Falling UP-Filters

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***Abstract*—Further properties of falling UP-ideals are considered. The concepts of a falling UP-filter and a I-fuzzy filter are presented, and many characteristics are examined. The relationship between the falling UP-filter and the falling UP-ideal is established, and it is demonstrated that the falling UP-filter is a generalization of the fuzzy UP-filter. The concept of falling inference relations is applied to UP-algebras, and a related result is obtained.**

***Index Terms*—UP-ideal, UP-filter, falling UP-ideal, falling UP-filter, I-fuzzy filter.**

I. Introduction

WANG and Sanchez [1] proposed the idea of falling shadows, which connects probability notions to the membership function of fuzzy sets directly. Wang [2] formulates the mathematical structure of the theory of falling shadows. On the basis of the idea of falling shadows, Tan et al. [3], [4] developed a theoretical method to construct a fuzzy inference relation and fuzzy set operations. The notion of falling shadows was used by Jun and Kang [5] to analyze positive implicative ideals of BCK-algebras. Iampan [6] introduced a new algebraic structure called UP-algebras, and investigated several properties. Based on the notion of falling shadows, Jun et al. [7] developed a theoretical approach for defining fuzzy UP-subalgebras and fuzzy UP-ideals in a UP-algebra. They provided relations between falling UP-subalgebras and falling UP-ideals. They also looked at the relationships between fuzzy UP-subalgebras (resp., fuzzy UP-ideals) and falling UP-subalgebras (resp., falling UP-ideals), as well as a number other characteristics.

The idea of falling shadows as applied to UP-filters is discussed in this paper. We first investigate some properties of falling UP-ideals. We define falling UP-filter and I-fuzzy filter, and investigate several properties. We establish the relation between falling UP-filter and falling UP-ideal. We show that falling UP-filter is a generalization of fuzzy UP-filter. The idea of falling inference relations is applied to UP-algebras, and a related consequence is obtained.

II. Preliminaries

An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* (see [6]) it fulfills the following requirements.

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0), \tag{1}$$
$$(\forall x \in X)(0 \cdot x = x), \tag{2}$$
$$(\forall x \in X)(x \cdot 0 = 0), \tag{3}$$
$$(\forall x, y \in X)(x \cdot y = 0 = y \cdot x \ \Rightarrow\ x = y). \tag{4}$$

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The following statements are true in a UP-algebra X (see [6], [8]).

$$(\forall x \in X)(x \cdot x = 0), \tag{5}$$
$$(\forall x, y, z \in X)(x \cdot y = 0,\ y \cdot z = 0 \ \Rightarrow\ x \cdot z = 0), \tag{6}$$
$$(\forall x, y \in X)(x \cdot y = 0 \ \Rightarrow\ (z \cdot x) \cdot (z \cdot y) = 0), \tag{7}$$
$$(\forall x, y \in X)(x \cdot y = 0 \ \Rightarrow\ (y \cdot z) \cdot (x \cdot z) = 0), \tag{8}$$
$$(\forall x, y \in X)(x \cdot (y \cdot x) = 0, \text{ in particular, } (y \cdot z) \cdot (x \cdot (y \cdot z)) = 0) \tag{9}$$
$$(\forall x, y \in X)((y \cdot x) \cdot x = 0 \ \Leftrightarrow\ x = y \cdot x), \tag{10}$$
$$(\forall x, y \in X)(x \cdot (y \cdot y) = 0), \tag{11}$$
$$(\forall a, x, y, z \in X)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0), \tag{12}$$
$$(\forall a, x, y, z \in X)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0), \tag{13}$$
$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0), \tag{14}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0), \tag{15}$$
$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0), \tag{16}$$
$$(\forall a, x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0). \tag{17}$$

For more studies and examples of UP-algebras, see [8], [9], [10], [11], [12], [13].

A subset A of X is called a *UP-ideal* of X (see [6]) if the following conditions are valid.

$$0 \in A, \tag{18}$$
$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in A,\ y \in A \ \Rightarrow\ x \cdot z \in A). \tag{19}$$

A subset F of X is called a *UP-filter* of X (see [14]) if the following conditions are valid.

$$0 \in F, \tag{20}$$
$$(\forall x, y \in X)(x \in F,\ x \cdot y \in F \ \Rightarrow\ y \in F). \tag{21}$$

A fuzzy set λ in a UP-algebra X is called a *fuzzy UP-ideal* of X (see [14]) if the following condition is valid.

$$(\forall x \in X)\left(\lambda(0) \geq \lambda(x)\right), \tag{22}$$
$$(\forall x, y, z \in X)\left(\lambda(x \cdot z) \geq \min\{\lambda(x \cdot (y \cdot z)), \lambda(y)\}\right). \tag{23}$$

A fuzzy set λ in a UP-algebra X is called a *fuzzy UP-filter* of X (see [14]) if it satisfies (22) and

$$(\forall x, y \in X)\left(\lambda(y) \geq \min\{\lambda(x), \lambda(x \cdot y)\}\right). \tag{24}$$

The fundamentals of falling shadows are now displayed. For more information on the theory of falling shadows, we recommend reading the papers [1], [2], [3], [4], [15].

Let $\mathscr{P}(U)$ represent the power set of a discourse universe U. For any $u \in U$, let

$$\ddot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{25}$$

and for any $E \in \mathscr{P}(U)$, let

$$\ddot{E} := \{\ddot{u} \mid u \in E\}. \tag{26}$$

An ordered pair $(\mathscr{P}(U), \mathscr{B})$ is said to be a *hyper-measurable structure* on U if $\mathscr{B}$ is a σ-field in $\mathscr{P}(U)$ and $\ddot{U} \subseteq \mathscr{B}$. Given a probability space $(\Omega, \mathscr{A}, P)$ and a hyper-measurable structure $(\mathscr{P}(U), \mathscr{B})$ on U, a *random set* on U is defined to be a mapping $\xi : \Omega \to \mathscr{P}(U)$ which is $\mathscr{A}$-$\mathscr{B}$ measurable, that is,

$$(\forall C \in \mathscr{B})\,(\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathscr{A}). \tag{27}$$

Assume ξ is a random set on U. Let

$$\tilde{\alpha}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for any } u \in U.$$

Then $\tilde{\alpha}$ is a kind of fuzzy set in U. We call $\tilde{\alpha}$ a *falling shadow* of the random set ξ, and ξ is called a *cloud* of $\tilde{\alpha}$.

For example, $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{\alpha}$ be a fuzzy set in U and $\tilde{\alpha}_t := \{u \in U \mid \tilde{\alpha}(u) \geq t\}$ be a t-cut of $\tilde{\alpha}$. Then

$$\xi : [0,1] \to \mathscr{P}(U),\ t \mapsto \tilde{\alpha}_t$$

is a random set and ξ is a cloud of $\tilde{\alpha}$. We shall call ξ defined above as the *cut-cloud* of $\tilde{\alpha}$ (see [15]).

III. Properties of falling UP-ideals

Unless otherwise stated, let X indicate a UP-algebra.

Definition III.1 ([7])**.** Let $(\Omega, \mathscr{A}, P)$ be a probability space, and let

$$\xi : \Omega \to \mathscr{P}(X)$$

be a random set. If $\xi(\omega)$ is a UP-ideal of X for each $\omega \in \Omega$, then the falling shadow $\tilde{\alpha}$ of the random set ξ, i.e.,

$$\tilde{\alpha}(x) = P(\omega \mid x \in \xi(\omega)) \tag{28}$$

is called a *falling UP-ideal* of X.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $\tilde{\alpha}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(X)$. For each $x \in X$, let

$$\Omega_\xi(x) = \{\omega \in \Omega \mid x \in \xi(\omega)\}.$$

Then $\Omega_\xi(x) \in \mathscr{A}$.

Proposition III.2. *Every falling UP-ideal $\tilde{\alpha}$ satisfies the following condition.*

$$(\forall x \in X)(\Omega_\xi(x) \subseteq \Omega_\xi(0)). \tag{29}$$

Proof: Let $\tilde{\alpha}$ be a falling UP-ideal of X. For each $x \in X$, let $\omega \in \Omega_\xi(x)$. Then $\xi(\omega)$ is a UP-ideal of X, and so $0 \in \xi(\omega)$ by (18). It follows that $\omega \in \Omega_\xi(0)$. Therefore, the inclusion (29) is valid. ■

Proposition III.3. *For every falling UP-ideal $\tilde{\alpha}$ of X, we have the following results.*

$$(\forall x, y \in X)\,(\Omega_\xi(x \cdot y) \cap \Omega_\xi(x) \subseteq \Omega_\xi(y))\,, \tag{30}$$

$$(\forall x, y \in X)\,(\Omega_\xi(y) \subseteq \Omega_\xi(x \cdot y))\,, \tag{31}$$

$$(\forall a, b, x \in X)\,(\Omega_\xi(a) \cap \Omega_\xi(b) \subseteq \Omega_\xi((b \cdot (a \cdot x)) \cdot x))\,. \tag{32}$$

Proof: Let $\tilde{\alpha}$ be a falling UP-ideal of X. Then $\xi(\omega)$ is a UP-ideal of X. For each $x, y \in X$, let $\omega \in \Omega_\xi(x \cdot y) \cap \Omega_\xi(x)$. Then $x \cdot y \in \xi(\omega)$ and $x \in \xi(\omega)$. Using (2), we have $0 \cdot (x \cdot y) = x \cdot y \in \xi(\omega)$. It follows from (2) and (19) that $y = 0 \cdot y \in \xi(\omega)$ and so that $\omega \in \Omega_\xi(y)$. Thus (30) is true. Now, for each $x, y \in X$, let $\omega \in \Omega_\xi(y)$. Then $y \in \xi(\omega)$, which implies from (3) and (5) that $x \cdot (y \cdot y) = x \cdot 0 = 0 \in \xi(\omega)$. Hence, $x \cdot y \in \xi(\omega)$ by (19), and so $\omega \in \Omega_\xi(x \cdot y)$. Therefore, (31) holds. Finally, for each $a, b, x \in X$, let $\omega \in \Omega_\xi(a) \cap \Omega_\xi(b)$. Then $a, b \in \xi(\omega)$. Using (5) implies that $(a \cdot x) \cdot (a \cdot x) = 0 \in \xi(\omega)$. It follows from (19) that $(a \cdot x) \cdot x \in \xi(\omega)$, that is, $\omega \in \Omega_\xi((a \cdot x) \cdot x)$. Using (1), we have

$$((a \cdot x) \cdot x) \cdot ((b \cdot (a \cdot x)) \cdot (b \cdot x)) = 0 \in \xi(\omega),$$

and so $\omega \in \Omega_\xi(((a \cdot x) \cdot x) \cdot ((b \cdot (a \cdot x)) \cdot (b \cdot x)))$. Hence,

$$\omega \in \Omega_\xi(((a \cdot x) \cdot x) \cdot ((b \cdot (a \cdot x)) \cdot (b \cdot x))) \cap \Omega_\xi((a \cdot x) \cdot x) \subseteq \Omega_\xi(((b \cdot (a \cdot x)) \cdot (b \cdot x)))$$

by (30), and thus $(b \cdot (a \cdot x)) \cdot (b \cdot x) \in \xi(\omega)$. It follows from (19) that $(b \cdot (a \cdot x)) \cdot x \in \xi(\omega)$ and so that $\omega \in \Omega_\xi((b \cdot (a \cdot x)) \cdot x)$. This proves that (32) is valid. ■

Proposition III.4. *For every falling UP-ideal $\tilde{\alpha}$ of X, we have the following results.*

$$(\forall x, y \in X)(x \leq y \ \Rightarrow\ \Omega_\xi(x) \subseteq \Omega_\xi(y)), \tag{33}$$

$$(\forall a, b, x \in X)(b \leq a \cdot x \ \Rightarrow\ \Omega_\xi(a) \cap \Omega_\xi(b) \subseteq \Omega_\xi(x)). \tag{34}$$

Proof: Let $\tilde{\alpha}$ be a falling UP-ideal of X. Then $\xi(\omega)$ is a UP-ideal of X. For each $x, y \in X$ with $x \leq y$, let $\omega \in \Omega_\xi(x)$. Then $x \cdot y = 0 \in \xi(\omega)$ and so $\omega \in \Omega_\xi(x \cdot y)$. It follows from (30) that

$$\omega \in \Omega_\xi(x \cdot y) \cap \Omega_\xi(x) \subseteq \Omega_\xi(y)$$

Thus (33) is valid. Assume that $b \leq a \cdot x$ for each $a, b, x \in X$ and let $\omega \in \Omega_\xi(a) \cap \Omega_\xi(b)$. Then $b \cdot (a \cdot x) = 0 \in \xi(\omega)$, and so $\omega \in \Omega_\xi(b \cdot (a \cdot x))$. Since $\omega \in \Omega_\xi(b)$, we have

$$\omega \in \Omega_\xi(b \cdot (a \cdot x)) \cap \Omega_\xi(b) \subseteq \Omega_\xi(a \cdot x)$$

by (30). Since $\omega \in \Omega_\xi(a)$, it follows from (30) that

$$\omega \in \Omega_\xi(a \cdot x) \cap \Omega_\xi(a) \subseteq \Omega_\xi(x).$$

Therefore, $\Omega_\xi(a) \cap \Omega_\xi(b) \subseteq \Omega_\xi(x)$ for all $a, b, x \in X$ with $b \leq a \cdot x$. ■

IV. Falling UP-filters

Definition IV.1. Let $(\Omega, \mathscr{A}, P)$ be a probability space, and let

$$\xi : \Omega \to \mathscr{P}(X)$$

be a random set. If $\xi(\omega)$ is a UP-filter of X for each $\omega \in \Omega$, then the falling shadow $\tilde{\alpha}$ of the random set ξ, i.e.,

$$\tilde{\alpha}(x) = P(\omega \mid x \in \xi(\omega)) \tag{35}$$

is called a *falling UP-filter* of X.

Example IV.2. Consider a UP-algebra $X = \{0, a, b, c, d\}$ with the binary operation "$\cdot$" which is given in Table I.

TABLE I
TABULAR REPRESENTATION OF THE BINARY OPERATION "·"

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Let $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$ and let

$$\xi : [0,1] \to \mathscr{P}(X),\ t \mapsto \begin{cases} \{0,a,c\} & \text{if } \ t \in [0, 0.6), \\ X & \text{if } \ t \in [0.6, 1]. \end{cases} \tag{36}$$

Then $\xi(t)$ is a UP-filter of X for all $t \in [0,1]$. Hence, $\tilde{\alpha}$ is a falling UP-filter of X, and

$$\tilde{\alpha}(x) = \begin{cases} 1 & \text{if } \ x \in \{0,a,c\}, \\ 0.4 & \text{if } \ x \in \{b,d\}. \end{cases} \tag{37}$$

Theorem IV.3. *Every falling UP-ideal is a falling UP-filter.*

Proof: Let $\tilde{\alpha}$ be a falling UP-ideal of X. Then $\xi(\omega)$ is a UP-ideal of X. Let $x, y \in X$ be such that $x \cdot y \in \xi(\omega)$ and $x \in \xi(\omega)$. Then $0 \cdot (x \cdot y) = x \cdot y \in \xi(\omega)$ by (2), and so $y = 0 \cdot y \in \xi(\omega)$ by (2) and (19). Hence, $\xi(\omega)$ is a UP-filter of X, and therefore, $\tilde{\alpha}$ is a UP-filter. ■

The following example shows that the converse of Theorem IV.3 is not true in general.

Example IV.4. Let $X = \{0,1,2,3\}$ be a set with the binary operation "·" which is given in Table II.

TABLE II
TABULAR REPRESENTATION OF THE BINARY OPERATION "·"

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Then X is a UP-algebra (see [16]). Let $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$ and let

$$\xi : [0,1] \to \mathscr{P}(X),\ t \mapsto \begin{cases} \{0\} & \text{if } \ t \in [0, 0.3), \\ \{0,1\} & \text{if } \ t \in [0.3, 0.7), \\ X & \text{if } \ t \in [0.7, 1]. \end{cases} \tag{38}$$

Then $\xi(t)$ is a UP-filter of X for all $t \in [0,1]$. Hence, $\tilde{\alpha}$ is a falling UP-filter of X. Note that $1 \in \{0,1\}$ and $2 \cdot (1 \cdot 3) = 0 \in \{0,1\}$. But $2 \cdot 3 = 2 \notin \{0,1\}$. Hence, if $t \in [0.3, 0.7)$, then $\xi(t) = \{0,1\}$ is not a UP-ideal of X. Therefore, $\tilde{\alpha}$ is not a falling UP-ideal of X.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let

$$F(X) := \{f \mid f : \Omega \to X \ \text{ is a mapping}\}.$$

Define an operation $\odot$ on $F(X)$ by

$$(\forall \omega \in \Omega)\left((f \odot g)(\omega) = f(\omega) \cdot g(\omega)\right)$$

for all $f, g \in F(X)$. Let $\theta \in F(X)$ be defined by $\theta(\omega) = 0$ for all $\omega \in \Omega$. It can be easily checked that $(F(X); \odot, \theta)$ is a UP-algebra.

For each subset A of X and $f \in F(X)$, let

$$A_f := \{\omega \in \Omega \mid f(\omega) \in A\} \tag{39}$$

and

$$\xi : \Omega \to \mathscr{P}(F(X)),\ \omega \mapsto \{f \in F(X) \mid f(\omega) \in A\}. \tag{40}$$

Then $A_f \in \mathscr{A}$.

Theorem IV.5. *If A is a UP-filter of X, then*

$$\xi(\omega) = \{f \in F(X) \mid f(\omega) \in A\} \tag{41}$$

is a UP-filter of $F(X)$ for each $\omega \in \Omega$.

Proof: Let $\omega \in \Omega$. Assume that A is a UP-filter of X. Since $\theta(\omega) = 0 \in A$, we know that $\theta \in \xi(\omega)$. Let $f, g \in F(X)$ be such that $f \odot g \in \xi(\omega)$ and $f \in \xi(\omega)$. Then $f(\omega) \in A$ and

$$f(\omega) \cdot g(\omega) = (f \odot g)(\omega) \in A.$$

It follows from (21) that $g(\omega) \in A$ and so that $g \in \xi(\omega)$. Therefore, $\xi(\omega)$ is a UP-filter of $F(X)$. ■

Since

$$\begin{aligned} \xi^{-1}(\ddot{f}) &= \{\omega \in \Omega \mid f \in \xi(\omega)\} \\ &= \{\omega \in \Omega \mid f(\omega) \in A\} = A_f \in \mathscr{A}, \end{aligned} \tag{42}$$

we can see that ξ is a random set on $F(X)$. Let

$$\tilde{\alpha}(f) = P(\omega \mid f(\omega) \in A). \tag{43}$$

Then $\tilde{\alpha}$ is a falling UP-filter of $F(X)$.

Lemma IV.6 ([14]). *A fuzzy set λ in X is a fuzzy UP-filter (resp., fuzzy UP-ideal) of X if and only if the set*

$$\lambda_t := \{x \in X \mid \lambda(x) \geq t\}$$

is a UP-filter (resp., UP-ideal) of X for all $t \in [0,1]$ with $\lambda_t \neq \emptyset$.

Theorem IV.7. *Every fuzzy UP-filter of X is a falling UP-filter of X.*

Proof: Consider the probability space $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$ where $\mathscr{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let λ be a fuzzy UP-filter of X. Then λ_t is a UP-filter of X for all $t \in [0,1]$ with $\lambda_t \neq \emptyset$. Let

$$\xi : [0,1] \to \mathscr{P}(X),\ t \mapsto \lambda_t$$

be a random set. Then λ is a falling UP-filter of X. ■

Proposition IV.8. *Let $\tilde{\alpha}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(X)$. If $\tilde{\alpha}$ is a falling UP-filter of X, then*

$$(\forall x \in X)\left(\Omega_\xi(x) \subseteq \Omega_\xi(0)\right), \tag{44}$$

$$(\forall x, y \in X)\left(\Omega_\xi(x \cdot y) \cap \Omega_\xi(x) \subseteq \Omega_\xi(y)\right). \tag{45}$$

Proof: Let $\tilde{\alpha}$ be a falling UP-filter of X. Then $\xi(\omega)$ is a UP-filter of X for each $\omega \in \Omega$. The result (44) is clear. For each $x, y \in X$, let $\omega \in \Omega_\xi(x \cdot y) \cap \Omega_\xi(x)$. Then $x \cdot y \in \xi(\omega)$ and $x \in \xi(\omega)$. It follows from (21) that $y \in \xi(\omega)$. Hence, $\omega \in \Omega_\xi(y)$ which shows that (45) is valid. ■

For each $s, t \in [0,1]$, let $T_m(s,t) := \max\{0, s+t-1\}$.

Theorem IV.9. *Every falling UP-filter $\tilde{\alpha}$ of X satisfies the following conditions.*

$$(\forall x \in X)\left(\tilde{\alpha}(0) \geq \tilde{\alpha}(x)\right). \tag{46}$$

$$(\forall x, y \in X)\left(\tilde{\alpha}(y) \geq T_m(\tilde{\alpha}(x \cdot y), \tilde{\alpha}(x))\right). \tag{47}$$

Proof: Let $\tilde{\alpha}$ be a falling UP-filter of X. Then $\xi(\omega)$ is a UP-filter of X for each $\omega \in \Omega$. Note that $\Omega_\xi(x) \subseteq \Omega_\xi(0)$ for all $x \in X$ by (44). Hence,

$$\tilde{\alpha}(0) = P(\omega \mid 0 \in \xi(\omega)) \geq P(\omega \mid x \in \xi(\omega)) = \tilde{\alpha}(x).$$

Using (45), we have $\Omega_\xi(x \cdot y) \cap \Omega_\xi(x) \subseteq \Omega_\xi(y)$, that is,

$$\begin{aligned} &\{\omega \in \Omega \mid x \cdot y \in \xi(\omega)\} \cap \{\omega \in \Omega \mid x \in \xi(\omega)\} \\ &\subseteq \{\omega \in \Omega \mid y \in \xi(\omega)\}. \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{\alpha}(x) &= P(\omega \mid y \in \xi(\omega)) \\ &\geq P(\{\omega \mid x \cdot y \in \xi(\omega)\} \cap \{\omega \mid x \in \xi(\omega)\}) \\ &\geq P(\omega \mid x \cdot y \in \xi(\omega)) + P(\omega \mid x \in \xi(\omega)) \\ &\qquad - P(\omega \mid x \cdot y \in \xi(\omega) \text{ or } x \in \xi(\omega)) \\ &\geq \tilde{\alpha}(x \cdot y) + \tilde{\alpha}(x) - 1. \end{aligned}$$

Therefore,

$$\tilde{\alpha}(x) \geq \max\{0, \tilde{\alpha}(x \cdot y) + \tilde{\alpha}(x) - 1\} = T_m(\tilde{\alpha}(x \cdot y), \tilde{\alpha}(x)).$$

This completes the proof. ■

Now, we consider the falling inference relations in UP-algebras. In 1993, Tan et al. [4] established a theoretical approach to define a fuzzy inference relation based on the theory of falling shadows.

Let ξ and ζ be cut-clouds of A and B, respectively, where A and B are fuzzy sets in the universes U and V respectively. Note that the random sets ξ and ζ are initially defined on two distinct probability spaces $([0,1], \mathscr{B}_1, m_1)$ and $([0,1], \mathscr{B}_2, m_2)$ where $\mathscr{B}_1$ and $\mathscr{B}_2$ are Borel fields on $[0,1]$, and m_1 and m_2 are Lebesgue measures. Tan et al. [4] have redefined ξ and ζ on a unified probability space $([0,1]^2, \mathscr{B}^2, P)$, where P is a joint probability on $[0,1]^2$, by setting $\xi : [0,1]^2 \to U$ and $\zeta : [0,1]^2 \to V$ to be

$$\xi : (t, s) \mapsto t \mapsto A_t \tag{48}$$

and

$$\zeta : (t, s) \mapsto s \mapsto B_s \tag{49}$$

for each $(t, s) \in [0,1]^2$.

Note that $\xi(t, s)$ and $\zeta(t, s)$ are two crisp sets A_t and B_s on U and V, respectively, for all $(t, s) \in [0,1]^2$. From the usual notion of the implication $A_t \to B_s$, we can obtain the corresponding inference relation:

$$I_{A_t \to B_s} = (A_t \times B_s) \cup (A_t^c \times V), \tag{50}$$

which can be considered as a random set on $U \times V$. We may get the following definition of fuzzy inference relation by identifying the falling shadow of this random set.

Let ξ and ζ be clouds of A and B respectively. Then the fuzzy inference relation $I_{A \to B}$ of the implication $A \to B$ is defined by

$$\begin{aligned} I_{A \to B}(u, v) &= P((t, s) \mid (u, v) \in I_{A_t \to B_s}) \\ &= P((t, s) \mid (u, v) \in (A_t \times B_s) \cup (A_t^c \times V)). \end{aligned} \tag{51}$$

Note that P in (51) is a joint probability on $[0,1]^2$, and thus different probability distribution P will generate different formula for the fuzzy inference relation (see [4]).

Let P be the whole probability of (t, s) on $[0,1]^2$. If P is concentrated and uniformly distributed on $\{(t, t) \mid t \in [0,1]\}$ of the unit square $[0,1]^2$, then P is the diagonal distribution and

$$I_{A \to B}(u, v) = \min\{1 - A(u) + B(v), 1\}.$$

We now consider the concept of I-fuzzy UP-filters in UP-algebras.

Definition IV.10. Let I be a falling implication operator over $[0,1]$ and $t \in (0,1]$. A fuzzy set λ in X is called an *I-fuzzy filter* of X with respect to t if the following assertions are valid.

$$(\forall x \in X)(I(\lambda(x), \lambda(0)) \geq t), \tag{52}$$

$$(\forall x, y \in X)(I(\min\{\lambda(x \cdot y), \lambda(x)\}, \lambda(y)) \geq t). \tag{53}$$

Obviously, if P is the diagonal distribution, then the notion of I-fuzzy filter with respect to $t = 1$ is equivalent to the notion of fuzzy filter.

Theorem IV.11. *Let λ be a fuzzy set in X and $t = 0.5$. If P is the diagonal distribution, then the following are equivalent:*

(1) *λ is an I-fuzzy filter of X with respect to $t = 0.5$.*
(2) *λ satisfies the following conditions.*

$$(\forall x \in X)(\lambda(x) \leq \lambda(0) \text{ or } 0 < \lambda(x) - \lambda(0) \leq 0.5), \tag{54}$$

$$(\forall x, y \in X)\left(\begin{array}{l} \min\{\lambda(x \cdot y), \lambda(x)\} \leq \lambda(y) \text{ or} \\ 0 < \min\{\lambda(x \cdot y), \lambda(x)\} - \lambda(y) \\ \leq 0.5 \end{array}\right). \tag{55}$$

Proof: Let P be the diagonal distribution. Then

$$(\forall x \in X)\left(\ I(\lambda(x), \lambda(0)) = \min\{1 - \lambda(x) + \lambda(0), 1\}\ \right). \tag{56}$$

Assume that λ is an I-fuzzy filter of X with respect to $t = 0.5$. Then

$$\min\{1 - \lambda(x) + \lambda(0), 1\} \geq 0.5$$

by (52) and (56). If $\lambda(x) > \lambda(0)$, then $0 < \lambda(x) - \lambda(0) \leq 0.5$ and so (54) is valid. Since P is the diagonal distribution, (53) implies that

$$\begin{aligned} 0.5 &\leq I(\min\{\lambda(x \cdot y), \lambda(x)\}, \lambda(y)) \\ &= \min\{1 - \min\{\lambda(x \cdot y), \lambda(x)\} + \lambda(y), 1\} \\ &= \min\{1 - (\min\{\lambda(x \cdot y), \lambda(x)\} - \lambda(y)), 1\}. \end{aligned} \tag{57}$$

If $\min\{\lambda(x \cdot y), \lambda(x)\} > \lambda(y)$, then

$$0 < \min\{\lambda(x \cdot y), \lambda(x)\} - \lambda(y)$$

and $1 - (\min\{\lambda(x \cdot y), \lambda(x)\} - \lambda(y)) \geq 0.5$ by (57). Hence,

$$0 < \min\{\lambda(x \cdot y), \lambda(x)\} - \lambda(y) \leq 0.5,$$

and therefore, (55) is valid.

Conversely, assume that λ satisfies conditions (54) and (55). Since P is the diagonal distribution, we have

$$I(\lambda(x), \lambda(0)) = \min\{1 - \lambda(x) + \lambda(0), 1\}.$$

If $\lambda(x) \leq \lambda(0)$, then

$$I(\lambda(x), \lambda(0)) = \min\{1 - \lambda(x) + \lambda(0), 1\} = 1 \geq 0.5.$$

If $0 < \lambda(x) - \lambda(0) \leq 0.5$, then

$$\begin{aligned} I(\lambda(x), \lambda(0)) &= \min\{1 - \lambda(x) + \lambda(0), 1\} \\ &= 1 - \lambda(x) + \lambda(0) \geq 0.5. \end{aligned}$$

Also, if $\min\{\lambda(x \cdot y), \lambda(x)\} \leq \lambda(y)$, then

$$\begin{aligned} &I(\min\{\lambda(x \cdot y), \lambda(x)\}, \lambda(y)) \\ &= \min\{1 - \min\{\lambda(x \cdot y), \lambda(x)\} + \lambda(y), 1\} = 1 \geq 0.5. \end{aligned}$$

If $0 < \min\{\lambda(x \cdot y), \lambda(x)\} - \lambda(y) \leq 0.5$, then

$$\begin{aligned} &I(\min\{\lambda(x \cdot y), \lambda(x)\}, \lambda(y)) \\ &= \min\{1 - \min\{\lambda(x \cdot y), \lambda(x)\} + \lambda(y), 1\} \\ &= 1 - \min\{\lambda(x \cdot y), \lambda(x)\} + \lambda(y) \geq 0.5. \end{aligned}$$

So, λ is an I-fuzzy filter of X with respect to $t = 0.5$. ■

V. Conclusion

Through the fuzzy UP-filters of UP-algebras, we have discovered certain links between fuzzy mathematics and probability theory. We presented the concept of falling UP-filters in UP-algebras as an algebraic approach to the idea of falling shadows. We've shown how fuzzy UP-filters and falling UP-filters are related. A falling UP-filter has been proven to be a generalization of a fuzzy UP-filter. We found a related result by applying the concept of falling inference relations to UP-algebras. In a future research, we will extend the theory of falling shadows to additional types of ideals, filters, and deductive systems in BCK/BCI-algebras, KU-algebras, and SU-algebras, among others, based on these results. We also hope that these results can be applied to computer and information systems.

References

[1] P. Z. Wang and E. Sanchez, *Treating a fuzzy subset as a projectable random set, in: M. M. Gupta, E. Sanchez, Eds., "Fuzzy Information and Decision"*. Pergamon, New York, 1982.

[2] P. Z. Wang, *Fuzzy Sets and Falling Shadows of Random Sets*. Beijing Normal Univ. Press, People's Republic of China, 1985.

[3] S. K. Tan, P. Z. Wang, and E. S. Lee, "Fuzzy set operations based on the theory of falling shadows," *J. Math. Anal. Appl.*, vol. 174, pp. 242–255, 1993.

[4] S. K. Tan, P. Z. Wang, and X. Z. Zhang, "Fuzzy inference relation based on the theory of falling shadows," *Fuzzy Sets Syst.*, vol. 53, pp. 179–188, 1993.

[5] Y. B. Jun and M. S. Kang, "Fuzzy positive implicative ideals of BCK-algebras based on the theory of falling shadows," *Comput. Math. Appl.*, vol. 61, pp. 62–67, 2011.

[6] A. Iampan, "A new branch of the logical algebra: UP-algebras," *J. Algebra Relat. Top.*, vol. 5, no. 1, pp. 35–54, 2017.

[7] Y. B. Jun, K. J. Lee, and A. Iampan, "Falling shadow theory applied to UP-algebras," *Thai J. Math.*, vol. 17, no. 1, pp. 1–9, 2019.

[8] A. Iampan, "Introducing fully UP-semigroups," *Discuss. Math., Gen. Algebra Appl.*, vol. 38, no. 2, pp. 297–306, 2018.

[9] A. Iampan, M. Songsaeng, and G. Muhiuddin, "Fuzzy duplex UP-algebras," *Eur. J. Pure Appl. Math.*, vol. 13, no. 3, pp. 459–471, 2020.

[10] T. Klinseesook, S. Bukok, and A. Iampan, "Rough set theory applied to UP-algebras," *J. Inf. Optim. Sci.*, vol. 41, no. 3, pp. 705–722, 2020.

[11] A. Satirad, P. Mosrijai, and A. Iampan, "Formulas for finding UP-algebras," *Int. J. Math. Comput. Sci.*, vol. 14, no. 2, pp. 403–409, 2019.

[12] ——, "Generalized power UP-algebras," *Int. J. Math. Comput. Sci.*, vol. 14, no. 1, pp. 17–25, 2019.

[13] S. Sripaeng, K. Tanamoon, and A. Iampan, "On anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras," *J. Inf. Optim. Sci.*, vol. 39, no. 5, pp. 1095–1127, 2018.

[14] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, "Fuzzy sets in UP-algebras," *Annal. Fuzzy Math. Inform.*, vol. 12, no. 6, pp. 739–756, 2016.

[15] I. R. Goodman, *Fuzzy sets as equivalence classes of random sets, in "Recent Developments in Fuzzy Sets and Possibility Theory"(R. Yager, Ed.).* Pergamon, New York, 1982.

[16] T. Guntasow, S. Sajak, A. Jomkhan, and A. Iampan, "Fuzzy translations of a fuzzy set in UP-algebras," *J. Indones. Math. Soc.*, vol. 23, no. 2, pp. 1–19, 2017.