. When GM (1, 1) is applied to the practical engineering system with fractional-order characteristics, the problems such as low fitting accuracy of background value and inaccurate description of the system will occur. All these lead to the fact that the essential attributes and behaviors of the research object cannot be revealed. In order to analyze the research system with fractional-order characteristics, Wu et al. [14] have proposed the fractional-order cumulative grey model GM (α, n). Wu et al. [15] have also used the improved FAGM (1,1, t^{α}) to forecast the health expenditure of China. Then, a discrete grey model also has been proposed based on fractional-order accumulate [16]. Based on the matrix perturbation theory, Wu et al. proposed a fractional-order cumulative grey model suitable for discrete grey system to reduce the perturbation bound [17]. In order to make the grey theory widely used and popularized, a large number of research literature have extended the grey theory model, which has the characteristics of describing fractional-order. Wu et al. [18] established a direct grey model of fractional-order new information preference NIGM (1, 1). Yang et al. [19] established a fractional-order discrete grey power function model GM (1, 1). Wu et al. [20] established the fractional-order grey cumulative model GM (1, 1). These studies have a good effect on the analysis of systems with equidistant fractional-order characteristics, thus promoting the development of grey theory. With the deepening of research, a large number of non-equidistant research objects have been often encountered in scientific research and

Manuscript received June 10, 2020; revised May 20, 2021.This work is supported by the fund items as follows: the science and technology plan project of Changde science and Technology Bureau (CKH (2020) 75|No.2020C084 and No.2020S109), the Dr. Scientific research start project (NO. 403|E0 7016046), the scientific Research Foundation of Hunan Provincial Education Department (17A147), the project of Hunan Provincial Department of Education (19c1271), the provincial specialty disciplines of higher education institutions in Hunan Province (XJT [2018]469), the general project of ChangDe science and Technology Bureau (2018J051), the provincial specialty disciplines of higher education institutions in Hunan Province (XJT [2019]379), the Hunan provincial and Municipal Joint Fund Project (2020JJ6006), the key projects of Hunan Education Department (19A341).

Xiaogao Yang is an associate professor of the College of Mechanical Engineering, Hunan University of Arts and Science, ChangDe, 415000, China (e-mail: yxg_568@126.com).

Deqiong Ding* is an associate professor of the college of Mathematics and Statistics, Hunan University of Finance and Economics, ChangSha, 410205, China (e-mail: ddqshe2010@163.com).

Youxin Luo is a Professor of College of Mechanical Engineering, Hunan University of Arts and Science, ChangDe, 415000, China (e-mail: llyx123@126.com)

engineering application. Through the analysis, it is not difficult to find that the equidistant grey theory is only a special case of the non-equidistant problem. The existing fractional-order grey theory model was no longer suitable for solving such problems. In order to expand the application scope of grey theory and the accuracy of data analysis, it is necessary to reconstruct the background value. By taking fractional-order as design variable and minimum mean relative error and minimum mean square as an objective function, two kinds of optimization models were established in the present paper. The two NFGM (1, 1) models of non-equidistant fractional-order series established showed good background value reconstruction and fitting accuracy through example data fitting and comparative analysis. Therefore, it was of great significance to establish fractional-order cumulative grey models of NFGM (1, 1) for solving practical problems in scientific research and engineering and promoting the development of grey theory.

II. NON-EQUIDISTANCE FRACTIONAL ORDER ACCUMULATION GREY MODEL NFGM (1, 1)

A. Non-equidistance Sequence

Given a non-negative sequence such as

$$X^{(0)}=[x^{(0)}(t_1),\cdots,x^{(0)}(t_m)] \tag{1}$$

if the variable spacing of the sequence $\Delta t_i = t_i - t_{i-1} \neq$ constant (i=2… m), $X^{(0)}$ is a non-equidistance sequence [21].

B. First Order Cumulative Generating Operation Sequence

For a given non equidistant sequence as shown below:

$$X^{(1)} = [x^{(1)}(t_1), x^{(1)}(t_2), \ldots x^{(1)}(t_m)] \tag{2}$$

if $x^{(1)}(t_1) = x^{(0)}(t_1)$, $x^{(1)}(t_{k+1}) = x^{(1)}(t_k) + x^{(0)}(t_{k+1}) \cdot \Delta t_{k+1}$, in which k=1…m-1, then $X^{(1)}$ is the first order cumulative generating operation sequence of non-equidistance sequence.

C. R-th Accumulated Generation Sequence

Given a non-negative sequence $X^{(0)}=[x^{(0)}(t_1)\cdots x^{(0)}(t_m)]$, if $\Delta t_i = t_i - t_{i-1} \neq$ constant (i=2… m), $X^{(r)} = [x^{(r)}(t_1) \ldots x^{(r)}(t_m)]$ is the r-th accumulated generation sequence, where

$$x^{(r)}(t_k)=\sum_{i=1}^{k}x^{(r-1)}(t_i)\Delta t_i \tag{3}$$

Namely,

$$x^{(r)}(t_k)=\begin{cases} x^{(r)}(t_{k-1})+x^{(r-1)}(t_k)\Delta t_k, & k=2,3,\cdots,m \\ x^{(r)}(t_1)=x^{(0)}(t_1), & \mathrm{k}=1 \end{cases}$$

D. R-th Accumulated Generation Matrix A^r

Based on the accumulated generation sequence in definition 3, the following expression can be obtained by matrix operation

$$x^{(r)}=Ax^{(r-1)}=AAx^{(r-2)}=\cdots=A^r x^{(0)} \tag{4}$$

in which, the A is an accumulated generating matrix as below

$$A=\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix},$$

A^r is r-th accumulated generating matrix $A^r = (a^r_{i1j1})_{m\times m}$.

$$a^r_{i_1 j_1}=\begin{cases} \dfrac{(i_1-j_1+r-1)!}{(r-1)!(i-j)!} & i_1 \geq j_1 \\ 0 & i_1 < j \end{cases}$$

So, the A^r can be expressed as

$$\begin{vmatrix} 1 & 0 & \cdots & 0 \\ r & 0 & \cdots & 0 \\ \dfrac{r(r+1)}{2!} & r & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \dfrac{r(r+1)\cdots(r+m-2)}{(m-1)!} & \dfrac{r(r+1)\cdots(r+m-3)}{(m-2)!} & \cdots & 1 \end{vmatrix} \tag{5}$$

For r in the formula, the combination number is generalized from integer to fraction; then the fractional-order cumulative generating matrix is derived. So that the fractional-order cumulative generating matrix can be obtained. If A^r satisfies the conditions: $(A^r)^{-1} = A^{-r}$, $A^r A^{-r} = I$ and $x^{(0)} = A^{-r} A^r x^{(0)} = A^{-r} x^{(r)}$, the A^r can be defined as the r-th inverse accumulated generating matrix. In that case, the A^r can be evolved into the original sequence $x^{(0)}$. The coefficient of $A^{(0)}(t_i)$ in the $A^{(r)}(t_k)$ can be expressed with the Gamma function Γ [17]. Suppose the coefficient of $A^{(0)}(t_i)$ is a_k,

$$a_k=\frac{\Gamma(r+k-1)}{\Gamma(k-i+1)\Gamma(r)} \quad (i=1\cdots m) \tag{6}$$

The r-th inverse accumulated generating matrix can be expressed as follows.

$$x^{(r)}(t_k)=\sum_{i=1}^{k}\frac{\Gamma(r+k-1)}{\Gamma(k-i+1)\Gamma(r)}(x^{(0)}(t_i)\Delta t_i),\ k=1,2,\cdots,m \tag{7}$$

Note: when r is an integer, the calculation formula (3) holds. The formula (7) is the description form of fraction expanded by integer r. The fractional-order $x^{(r)}(t_k)$ can be calculated using the formula (7).

E. New Non-equidistance Fractional-order Accumulated Grey Model NFGM (1,1)

Suppose $x^{(r-1)}(t_k)=x^{(r)}(t_k)-x^{(r)}(t_{k-1})$ (where k=2, 3… m), the non-equidistance fractional order accumulated grey model GM (1, 1)

$$x^{(r-1)}(t_k)+aZ^{(r)}(t_k)=b \tag{8}$$

will evolve into a new non-equidistance fractional-order accumulation grey model marked as the NFGM (1, 1). The parameters a, b, and $z^{(r)}=[z^{(r)}(t_2)\ z^{(r)}(t_3)\cdots z^{(r)}(t_m)\ ]$ is the model derived coefficient, model actuating quantity, and background value, respectively. When the average value is generated, the $Z^{(r)}(t_k)$ can be expressed as

$$Z^{(r)}(t_k)=0.5x^{(r)}(t_{k-1})+0.5x^{(r)}(t_k) \tag{9}$$

If the parameters in the fractional-order accumulated grey model $x^{(r-1)}(t_k)+aZ^{(r)}(t_k)=b$ can be estimated with the least square method, the parameters can be expressed as formula (10),

$$P=\begin{pmatrix} a \\ b \end{pmatrix}=(B^{\mathrm{T}}B)^{\mathrm{T}}B^{\mathrm{T}}Y \tag{10}$$

where Y in the formula (10),

$$Y=\left|\begin{matrix} x^{(r-1)}(t_2) \\ x^{(r-1)}(t_3) \\ \mathrm{L} \\ x^{(r-1)}(t_m) \end{matrix}\right|$$

B in the formula is a matrix, which can be expressed as

$$B=\left|\begin{matrix} -z^{r}(t_2) & 1 \\ -z^{r}(t_3) & 1 \\ \mathrm{L} & \mathrm{L} \\ -z^{r}(t_m) & 1 \end{matrix}\right|$$

F. *Whitening Differential Equation*

If a differential equation can be expressed with the parameters a, b as shown below

$$\frac{dx^{(r)}}{dt}+ax^{(r)}=b \tag{11}$$

the differential equation is called the whitening differential equation of fractional order cumulative grey model. Given an initial condition $x^{(r)}(t_1)=x^{(0)}(t_1)$, the solutions of a whitening differential equation can be obtained:

$$\hat{x}^{(r)}(t_k)=(x^{(r)}(t_1)-\frac{b}{a})e^{-a(t_k-t_1)}+\frac{b}{a}(k=1,2\cdots m) \tag{12}$$

If the $\hat{x}^{(r)}$ can be reduced to the original sequence $\hat{x}^{(0)}$ by r times of subtraction generation operation, the model is derived [8].

$$\hat{x}^{(0)}(t_k)=\sum_{i=1}^{k}(-1)^{i+1}\frac{\Gamma(r+1)}{\Gamma(i)\Gamma(r-i+2)}\hat{x}^{(r)}(t_{k+1-i})/\Delta t_k \tag{13}$$

The relative fitting error (%) and the average relative error can be expressed as follows with obtained by solving the difference equation

$$e(t_k)=\frac{\hat{x}^{(0)}(t_k)-x^{(0)}(t_k)}{x^{(0)}(t_k)}\times 100\% \tag{14}$$

The average relative error of the fitting is

$$f(r)=\frac{1}{m}\sum_{k=1}^{m}\left|e(t_k)\right| \tag{15}$$

III. ORDER OPTIMIZATION

A. *Order optimization*

In order to make the fitting data more accurate, it is necessary to optimize the order of fractional order. Taken the number of fractional-order r as the variable and the minimum average relative error as the objective function, the optimisation resolution of the non-equidistance fractional-order accumulated grey model NFGM (1,1) can be established (optimisation model NFGM (1,1)- I):

$$\min \quad f(r)=\frac{1}{m}\sum_{k=1}^{m}\left|\frac{x^{(0)}(t_k)-\hat{x}^{(0)}(t_k)}{x^{(0)}(t_k)}\right| \tag{16}$$

Taken r as the variable and the minimum average square error as the objective function, the optimisation resolution of the non-equidistance fractional-order accumulated grey model NFGM (1,1) can be established (optimisation model NFGM (1,1)- II):

$$\min \quad f(r)=\frac{1}{m}\sum_{k=1}^{m}(x^{(0)}(t_k)-\hat{x}^{(0)}(t_k))^2 \tag{17}$$

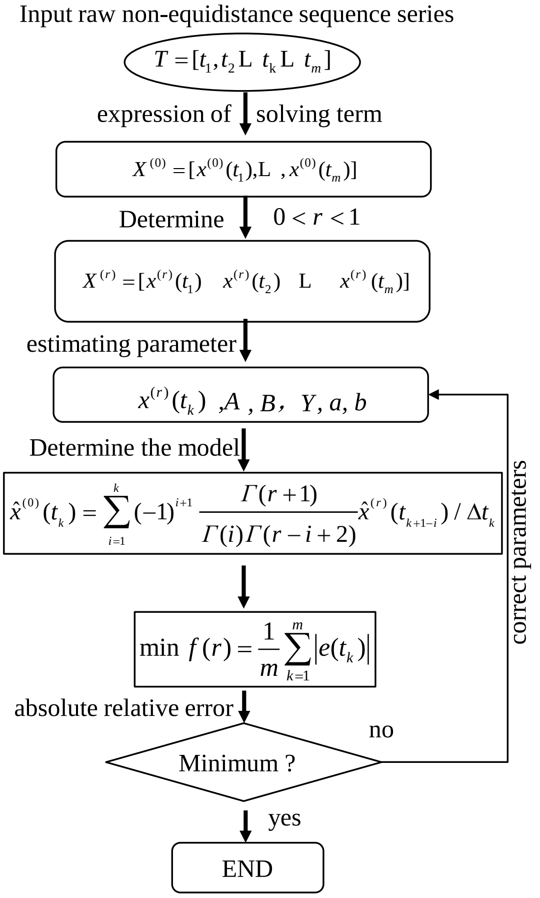

Fig.1 The flowchart of the solution

B. *Optimization solution*

In solving, the target set of error value also affects the accuracy of background value reconstruction and fitting. Taking the minimum error as the objective function and the known non-equidistant original data as the input, the program of non-equidistant fractional-order accumulation and reduction generation operator can be written. The objective function of the program is mainly used to optimize the design function of MATLAB. In order to optimize the method of solving the objective function, intelligent algorithm was used here (such as multi-parent genetic optimization algorithm [22]). Through the solution, the required parameters can be obtained. The solution process is shown in Figure 1.

IV. APPLICATIONS

In the section, some applications and verifications of the above results in fatigue strength analysis were introduced. The fatigue failure is the main form of damage of metal materials after infinite alternating deformation and breaking through the fatigue strength limit under load. The fatigue failure of mechanical parts will cause mechanical parts failure, equipment damage and major safety accidents. According to statistics, the probability of equipment damage caused by material fatigue failure is more than 80%. The material will not produce obvious deformation on the outer surface after fatigue failure, which becomes a hidden danger of major safety accidents. The necessary fatigue failure analysis of the key parts of mechanical equipment, such as shaft, blade, gear, spring and so on, is of great significance for efficient production, reducing safety accidents and property losses.

Because of the fractional order property of the system, the model description of the early modelling and simulation method is not accurate enough, which leads to the analysis conclusion inconsistent with reality. Therefore, the research on fatigue failure is mainly to obtain the analysis data through tedious test methods. The experimental data of fatigue strength of titanium alloy with temperature studied by G. Fore was shown in Tab.1 [23]. However, the infinite alternating load test is always a task that cannot be completed in the test analysis, which leads to the slow progress of the research on the fatigue failure of metal materials. It has become an urgent problem in scientific research to find a method to analyze non-equidistant fractional-order characteristic systems.

TABLE I
EXPERIMENTAL DATA OF FATIGUE STRENGTH OF Ti ALLOY ON DIFFERENT TEMPERATURE (MPa)

Ordinal	Temperature (t_k / $°C$)	fatigue strength of titanium alloy(σ_{-1})
1	100	560.00
2	130	557.54
3	170	536.10
4	210	516.10
5	240	505.60
6	270	486.10
7	310	467.40
8	340	453.80
9	380	436.40

In the present paper, a non-equidistant sequence fractional- order cumulative grey model NFGM (1, 1) was proposed to fit and analyze the fatigue strength test data. The validity of the model was verified by comparing with the literature, which opened up a new research method and way for the fatigue failure analysis of materials.

Through the model established in this paper, the system parameters were set. $x^{(0)}(t_k)$ refers to the fatigue strength σ_{-1} at the temperature t_k, $t=[t_1, t_2, \ldots t_k, \ldots t_m]$. The fatigue strength $\sigma_{-1}=[x^{(0)}(t_1), x^{(0)}(t_2), \cdots x^{(0)}(t_k), \cdots x^{(0)}(t_m)]$ .Obviously, the fatigue strength is a non-equidistance sequence with fractional order property. The parameters could be obtained with the optimization model based on the average generating background value. The optimized parameters were shown as the Tab.2

TABLE II
PARAMETERS OBTAINED WITH DIFFERENT OPTIMIZATION MODEL

Optimization model	Optimized parameters		
	A	*b*	*r*
Optimization model I	0.9962	0.0010	565.3382
Optimization model II	0.9933	0.0011	564.8235

The fitting values of the original data with the optimization model I, optimization model II and in [23] were shown in Tab.3

TABLE III
FITTING VALUES WITH DIFFERENT OPTIMIZATION MODELS

T/°C	Raw Data	Proposed models		In [23]
		NFGM (1, 1)- I	NFGM (1, 1)- II	
100	560.0	560.0	560.0	560.0
130	557.5	556.2	555.5	536.6
170	536.1	538.2	538.0	554.3
210	516.1	517.8	518.0	527.6
240	505.6	501.5	502.6	481.0
270	486.1	486.4	487.0	470.2
310	467.4	468.3	467.7	479.3
340	453.8	453.8	454.2	450.4
380	436.4	436.4	435.4	456.9

The fitting errors generated from the proposed models and [23] were shown in table 4.

TABLE IV
FITTING ERRORS WITH DIFFERENT OPTIMIZATION MODELS

T/°C	Raw Data	Proposed models		In [23]
		NFGM (1, 1)- I	NFGM (1, 1)- II	
100	560.0	0	0	0
130	557.5	0.2369	0.3693	14.3
170	536.1	-0.3831	-0.3560	-14.3
210	516.1	-0.3309	-0.3675	-13.5
240	505.6	0.8153	0.5978	16.5
270	486.1	-0.0525	-0.1918	16.0
310	467.4	-0.1857	-0.0661	-11.9
340	453.8	0.0081	-0.0979	16.9
380	436.4	0.0000	0.2350	-10.8

Using the model established in [23] to fit the data, the average value of the fitting data was quite different from the experimental data, and the error range is large. The data and error after fitting could be seen from the comparative analysis of NFGM with [23] (as shown in Fig.2).

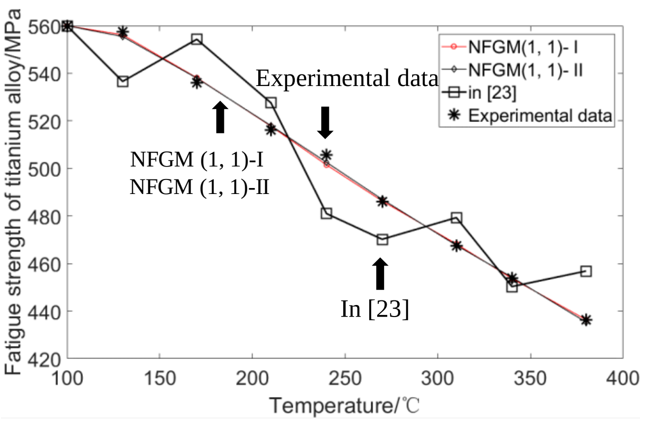

Fig.2 Comparative analysis error bar figure of NFGM with [23]

Through the statistical analysis, the histogram of fitting data of fatigue strength of titanium alloy at different temperatures was shown in Fig.3.

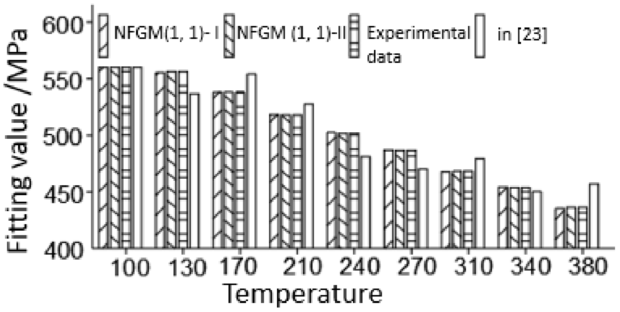

Fig.3 Histogram of fitting data with different models3

From the histogram of fatigue strength and fitting error of titanium alloy at different temperatures, it could be found that the data analysis model recommended in this paper has a high fitting degree and small residual error. The fitting value of the NFGM (1, 1)-I and NFGM (1, 1)-II were close to experimental data. By observing the enlarged area (as shown in Fig.4), it could be seen that the fitting accuracy of model-I is better for model- II.

From the residual histogram of fitting data with different models, it was easy to found that the error of the data analysis model used in [23] was more than ten times that of the model recommended in the present paper. It shows that the background value of this method is not accurate, which leads to the fitting value deviated from the experimental value much more than the NFGM (1, 1) – I and NFGM (1, 1) - II.

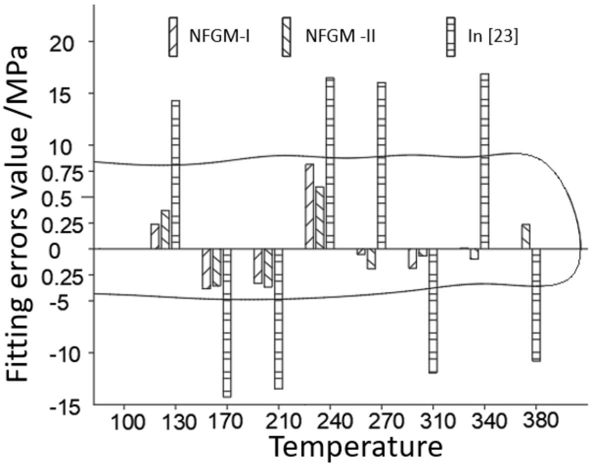

Fig.4 Residual histogram of fitting data with different models

This is mainly because the previous research cannot fit the parameters of non-equidistant research objects well, and the background value reconstruction ability is low, which leads to the low fitting accuracy.

V. CONCLUSION

In scientific research and engineering applications, there are a large number of research objects with non-equidistant fractional-order characteristics. Because the background value of the previous grey theory analysis models was not accurate, the analysis results could not meet the requirements, even contrary to the actual situation. If the experimental analysis method was used, it was often not the best solution because of the long cycle, high cost and high operational requirements. Given this urgent problem, this paper established a fractional-order cumulative grey model NFGM (1, 1). Taking the fractional-order, minimum mean relative error and minimum mean square error as design variables and objective functions, respectively, two optimization models were established. The models have the ability of high background reconstruction and accurate description of the non-equidistant fractional-order system. The effectiveness of the proposed models was verified by an example of fatigue strength calculation.

The establishment of the NFGM (1, 1) model opened up a new research method and way for the research objects with non-equidistant fractional-order characteristics. The application range of the grey theory analysis model has been extended from integer-order to fractional-order. The establishment of the model is of great significance in scientific research and engineering application, which greatly promotes the application and development of grey theory.

REFERENCES

[1] Chien Ping Chung, “An Optimization Model with Concerning Product Deterioration for Parameters Decision,” Engineering Letters, vol. 28, no. 3, pp676-680, 2020.

[2] Maulana Malik, Mustafa Mamat, Siti Sabariah Abas, Ibrahim Mohammed Sulaiman, and Sukono, “A New Coefficient of the Conjugate Gradient Method with the Sufficient Descent Condition and Global Convergence Properties,” Engineering Letters, vol. 28, no. 3, pp704-714, 2020

[3] Tao Ma, Heng Liu, and Yu Zhang, “A Method for Establishing Tropospheric Atmospheric Refractivity Profile Model Based on Multiquadric RBF and k-means Clustering,” Engineering Letters, vol. 28, no. 3, pp733-741, 2020

[4] XiaoGao Yang, Deqion Ding, Youxin Luo, "Non-equidistance Fractional Order Accumulation Grey Power Model NFGPM (1, 1) and Its Application," Engineering Letters, vol. 28, no. 3, pp827-832, 2020

[5] Zhenliang Zhu, Fengde Chen, Liyun Lai, and Zhong Li, “Dynamic Behaviors of a Discrete May Type Cooperative System Incorporating Michaelis-Menten Type Harvesting,” IAENG International Journal of Applied Mathematics, vol. 50, no. 3, pp458-467, 2020

[6] Titilayo Morenike Agbaje, Sandile Sydney Motsa, Peter Leach, Precious Sibanda, “Efficient Large Spectral Collocation Method for MHD Laminar Natural Convection Flow from a Vertical Permeable Flat Plate with Uniform Surface Temperature, Soret, Dufour, Chemical Reaction and Thermal Radiation,” IAENG International Journal of Applied Mathematics, vol. 50, no. 3, pp490-504, 2020

[7] Zhen Liu, Yumei Xie, Yaoguo Dang, “A progress analysis of Chinese poverty alleviation based on improved GM (1,1),” Systems Engineering- Theory & Practice, vol. 39, no. 10, pp2476-2486, 2019

[8] Junfang Wang, Dang Luo, “Fractional order discrete grey GM (1, 1) power model based on oscillation,” Control and Decision, vol. 32, no. 1, pp176-180, 2017

[9] Yuxiao Kang, Xinping Xiao, “Morbidity analysis of the grey GM (1,1) derived models,” Systems Engineering- Theory & Practice, vol. 39, no. 10, pp2610-2618, 2019

[10] Zhengxin Wang, Yufeng Zhao, “GM (1,1) model with seasonal dummy variables and its application,” Systems Engineering- Theory & Practice, vol. 40, no.11, pp2981-2990, 2020

[11] Dang Luo, Xiaolei Wang, “Grey GM (1,1, T) model coupled with trigonometric function and its application,” Systems Engineering- Theory & Practice, vol. 40, no. 7, pp1906-1917, 2020

[12] Wenzhan Dai, Junfeng Li, “Modeling Research on Non-equidistance GM (1,1) Model,” Systems Engineering - Theory & Practice, vol. 25, no. 9, pp 89-93, 2005

[13] Wang, Y. M., Dang Y. G., Wang Z. X. “The optimization of background value in non-equidistant GM (1, 1) model,” Chinese Journal of Management Science, vol. 16, no. 4, pp 159-162, 2008

[14] Wenqing Wu, Xin Ma, Yong Wang, Yuanyuan Zhang, Bo Zeng, “Research on a novel fractional GM (α, n) model and its applications,” Grey Systems: Theory and Application vol. 9, no. 3, pp 356-373, 2019

[15] Wenging Wu, Xin Ma, Yuanyuan Zhang, Yong Wang, Xinxing Wu, “Analysis of novel FAGM (1,1, t^{α}) model to forecast health expenditure of China,” Grey Systems: Theory and Application, vol. 9, no. 2, pp 232-250, 2019

[16] Lifeng Wu, Sifeng Liu, Ligen Yao, “Discrete grey model based on fractional order accumulate,” Systems Engineering - Theory & Practice, vol. 34, no. 7, pp 1822-1827, 2014

[17] Lifeng Wu, Sifeng Liu, Zhigeng Fang, Haiyan Xu, “Properties of the GM (1,1) with fractional order accumulation,” Applied Mathematics and Computation, vol. 252, pp287-293, 2015

[18] Lifeng Wu, Sifeng Liu, Ligen Yao, “Grey model with Caputo fractional order derivative,”. Systems Engineering - Theory & Practice, vol. 35, no. 5, pp1311-1316, 2015

[19] Baohua Yang, Jinshuai Zhao, “Fractional order discrete grey GM (1, 1) power model and its application,” Control and Decision, vol. 30, no. 7, pp1264-1268, 2015

[20] Lifeng Wu, Sifeng Liu, Jian Liu, “GM (1, 1) model based on fractional order accumulating method and its stability,” Control and Decision, vol. 29, no. 5, pp919-923, 2014

[21] Youxin Luo, “Non-equidistant new information GM (1, 1) model and its application”. Journal of Shen-yang University of Technology, vol. 32, no. 5, pp550-554, 2010

[22] Jianxin Chen, Yongyi Guo, Maixia Lv, “An Adaptive Genetic Algorithm Based on Multi-Population Parallel Evolutionary for Highway Alignment Optimization Model,” Applied Mechanics and Materials, vol. 58-60, pp1499-1503, 2011

[23] Youxin Luo, Jirong Zhou, “Non-equidistance GM (1, 1) model and its application in fatigue experimental data processing and on-line control,” Journal of Mechanical Strength, vol. 18, no. 3, pp60-63, 1996

XiaoGao Yang is working in College of Mechanical Engineering, Hunan University of Arts and Science. He is the Hunan Provincial General University Subject Leader. The main research areas include mechanical design and manufacturing, system research, uncertainty system theory in mechanical engineering, modern intelligent optimization method, computational mechanism and mechanical transmission. Email: yxg_568@126.com

DeQiong Ding is working in Hunan University of Finance and Economics. The main research areas include research work of mathematics statistics and modeling, data mining and financial and economic analysis. Email: ddqshe2010@163.com

YouXin Luo is working in College of Mechanical Engineering, Hunan University of Arts and Science. He is the Hunan Provincial General University Subject Leader, and the National Natural Science Foundation Letter Review Expert, in china. The main research areas include mechanical design and manufacturing, system research, uncertainty system theory in mechanical engineering, modern intelligent optimization method, computational mechanism. Email: llyx123@126.com