Adaptive Synchronization of Two Layers Coupled Network with Multi-weights and Time-varying Delay

Wenju Du

***Abstract*—The paper proposed a new two layers coupled network with multi-weights and time-varying delay. According to the method of network split, the complex network of multi-weights is divided into multiple sub-networks with different properties, so that each edge of each sub-network has only a single weight. Besides, we designed a new controller and then investigated the adaptive synchronization problem of the two layers coupled networks with multi-weights and time-varying delay. Finally, we take the Lorenz system as an example to prove the validity of the presented method.**

***Index Terms*—Two layers networks; Adaptive synchronization; Multi-weights; Time-varying delay**

I. INTRODUCTION

THERE are many complex systems in real life, and these systems can be abstracted as the complex networks, and the research of complex network has penetrated the various fields. At present, many scholars have been in-depth studied the synchronization problem of the complex network. In fact, the synchronization of complex network is not just limited to the single network. In recent years, the research of the synchronization problem of two layers complex network has attracted the attention of many scholars. Tang et al. [1] designed an effective adaptive controller and then addressed the theoretical analysis of synchronization between two complex networks with nonidentical topological structures. Chen et al. [2] presented a general model of two complex networks with time-varying delay and then derived a synchronization criterion by using the adaptive controller. Wang et al. [3] designed an adaptive controller to achieve synchronization between two different complex networks with time-varying delay. Sun et al. [4] investigated the linear generalized synchronization between two complex networks. Wang et al. [5] established an optimization model of urban public transit skeleton-network, and then the genetic algorithm and tabu search algorithm are designed. Shen et al. [6] investigated the cascading failure model of two layer complex networks based on betweenness analytical method. Yu [7] established a two layers coupled network model, and then identified the influential nodes and made the control policies of information spreading on two layers coupled network model. Ma et al. [8] proposed an improved global awareness routing strategy, and the traffic capacity problem of the multi-layer network is studied by using the proposed strategy. Due to the existence of time-varying delay is inevitable in real life and many synchronization phenomenon are affected by the time-varying delay, so the study of synchronization problem of the coupled network with time-varying delay is particularly important.

In addition, many networks are weighted networks in reality, and introduce the weight into the edge of complex networks can convenient for the connections between any two nodes, and the weight of the edge will affect the structure and function of the network. However, most of the studies of the complex network only focus on the complex network with single weight at present [9-15]. And the network with multi-weights can describe many real-world networks and the research on the complex network with multi-weights is more important. Lan et al. [16] modeled a network with double-weights by 93 stocks of coal & power sectors in China stock market and then the strong correlation among stocks and construct portfolio reasonably are analyzed. Zhang et al. [17] established a new complex network model with multi-weights, and then the globally adaptive synchronization of the network is investigated. An et al. [18] established a new public traffic network with multi-weights and then designed the criteria for the global synchronization of the new network. Dai et al. [19] introduced the double-weighted Koch network based on actual road networks, and the results show that the double-weighted Koch network are more efficient than classic Koch networks in receiving information. Du et al. [20] focus on the two different complex networks with multi-weights and they splits the complex network with multi-weights into several different single weighted two layers coupled networks based on the method of network split, and then investigate its global synchronization.

This paper innovatively established a two layers coupled network with multi-weights and time-varying delay, that is, there are many different weights on each edge of the network. Through the method of network splitting, the complex network with multi-weights are divided into multiple sub-networks with different properties, so that each edge of each sub-network has only a single weight. And then the synchronization problem of the above new network are investigated in detail.

The paper organizes as follows. In section 2, a new two layers coupled network with multi-weights is proposed. The synchronization criterion of the new network is designed in section 3. In section 4, some simulation results are given to

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show the validity of the presented method. We conclude the paper in section 5.

II. A NEW TWO LAYERS COUPLED NETWORK WITH MULTI-WEIGHTS

In the real world, there exist many two layers coupled networks with multi-weights which composed of multiple sub-networks of various properties. The following is a two layers coupled network with three-weights, and the topology map of the network as showing in Fig. 1.

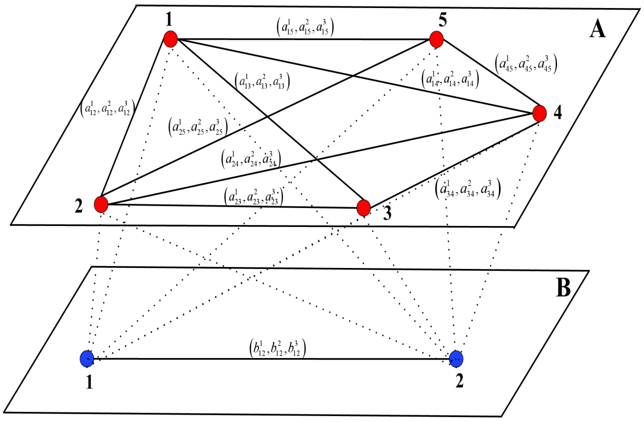

Fig. 1. The topology map of two layers coupled network model

Through the method of network split, the two layers coupled network with three-weights can be split into three single weighted two layers coupled network, and the split schematic diagram as shown in Fig. 2.

III. SYNCHRONIZATION BETWEEN TWO DIFFERENT NETWORKS WITH MULTI-WEIGHTS AND TIME-VARYING DELAY

Consider the two complex networks with multi-weights and time-varying delay and both of them consisting of l weights, and the models of the networks are described by

$$\begin{aligned}\dot{x}_i(t) &= f(x_i(t)) + \varepsilon_1^1 \sum_{j=1}^{N_1} a_{ij}^1 H_1^1 x_j(t-\tau(t)) + \varepsilon_2^1 \sum_{j=1}^{N_1} a_{ij}^2 \\ &\quad \times H_2^1 x_j(t-\tau(t)) + \cdots + \varepsilon_l^1 \sum_{j=1}^{N_1} a_{ij}^l H_l^1 x_j(t-\tau(t)) \\ &\quad + \mu \sum_{j=1}^{N_2} c_{ij} y_j(t-\tau(t)),\ \ i=1,2,\ldots,N_1\end{aligned} \tag{1}$$

$$\begin{aligned}\dot{y}_i(t) &= g(y_i(t)) + \varepsilon_1^2 \sum_{j=1}^{N_2} b_{ij}^1 H_1^2 y_j(t-\tau(t)) + \varepsilon_2^2 \sum_{j=1}^{N_2} b_{ij}^2 \\ &\quad \times H_2^2 y_j(t-\tau(t)) + \cdots + \varepsilon_l^2 \sum_{j=1}^{N_2} b_{ij}^l H_l^2 y_j(t-\tau(t)) \\ &\quad + \mu \sum_{j=1}^{N_1} d_{ij} x_j(t-\tau(t)) + u_i,\ \ i=1,2,\ldots,N_2\end{aligned} \tag{2}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^{\mathrm{T}} \in R^n, y_i = (y_{i1}, y_{i2}, \ldots, y_{in})^{\mathrm{T}} \in R^n$ are the state vectors of the i th node of network (1) and network (2), $\dot{x}_i(t), \dot{y}_i(t)$ are the dynamic equations of the single node, $f(\cdot), g(\cdot): R^n \to R^n$ are the nonlinear continuous differentiable vector functions, N_1, N_2 are the number of nodes of networks (1) and (2) respectively, $H_1^1, H_2^1, \cdots, H_l^1 \in R^{n\times n}$, $H_1^2, H_2^2, \cdots, H_l^2 \in R^{n\times n}$ are the internal coupling functions between the node state variables of two sub-networks, the constants $\varepsilon_l^1, \varepsilon_l^2$ denotes the inner coupling strengths of l sub-network in network (1) and network (2) respectively, the constant μ denotes the outer coupling strength, $\tau(t)$ is the time-varying delay. Coupling matrices $A^m = (a_{ij}^m) \in R^{N_1\times N_1}, B^m = (b_{ij}^m) \in R^{N_2 \times N_2}$ $(m=1,2,\cdots,l)$ are respectively the inner connection matrices of m sub-network in networks (1) and (2), where a_{ij}^m, b_{ij}^m are defined as follows: $a_{ij}^m = a_{ji}^m (i \neq j)$ is the weight of the edge between node i and j of m sub-network in network (1), and $b_{ij}^m = b_{ji}^m (i\neq j)$ is the weight of the edge between node i and j of m sub-network in network (2), if there is no connection from node i to node j, then $a_{ij}^m = a_{ji}^m (b_{ij}^m = b_{ji}^m) = 0 (i \neq j)$, and the diagonal element of matrices A^m and B^m respectively defined as: $a_{ii}^m = -\sum_{j=1, i\neq j}^{N_1} a_{ij}^m = -\sum_{j=1,i\neq j}^{N_1} a_{ji}^m (i=1,2,\cdots,N_1, m=1,2,\cdots,l)$ and $b_{ii}^m = -\sum_{j=1,i\neq j}^{N_2} b_{ij}^m = -\sum_{j=1,i\neq j}^{N_2} b_{ji}^m (i=1,2,\cdots,N_2, m=1,2,\cdots,l)$. The matrices $C=(c_{ij}) \in R^{N_1\times N_2}$, and $D=(d_{ij}) \in R^{N_2\times N_1}$ are the coupling matrices between network (1) and network (2), where c_{ij}, d_{ij} are defined as follows: if there is a connection from node i (belongs to network (1)) to node j (belongs to network (2)), then c_{ij} is equal to the weight of this edge, otherwise $c_{ij}=0$; and $d_{ji} = c_{ij} (i=1,2,\ldots,N_1, j=1,2,\ldots,N_2)$. And $u_i(t)$ is the controller of node i to be designed. Without loss of generality, we assume that $N_1 > N_2$, namely networks (1) and (2) has the different number of nodes. Besides, $(a_{ij}^1, a_{ij}^2, \cdots, a_{ij}^l)$ are the weights of the edge of sub-network (1), and a_{ij}^l is the l th weight between node i and node j of sub-network (1); $(b_{ij}^1, b_{ij}^2, \cdots, b_{ij}^l)$ are the weights of the edge of sub-network (2), and b_{ij}^l is the l th weight between node i and node j of sub-network (2). Therefore, the coupled complex networks with multi-weights are split into l sub-networks with single weight by the theory of network split.

Definition 1. Let $x_i(t, X_0)(i=1,2,\ldots,N_1)$ and $y_i(t, Y_0, u_i)$ $(i=1,2,\ldots,N_2)$ are the solutions of the networks (1) and (2), where $X_0 = (x_1^0, x_2^0, \ldots, x_{N_1}^0)^{\mathrm{T}} \in R^{nN_1}$, $Y_0 = (y_1^0, y_2^0, \ldots, y_{N_2}^0)^{\mathrm{T}} \in R^{nN_2}$, and $f, g: \Omega \to R^n$ are the continuously differentiable mappings with $\Omega \subseteq R^n$. If there exist a nonempty open subset $\Lambda \subseteq \Omega$, with $x_i^0, y_i^0 \in \Lambda$, so when $t \geq 0$, such that $x_i(t, X_0)(1\leq i \leq N_1)$, $y_i(t, Y_0, u_i)(1 \leq i \leq N_2) \in \Omega$, and

$$\lim_{t\to\infty} \| y_i(t, Y_0, u_i) - x_i(t, X_0) \| = 0,\ \ (i=1,2,\cdots,N_2) \tag{3}$$

then the complex networks (1) and (2) are achieved synchronization.

Assumption 1. For function $f(x)$, there exist a positive constant L such that

$$\| f(y(t)) - f(x(t)) \| \leq L \| y(t) - x(t) \|,$$

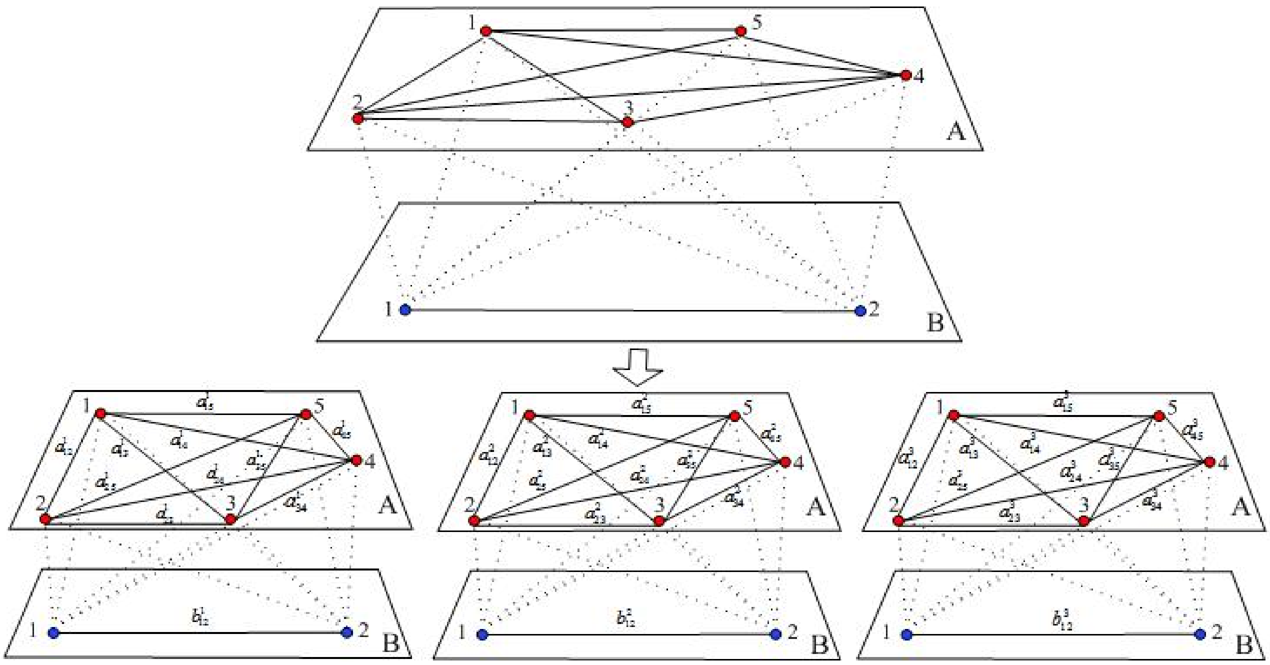

Fig. 2. The topology map of complex dynamical networks with three-weights and its split

where $\forall x(t), y(t) \in R^n$.

Theorem 1. Suppose that Assumption 1 holds, and we select the controllers as follows:

$$
\begin{aligned}
u_i = {} & f\left(y_i(t)\right) - g\left(y_i(t)\right) + \sum_{m=1}^{l}\sum_{j=1}^{N_2} \varepsilon^1_m a^m_{ij} H^1_m y_j\left(t-\tau(t)\right) - \\
& \sum_{m=1}^{l}\sum_{j=1}^{N_1} \varepsilon^2_m b^m_{ij} H^2_m x_j\left(t-\tau(t)\right) + \sum_{m=1}^{l}\sum_{j=N_2+1}^{N_1} \left(\varepsilon^1_m a^m_{ij} H^1_m + \varepsilon^2_m b^m_{ij} H^2_m\right) \\
& \times x_j\left(t-\tau(t)\right) - \mu \sum_{j=N_2+1}^{N_1} \left(c_{ij}+d_{ij}\right) x_j\left(t-\tau(t)\right) - g_i e_i(t),
\end{aligned} \tag{4}
$$

then the driving network (1) and the response network (2) can realize synchronization under the controller (4), where $\dot{g}_i = k_i \left\|e_i\right\|^2$, k_i are the positive constants, $i = 1,2,\ldots,N_2$.

Proof Define the error vector by $e_i(t) = y_i(t) - x_i(t), i = 1,2,\ldots,N_2$, and the error system can be described by:

$$
\begin{aligned}
\dot{e}_i(t) = {} & g\left(y_i(t)\right) - f\left(x_i(t)\right) + \varepsilon^2_1 \sum_{j=1}^{N_2} b^1_{ij} H^2_1 y_j\left(t-\tau(t)\right) - \\
& \varepsilon^1_1 \sum_{j=1}^{N_1} a^1_{ij} H^1_1 x_j\left(t-\tau(t)\right) + \varepsilon^2_2 \sum_{j=1}^{N_2} b^2_{ij} H^2_2 y_j\left(t-\tau(t)\right) - \\
& \varepsilon^1_2 \sum_{j=1}^{N_1} a^2_{ij} H^1_2 x_j\left(t-\tau(t)\right) + \cdots + \varepsilon^2_l \sum_{j=1}^{N_2} b^l_{ij} H^2_l y_j\left(t-\tau(t)\right) - \\
& \varepsilon^1_l \sum_{j=1}^{N_1} a^l_{ij} H^1_l x_j\left(t-\tau(t)\right) + \mu \sum_{j=1}^{N_1} d_{ij} x_j\left(t-\tau(t)\right) - \mu \sum_{j=1}^{N_2} c_{ij} y_j \times \\
& \left(t-\tau(t)\right) + f\left(y_i(t)\right) - g\left(y_i(t)\right) - \varepsilon^2_1 \sum_{j=1}^{N_1} b^1_{ij} H^2_1 x_j\left(t-\tau(t)\right) \\
& + \varepsilon^1_1 \sum_{j=1}^{N_2} a^1_{ij} H^1_1 y_j\left(t-\tau(t)\right) - \varepsilon^2_2 \sum_{j=1}^{N_1} b^2_{ij} H^2_2 x_j\left(t-\tau(t)\right) + \\
& \varepsilon^1_2 \sum_{j=1}^{N_2} a^2_{ij} H^1_2 y_j\left(t-\tau(t)\right) + \cdots + \varepsilon^1_l \sum_{j=1}^{N_2} a^l_{ij} H^1_l y_j\left(t-\tau(t)\right) \\
& - \varepsilon^2_l \sum_{j=1}^{N_1} b^l_{ij} H^2_l x_j\left(t-\tau(t)\right) - \mu \sum_{j=1}^{N_2} d_{ij} y_j\left(t-\tau(t)\right) + \\
& \mu \sum_{j=1}^{N_1} c_{ij} x_j\left(t-\tau(t)\right) + \varepsilon^2_1 \sum_{j=N_2+1}^{N_1} b^1_{ij} H^2_1 x_j\left(t-\tau(t)\right) + \\
& \varepsilon^1_1 \sum_{j=N_2+1}^{N_1} a^1_{ij} H^1_1 x_j\left(t-\tau(t)\right) + \varepsilon^2_2 \sum_{j=N_2+1}^{N_1} b^2_{ij} H^2_2 x_j\left(t-\tau(t)\right) + \\
& \varepsilon^1_2 \sum_{j=N_2+1}^{N_1} a^2_{ij} H^1_2 x_j\left(t-\tau(t)\right) + \cdots + \varepsilon^2_l \sum_{j=N_2+1}^{N_1} b^l_{ij} H^2_l x_j\left(t-\tau(t)\right) \\
& + \varepsilon^1_l \sum_{j=N_2+1}^{N_1} a^l_{ij} H^1_l x_j\left(t-\tau(t)\right) - \mu \sum_{j=N_2+1}^{N_1} d_{ij} x_j\left(t-\tau(t)\right) - \\
& \mu \sum_{j=N_2+1}^{N_1} c_{ij} x_j\left(t-\tau(t)\right) - g_i e_i(t) \\
= {} & f\left(y_i(t)\right) - f\left(x_i(t)\right) + \varepsilon^1_1 \sum_{j=1}^{N_2} a^1_{ij} H^1_1 e_j\left(t-\tau(t)\right) + \varepsilon^2_1 \sum_{j=1}^{N_2} b^1_{ij} \\
& \times H^2_1 e_j\left(t-\tau(t)\right) + \varepsilon^1_2 \sum_{j=1}^{N_1} a^2_{ij} H^1_2 e_j\left(t-\tau(t)\right) + \varepsilon^2_2 \sum_{j=1}^{N_2} b^2_{ij} H^2_2 \\
& \times e_j\left(t-\tau(t)\right) + \cdots + \varepsilon^1_l \sum_{j=1}^{N_2} a^l_{ij} H^1_l e_j\left(t-\tau(t)\right) + \varepsilon^2_l \sum_{j=1}^{N_2} b^l_{ij} H^2_l e_j \\
& \times\left(t-\tau(t)\right) - \mu \sum_{j=1}^{N_2} \left(c_{ij}+d_{ij}\right) e_j\left(t-\tau(t)\right) - g_i e_i(t) \\
= {} & f\left(y_i(t)\right) - f\left(x_i(t)\right) + \sum_{m=1}^{l}\sum_{j=1}^{N_2} \varepsilon^1_m a^m_{ij} H^1_m e_j\left(t-\tau(t)\right) + \sum_{m=1}^{l}\sum_{j=1}^{N_2} \varepsilon^2_m \\
& \times b^m_{ij} H^2_m e_j\left(t-\tau(t)\right) - \mu \sum_{j=1}^{N_2} \left(c_{ij}+d_{ij}\right) e_j\left(t-\tau(t)\right) - g_i e_i(t) \\
= {} & f\left(y_i(t)\right) - f\left(x_i(t)\right) + \sum_{m=1}^{l}\sum_{j=1}^{N_2} \left(\varepsilon^1_m a^m_{ij} H^1_m + \varepsilon^2_m b^m_{ij} H^2_m\right) e_j\left(t-\tau(t)\right) \\
& - \mu \sum_{j=1}^{N_2} \left(c_{ij}+d_{ij}\right) e_j\left(t-\tau(t)\right) - g_i e_i(t).
\end{aligned}
$$

Choose the Lyapunov candidates as follows:

$$
\begin{aligned}
V(t) = {} & \frac{1}{2}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) + \frac{1}{2}\sum_{i=1}^{N_2} \frac{1}{k_i}\left(g_i - \bar{g}\right)^2 \\
& + \frac{1}{2(1-\xi)} \sum_{i=1}^{N_2} \int_{t-\tau(t)}^{t} e_i^{\mathrm{T}}(\alpha) e_i(\alpha) d\alpha,
\end{aligned} \tag{6}
$$

where $\overline{g}$ is a sufficiently larger positive constant which is to be determined. By using Assumption 1 and derivation of the Eq. (5), we can get the following formula:

$$
\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)\dot{e}_i(t) + \frac{1}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)e_i(t) - \\
&\quad \frac{1-\dot{\tau}(t)}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t-\tau(t))e_i(t-\tau(t)) + \sum_{i=1}^{N_2}\frac{1}{k_i}(g_i-\overline{g})\dot{g}_i \\
&= \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)\Big[f(y_i(t)) - f(x_i(t)) + \sum_{m=1}^{l}\sum_{j=1}^{N_2}\left(\varepsilon_m^1 a_{ij}^m H_m^1 + \varepsilon_m^2 b_{ij}^m H_m^2\right) \\
&\quad \times e_j(t-\tau(t)) - \sum_{j=1}^{N_2}\mu\left(c_{ij}+d_{ij}\right)e_j(t-\tau(t)) - g_i e_i(t)\Big] + \\
&\quad \frac{1}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)e_i(t) - \frac{1-\dot{\tau}(t)}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t-\tau(t))e_i(t-\tau(t)) \\
&\quad + \sum_{i=1}^{N_2}(g_i-\overline{g})\left\|e_i(t)\right\|^2 \\
&\le \sum_{i=1}^{N_2}(L-\overline{g})\left\|e_i(t)\right\|^2 + \sum_{m=1}^{l}\sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)\left(\varepsilon_m^1 a_{ij}^m H_m^1 + \varepsilon_m^2 b_{ij}^m H_m^2\right) \\
&\quad \times e_j(t-\tau(t)) - \sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)\mu\left(c_{ij}+d_{ij}\right)e_j(t-\tau(t)) + \\
&\quad \frac{1}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)e_i(t) - \frac{1-\dot{\tau}(t)}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t-\tau(t))e_i(t-\tau(t)) \\
&= \sum_{i=1}^{N_2}(L-\overline{g})\left\|e_i(t)\right\|^2 + \sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)\varepsilon_1^1 a_{ij}^1 H_1^1 e_j(t-\tau(t)) + \\
&\quad \sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)\varepsilon_2^1 a_{ij}^2 H_2^1 e_j(t-\tau(t)) + \cdots + \sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)\varepsilon_l^1 a_{ij}^l H_l^1 \\
&\quad \times e_j(t-\tau(t)) + \sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)\varepsilon_1^2 b_{ij}^1 H_1^2 e_j(t-\tau(t)) + \\
&\quad \sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)\varepsilon_2^2 b_{ij}^2 H_2^2 e_j(t-\tau(t)) + \cdots + \sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)\varepsilon_l^2 b_{ij}^l H_l^2 \\
&\quad \times e_j(t-\tau(t)) - \sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)\mu\left(c_{ij}+d_{ij}\right)e_j(t-\tau(t)) + \\
&\quad \frac{1}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)e_i(t) - \frac{1-\dot{\tau}(t)}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t-\tau(t))e_i(t-\tau(t)) \\
&= \sum_{i=1}^{N_2}(L-\overline{g})\left\|e_i(t)\right\|^2 + \varepsilon_1^1\sum_{j=1}^{n} e_j^{\mathrm{T}}(t)\gamma_{1j}^1 A_1' e_j(t-\tau(t)) + \varepsilon_2^1 \times \\
&\quad \sum_{j=1}^{n} e_j^{\mathrm{T}}(t)\gamma_{2j}^1 A_2' e_j(t-\tau(t)) + \cdots + \varepsilon_l^1\sum_{j=1}^{n} e_j^{\mathrm{T}}(t)\gamma_{lj}^1 A_l' \times \\
&\quad e_j(t-\tau(t)) + \varepsilon_1^2\sum_{j=1}^{n} e_j^{\mathrm{T}}(t)\gamma_{1j}^2 B_1' e_j(t-\tau(t)) + \varepsilon_2^2\sum_{j=1}^{n} e_j^{\mathrm{T}}(t) \\
&\quad \times \gamma_{2j}^2 B_2' e_j(t-\tau(t)) + \cdots + \varepsilon_l^2\sum_{j=1}^{n} e_j^{\mathrm{T}}(t)\gamma_{lj}^2 B_l' e_j(t-\tau(t)) - \\
&\quad \mu\sum_{j=1}^{n} e_j^{\mathrm{T}}(t)(C'+D)e_j(t-\tau(t)) + \frac{1}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad - \frac{1-\dot{\tau}(t)}{2(1-\xi)}\sum_{j=1}^{n} e_j^{\mathrm{T}}(t-\tau(t))e_j(t-\tau(t)) \\
&\le \sum_{i=1}^{N_2}(L-\overline{g})\left\|e_i(t)\right\|^2 + \sum_{j=1}^{n}\frac{\left(\gamma_{1j}^1\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_1^1 A_1'\right)\left(\varepsilon_1^1 A_1'\right)^{\mathrm{T}} e_j(t) \\
&\quad + \sum_{j=1}^{n}\frac{\left(\gamma_{2j}^1\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_2^1 A_2'\right)\left(\varepsilon_2^1 A_2'\right)^{\mathrm{T}} e_j(t) + \cdots + \sum_{j=1}^{n}\frac{\left(\gamma_{lj}^1\right)^2}{2} \\
&\quad \times e_j^{\mathrm{T}}(t)\left(\varepsilon_l^1 A_l'\right)\left(\varepsilon_l^1 A_l'\right)^{\mathrm{T}} e_j(t) + \sum_{j=1}^{n}\frac{\left(\gamma_{1j}^2\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_1^2 B_1'\right) \\
&\quad \times\left(\varepsilon_1^2 B_1'\right)^{\mathrm{T}} e_j(t) + \sum_{j=1}^{n}\frac{\left(\gamma_{2j}^2\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_2^2 B_2'\right)\left(\varepsilon_2^2 B_2'\right)^{\mathrm{T}} e_j(t) \\
&\quad + \cdots + \sum_{j=1}^{n}\frac{\left(\gamma_{lj}^2\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_l^2 B_l'\right)\left(\varepsilon_l^2 B_l'\right)^{\mathrm{T}} e_j(t) - \sum_{j=1}^{n}\frac{1}{2} e_j^{\mathrm{T}}(t) \\
&\quad \times\left(\mu C' + \mu D\right)\left(\mu C' + \mu D\right)^{\mathrm{T}} e_j(t) + \frac{1}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad + \left(\frac{1}{2} - \frac{1-\dot{\tau}(t)}{2(1-\xi)}\right)\sum_{j=1}^{n} e_j^{\mathrm{T}}(t-\tau(t))e_j(t-\tau(t)) \\
&\le \sum_{i=1}^{N_2}(L-\overline{g})\left\|e_i(t)\right\|^2 + \sum_{j=1}^{n}\frac{\left(\gamma_{1j}^1\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_1^1 A_1'\right)\left(\varepsilon_1^1 A_1'\right)^{\mathrm{T}} e_j(t) \\
&\quad + \sum_{j=1}^{n}\frac{\left(\gamma_{2j}^1\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_2^1 A_2'\right)\left(\varepsilon_2^1 A_2'\right)^{\mathrm{T}} e_j(t) + \cdots + \sum_{j=1}^{n}\frac{\left(\gamma_{lj}^1\right)^2}{2} \\
&\quad \times e_j^{\mathrm{T}}(t)\left(\varepsilon_l^1 A_l'\right)\left(\varepsilon_l^1 A_l'\right)^{\mathrm{T}} e_j(t) + \sum_{j=1}^{n}\frac{\left(\gamma_{1j}^2\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_1^2 B_1'\right) \\
&\quad \times\left(\varepsilon_1^2 B_1'\right)^{\mathrm{T}} e_j(t) + \sum_{j=1}^{n}\frac{\left(\gamma_{2j}^2\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_2^2 B_2'\right)\left(\varepsilon_2^2 B_2'\right)^{\mathrm{T}} e_j(t) \\
&\quad + \cdots + \sum_{j=1}^{n}\frac{\left(\gamma_{lj}^2\right)^2}{2} e_j^{\mathrm{T}}(t)\left(\varepsilon_l^2 B_l'\right)\left(\varepsilon_l^2 B_l'\right)^{\mathrm{T}} e_j(t) - \sum_{j=1}^{n}\frac{1}{2} e_j^{\mathrm{T}}(t) \\
&\quad \times\left(\mu C' + \mu D\right)\left(\mu C' + \mu D\right)^{\mathrm{T}} e_j(t) + \frac{1}{2(1-\xi)}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)e_i(t) \\
&\le \sum_{i=1}^{N_2}\left(L-\overline{g}+\frac{1}{2(1-\xi)}\right)\left\|e_i(t)\right\|^2 + \gamma_j\sum_{j=1}^{n} e_j^{\mathrm{T}}(t)Pe_j(t) \\
&\le e^{\mathrm{T}}(t)\left\{\left[L-\overline{g}+\frac{1}{2(1-\xi)} + \max_{1\le j\le n}\left(\gamma_j\right)\lambda_{\max}(P)\right]I_{nN_2}\right\}e(t) \\
&= \boldsymbol{e}^{\mathrm{T}}(t)Q\boldsymbol{e}(t),
\end{aligned}
$$

where $\gamma_{1j}^1,\gamma_{2j}^1,\ldots,\gamma_{lj}^1$ are respectively the j th diagonal elements of $H_1^1,H_2^1,\cdots,H_l^1$, and $\gamma_{1j}^2,\gamma_{2j}^2,\ldots,\gamma_{lj}^2$ are the j th diagonal elements of $H_1^2,H_2^2,\cdots,H_l^2$ respectively, and

$$\boldsymbol{e}(t) = \left(e_1(t), e_2(t), \ldots, e_{N_2}(t)\right)^{\mathrm{T}} \in R^{nN_2},$$

$$\gamma_j = \max\left\{\frac{\left(\gamma_{1j}^1\right)^2}{2}, \frac{\left(\gamma_{2j}^1\right)^2}{2}, \ldots, \frac{\left(\gamma_{lj}^1\right)^2}{2}, \frac{\left(\gamma_{1j}^2\right)^2}{2}, \frac{\left(\gamma_{2j}^2\right)^2}{2}, \ldots, \frac{\left(\gamma_{lj}^2\right)^2}{2}, \frac{1}{2}\right\},$$

$$
\begin{aligned}
P &= \left(\varepsilon_1^1 A_1'\right)\left(\varepsilon_1^1 A_1'\right)^{\mathrm{T}} + \left(\varepsilon_2^1 A_2'\right)\left(\varepsilon_2^1 A_2'\right)^{\mathrm{T}} + \cdots + \left(\varepsilon_l^1 A_l'\right)\left(\varepsilon_l^1 A_l'\right)^{\mathrm{T}} \\
&\quad + \left(\varepsilon_1^2 B_1'\right)\left(\varepsilon_1^2 B_1'\right)^{\mathrm{T}} + \left(\varepsilon_2^2 B_2'\right)\left(\varepsilon_2^2 B_2'\right)^{\mathrm{T}} + \cdots + \\
&\quad \left(\varepsilon_l^2 B_l'\right)\left(\varepsilon_l^2 B_l'\right)^{\mathrm{T}} + \left(\mu C' + \mu D\right)\left(\mu C' + \mu D\right)^{\mathrm{T}},
\end{aligned}
$$

$$Q = \left[L-\overline{g}+\frac{1}{2(1-\xi)} + \max_{1\le j\le n}\left(\gamma_j\right)\lambda_{\max}(P)\right]I_{nN_2},$$

$\lambda_{\max}(P)$ is the largest eigenvalue of matrix P, I_{nN_2} is the unit matrix, and the matrices $A_1', A_2', \cdots, A_l', B_1', B_2', \cdots, B_l', C'$ are respectively the N_2 order principal minor determinant of matrices $A_1, A_2, \cdots, A_l, B_1, B_2, \cdots, B_l, C$. Obviously, there exist a sufficiently large positive constant $\bar{g}$ such that the symmetry matrix Q is negative definite, namely, $\dot{V}(t) < 0$. Here, the largest invariant set contained in set $E = \{\dot{V}(t) = 0\} = \{\mathbf{e}(t) = 0,\ i = 1, 2, \ldots, N_2\}$ and which can be described as:

$$M = \left\{(\mathbf{e}, \mathbf{g}) \in R^{nN_2} \times R^{N_2} : \mathbf{e} = 0, \dot{\mathbf{g}} = 0\right\},$$

where $\mathbf{g} = (g_1, g_2, \ldots, g_{N_2})^{\mathrm{T}}$. According to the LaSalle's invariance principle, all the solutions which starting with arbitrary initial values, it's trajectory asymptotically converges to the largest invariant M which implies that $\lim_{t\to\infty} e_i(t) = 0, i = 1, 2, \ldots, N_2$, so the network (1) and network (2) are realized synchronization.

IV. NUMERICAL SIMULATIONS

To verify the effectiveness of the method proposed in this paper, the synchronization problem was studied with computer simulation for networks (1) and (2). For simplicity, we investigate the two layers coupled network with three-weights as shown in Fig. 1. Suppose that the dynamical equation for a single node of networks (1) and (2) are Lorenz chaotic system, so the nodes dynamical equations can be described as follows:

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} + \begin{bmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{bmatrix}, \quad (7)$$

$$\begin{bmatrix} \dot{y}_{i1} \\ \dot{y}_{i2} \\ \dot{y}_{i3} \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix} \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \end{bmatrix} + \begin{bmatrix} 0 \\ -y_{i1}y_{i3} \\ y_{i1}y_{i2} \end{bmatrix}, \quad (8)$$

For any vectors x_i and y_i of the Lorenz chaotic system, there exist a positive constant R such that $\|x_{im}\| \le R, \|y_{im}\| \le R (m = 1, 2, 3)$, because of the Lorenz chaotic system is bounded in the certain region. So, we have

$$\begin{aligned} \|f(y_i) - f(x_i)\| &= \sqrt{(-y_{i1}y_{i3} - (-x_{i1}x_{i3}))^2 + (y_{i1}y_{i2} - x_{i1}x_{i2})^2} \\ &= \sqrt{(-y_{i3}(y_{i1} - x_{i1}) - x_{i1}(y_{i3} - x_{i3}))^2 + (y_{i2}(y_{i1} - x_{i1}) + x_{i1}(y_{i2} - x_{i2}))^2} \\ &\le \sqrt{2}R\|y_i - x_i\|, \end{aligned} \quad (9)$$

namely, the Assumption 1 is satisfied.

Assuming that

$H_1^1 = H_2^1 = H_3^1 = H_4^1 = H_5^1 = H_1^2 = H_2^2 = H_3^2 = \mathrm{diag}\{1,1,1\}$,

and the node of networks A and B have the same dynamical equations, then the controllers can be designed as follows:

$$\begin{aligned} u_i = &\sum_{m=1}^{3}\sum_{j=1}^{2} \varepsilon_m^1 a_{ij}^m y_j(t - \tau(t)) - \sum_{m=1}^{3}\sum_{j=1}^{5} \varepsilon_m^2 b_{ij}^m x_j(t - \tau(t)) \\ &+ \sum_{m=1}^{3}\sum_{j=4}^{5} \left(\varepsilon_m^1 a_{ij}^m + \varepsilon_m^2 b_{ij}^m\right) x_j(t - \tau(t)) \\ &- \mu \sum_{j=4}^{5} (c_{ij} + d_{ij}) x_j(t - \tau(t)) - g_i e_i(t), \end{aligned} \quad (10)$$

where $\dot{g}_i = k_i \|e_i\|^2$, k_i are the positive constants, $i = 1, 2$.

According to Eq. (1), the dynamical equations of node $i (1 \le i \le 5)$ for network A can be described by:

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} + \begin{bmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{bmatrix} + \begin{bmatrix} M_{i1} \\ M_{i2} \\ M_{i3} \end{bmatrix}, \quad (11)$$

$$\begin{aligned} M_{1j} = &\varepsilon_1^1 \big(a_{11}^1 x_{1j}(t - \tau(t)) + a_{12}^1 x_{2j}(t - \tau(t)) + a_{13}^1 x_{3j}(t - \tau(t)) \\ &+ a_{14}^1 x_{4j}(t - \tau(t)) + a_{15}^1 x_{5j}(t - \tau(t))\big) + \varepsilon_2^1 \big(a_{11}^2 x_{1j}(t - \tau(t)) \\ &+ a_{12}^2 x_{2j}(t - \tau(t)) + a_{13}^2 x_{3j}(t - \tau(t)) + a_{14}^2 x_{4j}(t - \tau(t)) \\ &+ a_{15}^2 x_{5j}(t - \tau(t))\big) + \varepsilon_3^1 \big(a_{11}^3 x_{1j}(t - \tau(t)) + a_{12}^3 x_{2j}(t - \tau(t)) \\ &+ a_{13}^3 x_{3j}(t - \tau(t)) + a_{14}^3 x_{4j}(t - \tau(t)) + a_{15}^3 x_{5j}(t - \tau(t)) \\ &+ \mu\big(c_{11} y_{1j}(t - \tau(t)) + c_{12} y_{2j}(t - \tau(t))\big), \quad j = 1, 2, 3 \end{aligned}$$

$$\begin{aligned} M_{2j} = &\varepsilon_1^1 \big(a_{21}^1 x_{1j}(t - \tau(t)) + a_{22}^1 x_{2j}(t - \tau(t)) + a_{23}^1 x_{3j}(t - \tau(t)) \\ &+ a_{24}^1 x_{4j}(t - \tau(t)) + a_{25}^1 x_{5j}(t - \tau(t))\big) + \varepsilon_2^1 \big(a_{21}^2 x_{1j}(t - \tau(t)) \\ &+ a_{22}^2 x_{2j}(t - \tau(t)) + a_{23}^2 x_{3j}(t - \tau(t)) + a_{24}^2 x_{4j}(t - \tau(t)) \\ &+ a_{25}^2 x_{5j}(t - \tau(t))\big) + \varepsilon_3^1 \big(a_{21}^3 x_{1j}(t - \tau(t)) + a_{22}^3 x_{2j}(t - \tau(t)) \\ &+ a_{23}^3 x_{3j}(t - \tau(t)) + a_{24}^3 x_{4j}(t - \tau(t)) + a_{25}^3 x_{5j}(t - \tau(t)) \\ &+ \mu\big(c_{21} y_{1j}(t - \tau(t)) + c_{22} y_{2j}(t - \tau(t))\big), \quad j = 1, 2, 3 \end{aligned}$$

$$\begin{aligned} M_{3j} = &\varepsilon_1^1 \big(a_{31}^1 x_{1j}(t - \tau(t)) + a_{32}^1 x_{2j}(t - \tau(t)) + a_{33}^1 x_{3j}(t - \tau(t)) \\ &+ a_{34}^1 x_{4j}(t - \tau(t)) + a_{35}^1 x_{5j}(t - \tau(t))\big) + \varepsilon_2^1 \big(a_{31}^2 x_{1j}(t - \tau(t)) \\ &+ a_{32}^2 x_{2j}(t - \tau(t)) + a_{33}^2 x_{3j}(t - \tau(t)) + a_{34}^2 x_{4j}(t - \tau(t)) \\ &+ a_{35}^2 x_{5j}(t - \tau(t))\big) + \varepsilon_3^1 \big(a_{31}^3 x_{1j}(t - \tau(t)) + a_{32}^3 x_{2j}(t - \tau(t)) \\ &+ a_{33}^3 x_{3j}(t - \tau(t)) + a_{34}^3 x_{4j}(t - \tau(t)) + a_{35}^3 x_{5j}(t - \tau(t)) \\ &+ \mu\big(c_{31} y_{1j}(t - \tau(t)) + c_{32} y_{2j}(t - \tau(t))\big), \quad j = 1, 2, 3 \end{aligned}$$

$$\begin{aligned} M_{4j} = &\varepsilon_1^1 \big(a_{41}^1 x_{1j}(t - \tau(t)) + a_{42}^1 x_{2j}(t - \tau(t)) + a_{43}^1 x_{3j}(t - \tau(t)) \\ &+ a_{44}^1 x_{4j}(t - \tau(t)) + a_{45}^1 x_{5j}(t - \tau(t))\big) + \varepsilon_2^1 \big(a_{41}^2 x_{1j}(t - \tau(t)) \\ &+ a_{42}^2 x_{2j}(t - \tau(t)) + a_{43}^2 x_{3j}(t - \tau(t)) + a_{44}^2 x_{4j}(t - \tau(t)) \\ &+ a_{45}^2 x_{5j}(t - \tau(t))\big) + \varepsilon_3^1 \big(a_{41}^3 x_{1j}(t - \tau(t)) + a_{42}^3 x_{2j}(t - \tau(t)) \\ &+ a_{43}^3 x_{3j}(t - \tau(t)) + a_{44}^3 x_{4j}(t - \tau(t)) + a_{45}^3 x_{5j}(t - \tau(t)) \\ &+ \mu\big(c_{41} y_{1j}(t - \tau(t)) + c_{42} y_{2j}(t - \tau(t))\big), \quad j = 1, 2, 3 \end{aligned}$$

$$\begin{aligned} M_{5j} = &\varepsilon_1^1 \big(a_{51}^1 x_{1j}(t - \tau(t)) + a_{52}^1 x_{2j}(t - \tau(t)) + a_{53}^1 x_{3j}(t - \tau(t)) \\ &+ a_{54}^1 x_{4j}(t - \tau(t)) + a_{55}^1 x_{5j}(t - \tau(t))\big) + \varepsilon_2^1 \big(a_{51}^2 x_{1j}(t - \tau(t)) \\ &+ a_{52}^2 x_{2j}(t - \tau(t)) + a_{53}^2 x_{3j}(t - \tau(t)) + a_{54}^2 x_{4j}(t - \tau(t)) \\ &+ a_{55}^2 x_{5j}(t - \tau(t))\big) + \varepsilon_3^1 \big(a_{51}^3 x_{1j}(t - \tau(t)) + a_{52}^3 x_{2j}(t - \tau(t)) \\ &+ a_{53}^3 x_{3j}(t - \tau(t)) + a_{54}^3 x_{4j}(t - \tau(t)) + a_{55}^3 x_{5j}(t - \tau(t)) \\ &+ \mu\big(c_{51} y_{1j}(t - \tau(t)) + c_{52} y_{2j}(t - \tau(t))\big), \quad j = 1, 2, 3 \end{aligned}$$

And based on Eq. (2), the dynamical equations of node $i (1 \le i \le 2)$ for network B can be described by:

$$\begin{bmatrix} \dot{y}_{i1} \\ \dot{y}_{i2} \\ \dot{y}_{i3} \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix} \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \end{bmatrix} + \begin{bmatrix} 0 \\ -y_{i1}y_{i3} \\ y_{i1}y_{i2} \end{bmatrix} + \begin{bmatrix} N_{i1} \\ N_{i2} \\ N_{i3} \end{bmatrix} + u_i, \quad (12)$$

$$\begin{aligned}N_{1j} &= \varepsilon_1^2\left(b_{11}^1 y_{1j}\left(t-\tau(t)\right)+b_{12}^1 y_{2j}\left(t-\tau(t)\right)\right)+\varepsilon_2^2\left(b_{11}^2 y_{1j}\left(t-\tau(t)\right)\right.\\ &\quad \left.+b_{12}^2 y_{2j}\left(t-\tau(t)\right)\right)+\varepsilon_3^2\left(b_{11}^3 y_{1j}\left(t-\tau(t)\right)+b_{12}^3 y_{2j}\left(t-\tau(t)\right)\right)\\ &\quad +\mu\left(d_{11}x_{1j}\left(t-\tau(t)\right)+d_{12}x_{2j}\left(t-\tau(t)\right)+d_{13}x_{3j}\left(t-\tau(t)\right)\right.\\ &\quad \left.+d_{14}x_{4j}\left(t-\tau(t)\right)+d_{15}x_{5j}\left(t-\tau(t)\right)\right),\quad j=1,2,3\\ N_{2j} &= \varepsilon_1^2\left(b_{21}^1 y_{1j}\left(t-\tau(t)\right)+b_{22}^1 y_{2j}\left(t-\tau(t)\right)\right)+\varepsilon_2^2\left(b_{21}^2 y_{1j}\left(t-\tau(t)\right)\right.\\ &\quad \left.+b_{22}^2 y_{2j}\left(t-\tau(t)\right)\right)+\varepsilon_3^2\left(b_{21}^3 y_{1j}\left(t-\tau(t)\right)+b_{22}^3 y_{2j}\left(t-\tau(t)\right)\right)\\ &\quad +\mu\left(d_{21}x_{1j}\left(t-\tau(t)\right)+d_{22}x_{2j}\left(t-\tau(t)\right)+d_{23}x_{3j}\left(t-\tau(t)\right)\right.\\ &\quad \left.+d_{24}x_{4j}\left(t-\tau(t)\right)+d_{25}x_{5j}\left(t-\tau(t)\right)\right),\quad j=1,2,3\end{aligned}$$

According to Theorem 1, the network A and network B are realized synchronization under the controllers (10).

For the two layers coupled network with three-weights and time-varying delay, we assume that

$$\begin{aligned}&a_{12}^1=1,a_{13}^1=2,a_{14}^1=3,a_{15}^1=2,a_{23}^1=1,a_{14}^1=4,a_{25}^1=2,b_{12}^1=1,\\ &a_{34}^1=1,a_{35}^1=0,a_{45}^1=3,a_{ji}^1=a_{ij}^1(i\neq j,i,j=1,2,\ldots,5),b_{21}^1=1,\\ &a_{ii}^1=-\sum_{i=1,i\neq j}^{5}a_{ij}^1(i,j=1,2,\ldots,5),b_{11}^1=-1,b_{22}^1=-1,\end{aligned}\tag{13}$$

$$\begin{aligned}&a_{12}^2=2,a_{13}^2=1,a_{14}^2=3,a_{15}^2=1,a_{23}^2=4,a_{14}^2=1,a_{25}^2=3,b_{12}^2=2,\\ &a_{34}^2=2,a_{35}^2=0,a_{45}^2=1,a_{ji}^2=a_{ij}^2(i\neq j,i,j=1,2,\ldots,5),b_{21}^2=2,\\ &a_{ii}^2=-\sum_{i=1,i\neq j}^{5}a_{ij}^2(i,j=1,2,\ldots,5),b_{11}^2=-2,b_{22}^2=-2\end{aligned}\tag{14}$$

$$\begin{aligned}&a_{12}^3=2,a_{13}^3=4,a_{14}^3=2,a_{15}^3=1,a_{23}^3=4,a_{14}^3=2,a_{25}^3=1,b_{12}^3=3,\\ &a_{34}^3=3,a_{35}^3=0,a_{45}^3=2,a_{ji}^3=a_{ij}^3(i\neq j,i,j=1,2,\ldots,5),b_{21}^3=3,\\ &a_{ii}^3=-\sum_{i=1,i\neq j}^{5}a_{ij}^3(i,j=1,2,\ldots,5),b_{11}^3=-3,b_{22}^3=-3\end{aligned}\tag{15}$$

$$\begin{aligned}&c_{11}=0.2,c_{12}=0.1,c_{21}=0.3,c_{22}=0.6,c_{31}=0.4,\\ &c_{32}=0.5,c_{41}=0.8,c_{42}=0.7,c_{51}=0.4,c_{52}=0.5,\\ &d_{ji}=c_{ij}(i=1,2,\ldots,5,j=1,2)\end{aligned}\tag{16}$$

And the initial value conditions are selected as follows:

$$k_i=10,(1\le i\le 2),\varepsilon_1^1=\varepsilon_2^1=\varepsilon_3^1=\varepsilon_1^2=\varepsilon_2^2=\varepsilon_3^2=\mu=0.3\,,$$
$$x_i(0)=\left(0.1+0.3i,0.2+0.3i,0.3+0.3i\right)^{\mathrm{T}},(1\le i\le 5)\,,$$
$$y_i(0)=\left(1.6+0.3i,1.7+0.3i,1.8+0.3i\right)^{\mathrm{T}},(1\le i\le 2)\,,$$
$$g_i(0)=2.4+0.1i,(1\le i\le 2)\,.$$

then we take $\tau(t)=0.03$ and draw the synchronization error for the two layers coupled network with three-weights and time-varying delay, as shown in Fig. 3. Obviously, the two layers coupled network achieves synchronization at 15 time units.

We take $\tau(t)=0.06$ and keep other parameters unchanged, then draw the synchronization error of two layers coupled network with three-weights and time-varying delay, as shown in Fig. 4. From the simulation results, we found that the two layers coupled network achieves synchronization at 22 time units when $\tau(t)=0.06$. Obviously, the smaller the value of coupling delay $\tau(t)$, the shorter the time that the two layers coupled network reaches synchronization. Accordingly, the coupling delays have a certain influence on the synchronization of the two layers coupled network with multi-weights and time-varying delay.

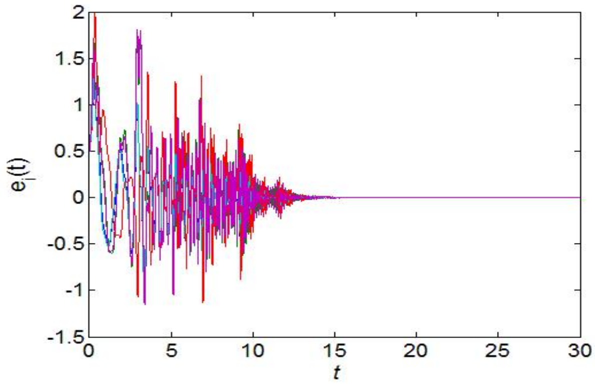

Fig. 3. Synchronization error for two layers coupled networks with $\tau(t)=0.03$

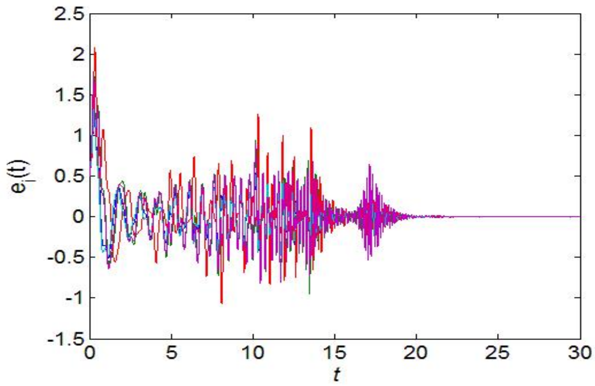

Fig. 4. Synchronization error for two layers coupled networks with $\tau(t)=0.06$

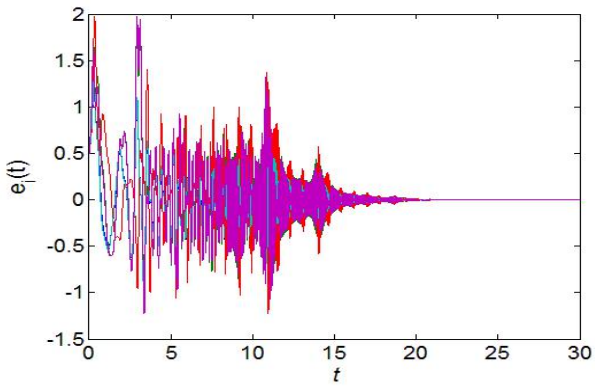

Fig. 5. Synchronization error for two layers coupled networks after increase the weight

Fixed $\tau(t)=0.03$, and if the weights shown in formula (13) are changed to:

$$\begin{aligned}&a_{12}^1=4,a_{13}^1=5,a_{14}^1=6,a_{15}^1=5,a_{23}^1=4,a_{14}^1=7,a_{25}^1=5,\\ &a_{34}^1=4,a_{35}^1=0,a_{45}^1=6,a_{ji}^1=a_{ij}^1(i\neq j,i,j=1,2,\ldots,5),\\ &a_{ii}^1=-\sum_{i=1,i\neq j}^{5}a_{ij}^1(i,j=1,2,\ldots,5),\\ &b_{11}^1=-10,\;\; b_{12}^1=10,\;\; b_{21}^1=10,\;\; b_{22}^1=-10,\end{aligned}\tag{17}$$

then we get the synchronization error as shown in Fig. 5. And Fig. 5 shows that the two layers coupled network with three-weights and time-varying delay achieves synchronization at 21 time units, and compared with Fig. 3 the synchronization time increased 6 time units. This suggests that the weight will affect the synchronization process of the two layers coupled network with multi-weights and time-varying delay.

V. CONCLUSIONS

The paper established a new two layers coupled network with multi-weights and time-varying delay, and then the synchronization problem of the new network is investigated in detail. Through numerical simulation, the impact of time-varying delay and weight on the synchronization of the two layers coupled network with three-weights and time-varying delay are investigated in detail. It is noteworthy that this two layers coupled network with three-weights and time-varying delay is very practical in reality, and the application of the new network will be the focus of the next research.

REFERENCES

[1] H. Tang, L. Chen, J. Lu, et al, "Adaptive synchronization between two complex networks with nonidentical topological structures," *Physica A: Statistical Mechanics and its Applications*, vol. 387, no. 22, pp. 5623-5630, Sep. 2008.

[2] J. R. Chen, L. C. Jiao, J. S. Wu, et al. "Adaptive synchronization between two different complex networks with time-varying delay coupling," *Chinese Physics Letters*, vol. 26, no. 6, pp. 1-4, Feb. 2009.

[3] J. S. Wang, H. P. Liu, X. Shi, "Adaptive synchronization between two different complex networks with time-varing delay coupling," *Journal of University of Science and Technology Beijing*, vol. 32, no. 6, pp. 61-64, Oct. 2010.

[4] M. Sun, C. Y. Zeng, L. X. Tian, "Linear generalized synchronization between two complex networks," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 8, pp. 2162-2167, Aug. 2010.

[5] J. Wang, Z. Fu, "Research on design and optimum-method of double-layer urban transit network," *Computer Engineering and Applications*, vol. 48, no. 36, pp. 1-8, Feb. 2012.

[6] D. Shen, J. H. Li, J. S. Xiong, et al, "A cascading failure model of double layer complex networks based on betweenness," *Complex Systems and Complexity Science*, vol. 11, no. 3, pp. 12-18, Sep. 2014.

[7] K. Yu, *Research on Information Spreading Model on Two layers Coupled Networks.* Dalian: Dalian University of Technology, 2015

[8] J. L. Ma, W. Z. Han, Q. Guo, et al, "Improved efficient routing strategy on two-layer complex networks," *International Journal of Modern Physics C*, vol. 27, no. 4, pp. 1650044, Oct. 2016.

[9] S. T. Tang, S. Yan, S. Pei, et al, "Evolutionary dynamics of the weighted voter model with opinion strength on complex networks," *Journal of the Korean Physical Society*, vol. 66, no. 11, pp. 1783-1788, Jun. 2015.

[10] R. Franke, "CHIMERA: Top-down model for hierarchical, overlapping and directed cluster structures in directed and weighted complex networks," *Physica A: Statistical Mechanics & Its Applications*, vol. 461, no. 1, pp. 384-408, Nov. 2016.

[11] Y. Y. Yang, G. Xie, J. Xie, "Mining important nodes in directed weighted complex networks," *Discrete Dynamics in Nature and Society*, vol. 2017, no. 5, pp. 1-7, Jan. 2017.

[12] W. J. Du, Y. Z. Li, J. G. Zhang, et al, "Balance Analysis of Two Layers Coupled Public Traffic Network with Dual Time-varying Delays," Engineering Letters, vol. 26, no. 1, pp. 100-106, 2018.

[13] X. L. Zhang, Y. B. Ren, B. X. Huang, et al, "Analysis of time-varying characteristics of bus weighted complex network in Qingdao based on boarding passenger volume," *Physica A Statistical Mechanics & Its Applications*, vol. 506, no. 15, pp. 376-394, Sep. 2018.

[14] X. Wang, L. Liang, M. Che, "Finite-time convergent complex-valued neural networks for the time-varying complex linear matrix equations," *Engineering Letters*, vol. 26, no. 4, pp. 432-440, 2018.

[15] W. J. Du, Y. Z. Li, J. G. Zhang, et al, "Stochastic Synchronization between Different Networks and Its Application in Bilayer Coupled Public Traffic Network," *IAENG International Journal of Computer Science*, vol. 46, no. 1, pp. 102-108, 2019.

[16] W. S. Lan, G. H. Zhao, "Strong correlation analysis of stocks based double weighted networks," *Mathematics in Practice and Theory*, vol. 41, no. 13, pp. 45-51, Feb. 2011.

[17] L. Zhang, X. L. An, J. K. Peng, "Synchronization analysis of a new multi-weights complex network model," *Journal of Henan Normal University (Natural Science Edition)*, vol. 42, no. 3, pp. 34-38, Jan. 2014.

[18] X. L. An, L. Zhang, Y. Z. Li, et al, " Synchronization analysis of complex networks with multi-weights and its application in public traffic network," *Physica A: Statistical Mechanics and its Applications*, vol. 412, no. 15, pp. 149-156, Oct. 2014.

[19] M. Dai, D. Ye, J. Hou, et al, "Scaling of average weighted receiving time on double-weighted Koch networks," *Fractals*, vol. 23, no. 2, pp. 1-7, Mar. 2015.

[20] W. J. Du, Y. Z. Li, J. G. Zhang, et al, "Synchronisation between two different networks with multi-weights and its application in public traffic network," *International Journal of Systems Science*, vol. 50, no. 3, pp. 534-545, Jan. 2019.