On the Wiener Index of the Strong Product of Paths

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***Abstract*—The sum of the distances between all unordered pairs of vertices of a connected graph G is called the Wiener index of G and denoted by $W(G)$. In this paper, we researched the Wiener index of the strong product of a class of paths with given orders. By using the structural characteristics of the strong product graph, we derive the exact Wiener index of the strong product of two paths while their lengths have different parities. In addition, we also obtain an upper bound of the edge-forwarding index of the strong product of two paths.**

***Index Terms*—Wiener index, Strong product graph, Topological structure, Path, Edge-forwarding index.**

I. Introduction

IN 1959, Sabidussi defined the strong product graph, Cartesian product graph and direct product graph for the first time in Reference [1], and they were widely used in network design. In this paper, we are interested in the strong product of two graphs, which is defined as follows:
For two random graphs G_1 and G_2, their vertex sets and edge sets are $V(G_i)$ and $E(G_i)$, respectively, where $i = 1, 2$. The strong product of G_1 and G_2 is denoted by $G_1 \boxtimes G_2$, with vertex set

$$\begin{aligned} V(G_1 \boxtimes G_2) &= V(G_1) \times V(G_2) \\ &= \{(x_i, y_j) : x_i \in V(G_1), y_j \in V(G_2)\}. \end{aligned}$$

The connection rules of the two vertices (x_i, y_j) and (x_k, y_h) of $G_1 \boxtimes G_2$ are: If and only if $i = k$ and $(y_j, y_h) \in E(G_2)$, or $j = h$ and $(x_i, x_k) \in E(G_1)$, or $(x_i, x_k) \in E(G_1)$ and $(y_j, y_h) \in E(G_2)$. We call graphs G_1 and G_2 the factor graphs of $G_1 \boxtimes G_2$, and for convenience, the vertex (x_i, y_j) is usually written as $x_i y_j$.

The strong product is a method available for the construction "large" networks from existing "small" networks. For any random network, we usually focus on its topology, and the network topology is essentially a graph. Thus, in the following content, we will interchange "graph" and "network". A graph constructed by the strong product method can retain some good properties of the factor graphs, such as symmetry, transitivity and connectivity. As early as 1973, by using the extreme value method, Hales [2] determined the domination number and the cliquecovering number of strong product graphs. In 1992, Imrich and Klavzar [3] proved that the contract of the strong product of two graphs is the strong product of the subgraphs of these two graphs. In 1998, Vesel obtained the independent number and chromatic number of the strong product of some odd cycles [4]. Bresar et al. [5] calibrated the edge connectivity of strong product graphs based on the edge connectivities, orders, sizes of factor graphs and the minimum vertex degree of the strong product graphs. More on the theory and application of the product graphs are present in other works [6-8].

In graph theory, the Wiener index is a graph invariant based on distances, denoted by $W(G)$, defined as $W(G) = \sum\limits_{x,y \in V(G)} d_G(x, y)/2$, where $d_G(x, y)$ refers to the distance between two different vertices $x, y \in V(G)$. The Wiener index can be widely studied in various fields and disciplines [10-13], originating from the pioneering article of chemist H. Wiener in 1947 [9]. In terms of graph theory, many scholars have obtained the exact Wiener indices of some graphs with special structures, such as polygonal systems [14], unicyclic graphs [15], trees [16-17], and other special graphs [18-19]. In addition, mathematicians and scholars have obtained many results by exploring the universal results. Gutman et al. used the orders, sizes and the Wiener indices of factor graphs to derive the universal results of Wiener indices of several composite graphs [20]. Plesnik proved that in all 2-connected graphs with order n, the cycle of C_n has the largest Wiener index [21]. By using the nondecreasing sequence of the distances between all different vertex pairs of factor graphs, Casablanca et al. obtained some upper and lower bounds of the Wiener indices of some strong product graphs [22]. In reference [23], by limiting the eccentricities of the two factor graphs, Peterin et al. obtained the Wiener index of the strong product of two-factor graphs when they both have constant eccentricities. For more results on the Wiener indices, please refer to [24-26].

Paths have convenient constructions and excellent properties, in most virtual and real networks, there are paths as their subgraphs. Donald gave an upper bound on the number of paths of a random graph [27]. Barovich studied the paths of 2-connected graphs [28]. In [29-30], Pattabiraman et al. relied on the transitivity of the vertices of cycles to minimize the computational complexity, and by using only the orders of two-factor graphs, they obtained the Wiener indices of the strong product of a path and a cycle and the Wiener index of the strong product of two cycles. Therefore, it is great significance to study the strong product graph of paths.

Since the path is not vertex-transitivity, so we classify the vertices at different positions of $G = P_{2m} \boxtimes P_{2n+1}$. By making full use of the construction properties of $G = P_{2m} \boxtimes P_{2n+1}$, the shortest path selection between two vertices of $V(G)$ are defined. Based on the comprehensive above, the Wiener indices of $G = P_{2m} \boxtimes P_{2n+1}$ are given, where $m > n$ and $n > m$. In Section II, we introduce the symbols

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and concepts used in this paper. In Section III, we give the main results and their proofs of this paper. In Section IV, by using the shortest path selection of Section III, we determine an upper bound of the edge-forwarding index of a class of strong product networks. In addition, Section V presents the summary of this paper.

II. Symbols and Definitions

The symbols and their definitions to be used in this paper are as follows:

Let G be a graph, while two vertices $u, v \in V(G)$ and the edge $\{u, v\} \in E(G)$, then the vertices u and v are said to be adjacent to each other, denoted by $u \sim v$. G is called a connected graph, if there is at least one adjacent vertex for each vertex of $V(G)$. For two given graphs G and H, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, then H is called a subgraph of G.

A finite nonempty sequence of $A = (v_0 e_1 v_1 e_2 v_2 \ldots e_n v_n)$ is composed of vertices and edges, whose items are v and e, respectively, and the endpoints of e_i are v_{i-1} and $v_i (1 \leq i \leq n)$. If all vertices and edges of A are different from each other, then the graph composed of the vertices and edges of A is called $v_0 v_n$-path, denoted by P_{n+1}, where the length of the is n. In addition, if $v_0 = v_n$ of the sequence A, then the graph composed of the vertices and edges of A is an n-order cycle, denoted by C_n. The distance between two vertices u and v of a connected graph G is defined as the length of a shortest uv-path in the graph G, denoted by $d_G(u, v)$.

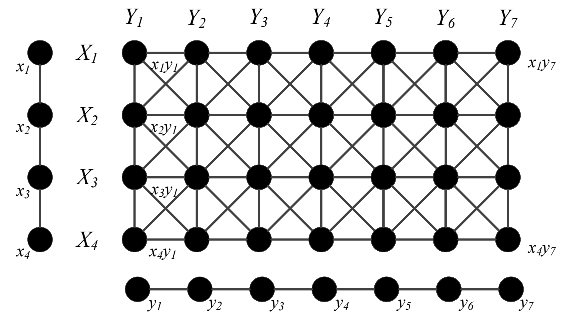

Fig. 1. Strong product graph $P_{2\times 2} \boxtimes P_{2\times 3+1}$

To facilitate the proofs of the main results, we introduce some new symbols and their definitions. To clarify the concepts of rows (also called layers) and columns of $P_{2m} \boxtimes P_{2n+1}$, in this paper, we use the following two new forms to express $V(P_{2m} \boxtimes P_{2n+1})$, i.e.,

$$1)\ V(P_{2m} \boxtimes P_{2n+1}) = \bigcup_{i=1}^{2m} X_i, where$$
$$X_i = \{x_i \in V(P_{2m})\} \times V(P_{2n+1}).$$

$$2)\ V(P_{2m} \boxtimes P_{2n+1}) = \bigcup_{i=1}^{2n+1} Y_j, where$$
$$Y_j = \{y_j \in V(P_{2n+1})\} \times V(P_{2m}).$$

We call X_i is the i-th layer of $G = P_{2m} \boxtimes P_{2n+1}$, and Y_j is the j-th column of G. As illustrated in Fig. 1, we enumerate the strong product of two paths P_{2m} and P_{2n+1}, where $m = 2, n = 3$, and the number of layers and columns of Fig. 1 are $2m$ and $2n + 1$, respectively.

For a vertex $x_i y_j$ of $X_i (1 \leq i \leq 2m)$ of $P_{2m} \boxtimes P_{2n+1}$, $X_i (1 \leq i \leq 2m)$ is "this layer" of $x_i y_j$, $X_1, X_2, ..., X_{i-1}$ are called "upper layers" of $x_i y_j$, and $X_{i+1}, X_{i+2}, ..., X_{2m}$ are called "lower layers" of $x_i y_j$. For the sake of convenience, in the following sections, we directly use the terms of "this layer","upper layers" and "lower layers". Please note that X_i has no upper layer when $i = 1$, and when $i = 2m$, X_i has no lower layer.

For a vertex $x_i y_j (1 \leq i \leq 2m, 1 \leq j \leq 2n + 1)$, we use LV_u to represent the vertices of 1st to $j-1$-th columns of their upper layers, and the vertices of j-th to $2m$-th columns of their upper layers are represented by RV_u. For the lower layers of $x_i y_j$, all the vertices of the 1st to $j-1$-th columns are denoted by LV_d, and the j-th to $2m$-th are denoted by RV_d(see Fig. 2).

Let $G = P_{2m} \boxtimes P_{2n+1}$, $D_{(i,j)}$ is defined as the sum of the distances from $x_i y_j (1 \leq i \leq 2m, 1 \leq j \leq 2n+1)$ to $V(G)$, i.e., $D_{(i,j)} = \sum\limits_{v \in V(G)} d_G(x_i y_j, v)$. Undefined symbols can be referenced in [6].

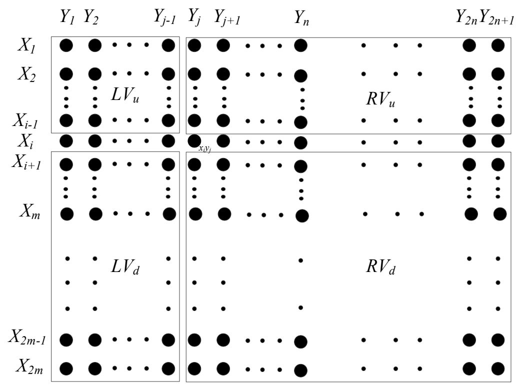

Fig. 2. LV_u, RV_u, LV_d and RV_d of $x_i y_j$

III. Main Results

Wiener index describes the situation of a graph distances, it is one of the most important graph invariants with prominent application background. Strong product of two paths is a very comeliness mesh shape, and the vertices at corresponding positions with the same characteristics. The following two lemmas need to be used in the proofs of main results.

Lemma 1[6]: For two random graphs G and H, their strong product $G \boxtimes H$ is commutative, i.e.,

$$G \boxtimes H \cong H \boxtimes G.$$

Lemma 2[6]: Let G and H be two random simple connected graphs, $G \boxtimes H$ is their strong product graph, for any two vertices $x_i y_j, x_k y_h \in V(G \boxtimes H)$, where $x_i, x_k \in V(G)$ and $y_j, y_h \in V(H)$, then,

$$d_{G \boxtimes H}(x_i y_j, x_k y_h) = \max\{d_G(x_i, x_k), d_H(y_j, y_h)\}.$$

Theorem 1: Suppose that P_{2m} and P_{2n+1} are two paths with orders $2m$ and $2n + 1 (n < m)$, respectively, and the Wiener index of their strong product $G = P_{2n+1} \boxtimes P_{2m}$ is:

$$W(G) = \begin{cases} w(G_{m>2n}) & if\ 2n < m, \\ w(G_{n<m<2n}) & if\ n < m \leq 2n. \end{cases}$$

Where

$$
\begin{aligned}
w(G_{m>2n}) = &-[n^5 + (5-10m)n^4 + (-70m^2 + 30m+\\
&5)n^3 + (-80m^3 - 90m^2 + 60m - 5)n^2+\\
&(-5m^4 - 150m^3 - 15m^2 + 30m - 6)n-\\
&160m^3 + 40m]/60,\\
w(G_{n<m<2n}) = &-[8n^5 + (20-40m)n4 + (10-80m)n^3+\\
&(-80m^3 - 30m - 5)n^2 + (-60m^3+\\
&5m - 3)n - 40m^3 + 10m]/15.
\end{aligned}
$$

Proof: As shown in Fig. 3, by using the structural properties of G, we find that the distances from the corresponding vertices in the four box(real line) to other vertices of $V(G)$ are the same, and the distances from the corresponding vertices in the two box(dotted line) to other vertices of $V(G)$ are the same. It is worth noting that the distance from a vertex to itself is 0. Therefore, we have,

$$
\begin{aligned}
W(G) &= (4\sum_{i=1}^{n}\sum_{j=1}^{m} D_{(i,j)} + 2\sum_{j=1}^{m} D_{(n+1,j)})/2\\
&= 2\sum_{i=1}^{n}\sum_{j=1}^{m} D_{(i,j)} + \sum_{j=1}^{m} D_{(n+1,j)}. \qquad (1)
\end{aligned}
$$

The entire process of proof is divided into two situations, correspond to the two addition factors of Eqn (1) respectively. In addition, the two situations are divided into three sub-situations respectively.

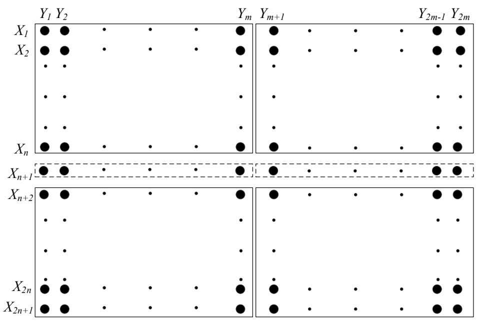

Fig. 3. The symmetry of $P_{2m} \boxtimes P_{2n+1}$

Situation 1: Combined the structural properties of strong product graphs with the symbols defined in Section II, Situation 1 can be divided into the following three sub-situations:
Sub-situation 1.1, calculate the sum of the distances from $x_{n+1}y_j(1 \le j \le m)$ to all vertices of this layer, i.e., $\sum_{j=1}^{m}\sum_{h=1}^{2m} d_G(x_{n+1}y_j, x_{n+1}y_h)$.
Sub-situation 1.2, calculate the sum of the distances from $x_{n+1}y_j(1 \le j \le m)$ to all the vertices of upper layers, i.e., $\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{h=1}^{2m} d_G(x_{n+1}y_j, x_k y_h)$.
Sub-situation 1.3, calculate the sum of the distances from $x_{n+1}y_j(1 \le j \le m)$ to all the vertices of lower layers, i.e., $\sum_{j=1}^{m}\sum_{k=n+2}^{2n+1}\sum_{h=1}^{2m} d_G(x_{n+1}y_j, x_k y_h)$. Notice that the sum of the above three sub-situations is the final result of Situation 1.

Sub-situation 1.1: Since each layer of G is a path with order $2m$, thus the sum of the distances from $x_{n+1}y_j$ to this layer is expressed as follows:

$$
\begin{aligned}
&D = d_G(x_{n+1}y_j, x_{n+1}y_h)\\
\sum_{j=1}^{m}\sum_{h=1}^{2m} D &= \sum_{j=1}^{m}\{2[1+2+\cdots+(j-1)] + j + (j+1)+\\
&\cdots + (2m-j)\}\\
&= \sum_{j=1}^{m}[j(j-1) + (2m-j)(2m-j+1)]/2\\
&= m(4m^2-1)/3. \qquad (2)
\end{aligned}
$$

Eqn (2) gives the sum of the distances from the first m vertices of X_{n+1} to this layer. For Eqn (2), under the premise of other conditions unchanged, no matter what the subscripts of the two x of $d_G(x_{n+1}y_j, x_{n+1}y_h)$ are, as long as the subscripts of the two x are the same, the distance is always satisfy $m(4m^2-1)/3$. Thus in some later situations, we will call the result of Eqn (2) directly.

Sub-situation 1.2: For the convenience of calculations, by using symbols LV_u and RV_u, we divide the sum of the distances from $x_{n+1}y_j$ to upper layers into two parts,

$$
\begin{aligned}
&D = d_G(x_{n+1}y_j, x_k y_h)\\
\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{h=1}^{2m} D &= \sum_{j=1}^{m}\{\sum_{u\in LV_u}\{d_G(x_{n+1}y_j, u)+\\
&\sum_{j=1}^{m}\sum_{v\in RV_u} d_G(x_{n+1}y_j, v)\}\}. \qquad (3)
\end{aligned}
$$

Next, we calculate the two addition factors of Eqn (3), respectively,

$$
\begin{aligned}
&L = d_G(x_{n+1}y_j, u)\\
\sum_{j=1}^{m}\sum_{u\in LV_u} L &= \sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{h=1}^{j-1} d(x_{n+1}y_j, x_k y_h)\\
&= \sum_{k=1}^{n}\sum_{j=1}^{n+2-k}[(j-1)(n+1-k)]+\\
&\sum_{k=1}^{n}\sum_{j=n+3-k}^{m}\{[(n+1-k)^2 + (n+2-\\
&k) + \cdots + (j-1)]\}\\
&= [-n^4 + (2-4m)n^3 - n^2 + (-4m^3+\\
&8m-2)n]/24, \qquad (4)
\end{aligned}
$$

The explanations of the terms appearing in Eqn (4) are as follows:
For the distances from $x_{n+1}y_j$ to all vertices of LV_u, when the value j of $x_{n+1}y_j$ is between 1 and $n+2-k$, the distances from $x_{n+1}y_j$ to all vertices of k-th layer of LV_u are all $n+1-k$, and there are $j-1$ vertices like this. When $n+3-k \le j \le m$, the sum of the distances from $x_{n+1}y_j$ to all vertices of k-th layer of LV_u is $(n+1-k)^2 + (n+2-k) + \cdots + (j-1)$. The representative shortest paths from $x_i y_j$ to LV_u and LV_d are shown in Fig. 4. The shortest paths to RV_u and RV_d are similar to Fig. 4, except the directions

of LV_u and LV_d are to the left, but the directions of RV_u and RV_d are to the right.

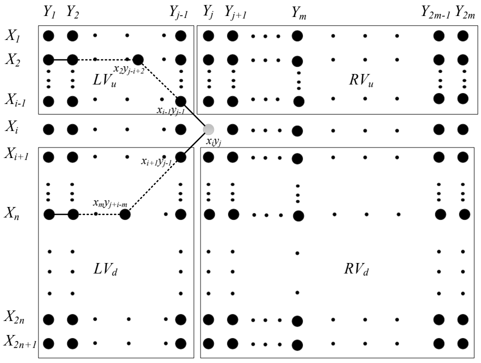

Fig. 4. The representative shortest path from x_iy_j to LV_u and RV_u

Next, we calculate the second addition factor of Eqn (3),

$$
\begin{aligned}
& R = d_G(x_{n+1}y_j, v) \\
\sum_{j=1}^{m}\sum_{v\in RV_u} R &= \sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{h=j}^{2m} d(x_{n+1}y_j, x_ky_h) \\
&= \sum_{k=1}^{n}\sum_{j=1}^{m}[(n+1-k)^2 + (n+1-k) + \\
&\quad \cdots + (2m-j)] \\
&= [mn^3 + 3mn^2 + (7m^3+m)n]/6.
\end{aligned} \tag{5}
$$

The explanations of Eqn (5) are similar to the second addition factor of Eqn (4). The differences between them are that the distance from $x_{n+1}y_j$ to x_ky_j is $n+1-k$, and the distance from vertex $x_{n+1}y_j$ to these $n+1-k$ vertices $x_ky_{j+1}, x_ky_{j+2}, \ldots, x_ky_{j+n+2-k}$ are all $n+1-k$, the distance from $x_{n+1}y_j$ to x_ky_{2m} is $2m-j$.

By using Eqn (3-5) and the construction characteristics of the strong product graph, we have

$$
\begin{aligned}
& L = d_G(x_{n+1}y_j, x_ky_h) \\
\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{h=1}^{2m} L &= \sum_{j=1}^{m}\sum_{u\in LV_u} d_G(x_{n+1}y_j, u) + \\
&\quad \sum_{j=1}^{m}\sum_{v\in RV_u} d_G(x_{n+1}y_j, v) \\
&= -[n^4 + (2-8m)n^3 + (-12m-1)n^2 + \\
&\quad (-32m^3 + 4m - 2)n]/24.
\end{aligned} \tag{6}
$$

Eqn (6) calculates the sum of the distances from $x_{n+1}y_j (1 \le j \le m)$ to their upper layers. Next, we calculate the sum of the distances from $x_{n+1}y_j (1 \le j \le m)$ to their lower layers.

Sub-situation 1.3: Similar to Sub-situation 1.2, we can use the symbols LV_d and RV_d to divide the sum of the distances from $x_{n+1}y_j$ to their lower layers into two parts, then calculate them respectively. It's not difficult to find that the upper layers and lower layers of $x_{n+1}y_j$ are completely symmetrical about X_{n+1}. Therefore, the sum of the distances from $x_{n+1}y_j$ to their upper layers and lower layers are equal. Therefore, in this section, we can directly use the result of Sub-situation 1.2.

Taking a comprehensive consideration of above three sub-situations, we have obtained the sum of the distances from $x_{n+1}y_j (1 \le j \le m)$ to all the vertices of their this layer, upper layers and lower layers. From Eqn(2) and Eqn(6), we get the result of Situation 1 as follows:

$$
\begin{aligned}
\sum_{j=1}^{m} D_{(n+1,j)} &= \sum_{j=1}^{m}\sum_{h=1}^{2m} d_G(x_{n+1}y_j, x_{n+1}y_h) + \\
&\quad \sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{h=1}^{2m} d_G(x_{n+1}y_j, x_ky_h) + \\
&\quad \sum_{j=1}^{m}\sum_{k=n+2}^{2n+1}\sum_{h=1}^{2m} d_G(x_{n+1}y_j, x_ky_h) \\
&= \sum_{j=1}^{m}\sum_{h=1}^{2m} d_G(x_{n+1}y_j, x_{n+1}y_h) + \\
&\quad 2\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{h=1}^{2m} d_G(x_{n+1}y_j, x_ky_h) \\
&= -[n^4 + (2-8m)n^3 + (-12m-1)n^2 + \\
&\quad (-16m^3 - 2)n - 32m^3 + 8m]/12.
\end{aligned} \tag{7}
$$

Situation 2: For $2\sum_{i=1}^{n}\sum_{j=1}^{m} D_{(i,j)}$, namely the sum of the distances from $x_iy_j (1 \le i \le n, 1 \le j \le m)$ to all vertices of $V(G)$, we also divide it into the following three sub-situations:

Sub-situation 2.1, calculate the sum of the distances from $x_iy_j (1 \le i \le n, 1 \le j \le m)$ to this layer, i.e., $\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{h=1}^{2m} d_G(x_iy_j, x_iy_h)$.

Sub-situation 2.2, calculate the sum of the distances from $x_iy_j (1 \le i \le n, 1 \le j \le m)$ to upper layers, i.e., $\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{h=1}^{2m} d_G(x_iy_j, x_iy_h)$.

Sub-situation 2.3, calculate the sum of the distances from $x_iy_j (1 \le i \le n, 1 \le j \le m)$ to lower layers, i.e., $\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}\sum_{h=1}^{2m} d_G(x_iy_j, x_iy_h)$.

Sub-situation 2.1: Since each layer of G is a path with order $2m$, we have

$$
\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{h=1}^{2m} d_G(x_iy_j, x_iy_h) = mn(4m^2-1)/3. \tag{8}
$$

Sub-situation 2.2: By using the symbols LV_u and RV_u, we have

$$
\begin{aligned}
& D = d_G(x_iy_j, x_iy_h) \\
\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{h=1}^{2m} D &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{u\in LV_u} d_G(x_iy_j, u) + \\
&\quad \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{v\in RV_u} d_G(x_iy_j, v).
\end{aligned} \tag{9}
$$

Eqn (9) used the same technique as Eqn (3). Therefore, we calculate the two addition factors of Eqn (9), respectively,

$$L = d_G(x_iy_j, u), R = d_G(x_iy_j, v)$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{u\in LV_u} L &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{h=1}^{j-1} d_G(x_iy_j, x_ky_h)\\ &= \sum_{i=1}^{n}\sum_{k=1}^{i-1}\sum_{j=1}^{i-k+1}(j-1)(i-k)+\\ &\quad \sum_{i=1}^{n}\sum_{k=1}^{i-1}\sum_{j=i-k+2}^{m}[(i-k)^2+(i-k+\\ &\quad 1)+\cdots+(j-1)]\\ &= -[n^5-5mn^4+(10m-5)n^3+(15m-\\ &\quad 10m^3)n^2+(10m^3-20m+4)n]/120,\end{aligned} \tag{10}$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{v\in RV_u} R &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{h=j}^{2m} d_G(x_iy_j, x_ky_h)\\ &= \sum_{i=1}^{n}\sum_{k=1}^{i-1}\sum_{j=1}^{m}[(i-k)+(i-k)^2+(i-\\ &\quad k+1)+\cdots+(2m-j)]\\ &= [mn^4+2mn^3+(14m^3-3m)n^2-\\ &\quad 14m^3n]/24.\end{aligned} \tag{11}$$

The explanations of Eqn (10-11) are similar to Eqn (4-5), namely their distance formulas follow the same rule. From Eqn (10-11), we get the final result of Sub-situation 2.2

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{h=1}^{2m} D &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{u\in LV_u} d_G(x_iy_j, u)+\\ &\quad \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{v\in RV_u} d_G(x_iy_j, v)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{h=1}^{j-1} d_G(x_iy_j, x_ky_h)+\\ &\quad \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{h=j}^{2m} d_G(x_iy_j, x_ky_h)\\ &= -[n^5-10mn^4-5n^3+30mn^2-\\ &\quad 80m^3n^2+(80m^3-20m+4)n]/120.\end{aligned} \tag{12}$$

Sub-situation 2.3: Since the distances from $x_iy_j(1 \le i \le n, 1 \le j \le m)$ to their upper and lower layers are not completely symmetrical, in this situation, we cannot call the value of Eqn (2). By using the symbols LV_d and RV_d, Sub-situation 2.3 is divided into the following two addition factors,

$$D = d_G(x_iy_j, x_iy_h)$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}\sum_{h=1}^{2m} D &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{u\in LV_d} d_G(x_iy_j, u)+\\ &\quad \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{v\in RV_d} d_G(x_iy_j, v).\end{aligned} \tag{13}$$

Next we calculate the two addition factors of Eqn (13), respectively.

$$L = d_G(x_iy_j, v)$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{v\in LV_d} L &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}\sum_{h=1}^{j-1} d_G(x_iy_j, x_ky_h)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{i+j-2}[(k-i)^2+(k-i+1)+\\ &\quad \cdots+(j-2)+(j-1)]+\\ &\quad \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+j-1}^{2n+1}[(j-1)(k-i)]\\ &= [(14m^2-14m)n^3+(18m^2-18m)n^2\\ &\quad +(m^4-2m^3+3m^2-2m)n]/24,\end{aligned} \tag{14}$$

Eqn (14) calculates the sum of the distances from $x_iy_j(1 \le i \le n, 1 \le j \le m)$ to their corresponding LV_d, and the distance formula follows the same rules as Eqn (4).

For the sum of the distances from $x_iy_j(1 \le i \le n, 1 \le j \le m)$ to their corresponding RV_d, according to the known condition of Theorem 1, namely $m > n$, the distance formula changes while the D-value of m and n is different. Therefore, the distances from $x_iy_j(1 \le i \le n, 1 \le j \le m)$ to their corresponding RV_d is divided into two cases of A and B.

Case A: When $m > 2n$, the distance rule from $x_iy_j(1 \le i \le n, 1 \le j \le m)$ to their corresponding RV_d is same as Eqn(5), i.e.,

$$R = d_G(x_iy_j, v)$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{v\in RV_d} R &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}\sum_{h=j}^{2m} d_G(x_iy_j, x_ky_h)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}[(k-i)+(k-i)^2+\\ &\quad (k-i+1)+\cdots+(2m-j)]\\ &= [15mn^4+42mn^3+(42m^3+\\ &\quad 27m)n^2+(14m^3+4m)n]/24.\end{aligned} \tag{15}$$

Case B: When $n < m \le 2n$,

$$R = d_G(x_iy_j, v)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{v\in RV_d} R = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}\sum_{h=j}^{2m} d_G(x_iy_j, x_ky_h)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{i+2(m-n-1)}\sum_{k=i+1}^{2n+1}[(k-i)^2+ (k-i)+\cdots+(2m-j)]+ \sum_{i=1}^{n}\sum_{j=i+2(m-n)-1}^{m}\sum_{k=i+1}^{2m+i-j-1}[(k- i)^2+(k-i)+\cdots+(2m-j)]+ \sum_{i=1}^{n}\sum_{j=i+2(m-n)-1}^{m}\sum_{k=2m+i-j}^{2n+1}[(2m- j+1)(k-i)]$$
$$= -[31n^5+(75-150m)n^4+(70m^2- 350m+35)n^3+(-240m^3+90m^2- 180m-15)n^2+(5m^4-90m^3+ 15m^2-10m-6)n]/120. \tag{16}$$

For the explanations of Eqn (16), as illustrated in Fig. 5, we enumerate the situations when $m = 6, n = 3, 4, 5$, and what we need to find is the sum of the distances form the white vertices and the gray vertices to their corresponding RV_d. For the white vertices of Fig. 5, the distance formula from them to their corresponding RV_d is the same as Eqn (15). For the gray vertices, we found that while the value range of k is $i+1 \le k \le 2m+i-j-1$, the distance formula is the same as the white vertices. While $2m+i-j-1 \le k \le 2n+1$, the distance from the gray vertices to X_k is all $k-i$, and the number of the vertices with distance $k-i$ is $2m-j+1$, so the distance formula is $(2m-j+1)(k-i)$. From Eqn (13-16), we have

$$m > 2n, D = d_G(x_iy_j, x_iy_h)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}\sum_{h=1}^{2m} D = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{u\in LV_d} d_G(x_iy_j, u)+ \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{v\in RV_d} d_G(x_iy_j, v)$$
$$= [15mn^4+(14m^2+28m)n^3+ (42m^3+18m^2+9m)n^2+(m^4+ 12m^3+3m^2+2m)n]/24, \tag{17}$$

$$n < m \le 2n$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}\sum_{h=1}^{2m} D = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{u\in LV_d} d_G(x_iy_j, u)+ \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{v\in RV_d} d_G(x_iy_j, v)$$
$$= -[31n^5+(75-150m)n^4+(35- 280m)n^3+(-240m^3-90m- 15)n^2+(-80m^3-6)n]/120. \tag{18}$$

In summary, by using Eqn (8), (12), (17), (18), the final result of Situation 2 is also divided into two cases, i.e., when $m > 2n$,

$$2\sum_{i=1}^{n}\sum_{j=1}^{m} D_{(i,j)} = 2\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{h=1}^{2m} d_G(x_iy_j, x_iy_h)+ 2\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{h=1}^{2m} d_G(x_iy_j, x_ky_h)+ 2\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}\sum_{h=1}^{2m} d_G(x_iy_j, x_ky_h)$$
$$= -[n^5-10mn^4+(-70m^2+70m -5)n^3+(-80m^3-90m^2+ 120m)n^2+(-5m^4-70m^3 -15m^2+30m+4)n]/60. \tag{19}$$

When $n < m \le 2n$,

$$2\sum_{i=1}^{n}\sum_{j=1}^{m} D_{(i,j)} = 2\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{h=1}^{2m} d_G(x_iy_j, x_iy_h)+ 2\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{h=1}^{2m} d_G(x_iy_j, x_ky_h) + 2\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i+1}^{2n+1}\sum_{h=1}^{2m} d_G(x_iy_j, x_ky_h)$$
$$= -[32n^5+(75-160m)n^4+(30- 280m)n^3+(-320m^3-60m-15)n^2+ (-160m^3+20m-2)n]/60. \tag{20}$$

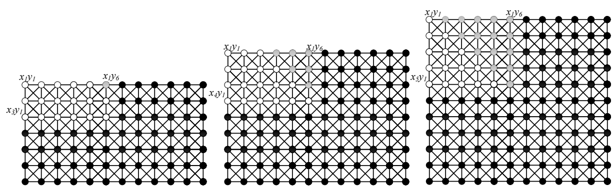

Fig. 5. $P_{2n+1} \boxtimes P_{2m}$ $m = 6, n = 3, 4, 5$

Finally, from Eqn (1) and the final result of Situation 1 and Situation 2, when $m > n$, we find the Wiener index of

$$
\begin{aligned}
&G = P_{2n+1} \boxtimes P_{2m},\\
&W(G) = 2\sum_{i=1}^{n}\sum_{j=1}^{m} D_{(i,j)} + \sum_{j=1}^{m} D_{(n+1,j)}\\
&= \begin{cases} -[n^5 + (5-10m)n^4 + (-70m^2+30m+ \\ 5)n^3 + (-80m^3 - 90m^2 + 60m - 5)n^2 + \\ (-5m^4 - 150m^3 - 15m^2 + 30m - 6)n \\ -160m^3 + 40m]/60, \quad if \quad 2n < m, \\ -[8n^5 + (20-40m)n4 + (10-80m)n^3 + \\ (-80m^3 - 30m - 5)n^2 + (-60m^3 + 5m- \\ 3)n - 40m^3 + 10m]/15, \quad if \quad n < m \le 2n. \end{cases}
\end{aligned}
$$

□

Next, we use the same method as Theorem 1 to derive the Wiener index of the strong product of $P_{2m} \boxtimes P_{2n+1}, m \le n$.

Theorem 2: Suppose that P_{2m} and $P_{2n+1} (m \le n)$ are two paths with orders $2m$ and $2n+1$, respectively, the Wiener index of their strong product $G = P_{2m} \boxtimes P_{2n+1}$ is:

$$
W(G) = \begin{cases} [5mn^4 + (290m^2 + 20m)n^3 + (70m^3 + \\ 390m^2 + 15m)n^2 + (85m^4 + 140m^3 + \\ 75m^2 - 10m)n - m^5 + 5m^4 + 75m^3 - \\ 5m^2 - 14m]/60, \qquad m > 1, \\ (16n^3 + 24n^2 + 14n + 3)/3, \qquad m = 1. \end{cases}
$$

Proof: When $m = 1$, there are only two layers in G, which don't satisfy most of the distance rules, but we solve this case separately at the end of the proof. Next we calculate the Wiener index of G when $1 < m \le n$. Once again, we use the symmetry of G to reduce our calculations, and get the following equation,

$$
\begin{aligned}
W(G) &= (\sum_{i=1}^{m} D_{(i,n+1)} + 4\sum_{i=1}^{m}\sum_{j=1}^{n} D_{(i,j)})/2 \\
&= \sum_{i=1}^{m} D_{(i,n+1)} + 2\sum_{i=1}^{m}\sum_{j=1}^{n} D_{(i,j)}.
\end{aligned} \tag{21}
$$

Compared Fig. 6 and Fig. 3, we find that the symmetry of the two graphs is different, which leads to the different of some distance rules between the two theorems, but the proof idea is same.

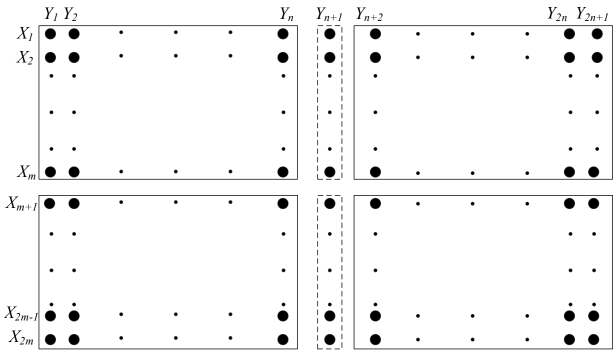

Fig. 6. Symmetry of $P_{2m} \boxtimes P_{2n+1}$

Situation 1: Initially we find the sum of the distances from $x_i y_{n+1} (1 \le i \le m)$ to this layer, upper layers and lower layers. The upper and lower layers can be divided into four parts LV_u, RV_u and LV_d, RV_d, respectively. For this layer, the distance rule is same as Theorem 1. As show in Fig. 6, the sum of the distances from $x_i y_{n+1}$ to all vertices on their left is equal to the sum of the distances from $x_i y_{n+1}$ to all vertices on their right, so we have

$$
\begin{aligned}
\sum_{i=1}^{m} D_{(i,n+1)} &= \sum_{i=1}^{m}\sum_{h=1}^{2n+1} d_G(x_i y_{n+1}, x_i y_h) + \\
&\quad \sum_{i=1}^{m}\sum_{k=1}^{2m}\sum_{h=1}^{2n+1} d_G(x_i y_{n+1}, x_k y_h) \\
&= m\sum_{h=1}^{2n+1} d_G(x_1 y_{n+1}, x_1 y_h) + \\
&\quad \sum_{i=1}^{m}\sum_{k=1}^{2m} d_G(x_i y_{n+1}, x_k y_{n+1}) + \\
&\quad 2(\sum_{i=1}^{m}\sum_{k=1}^{i-1}\sum_{h=1}^{n} d_G(x_i y_{n+1}, x_k y_h) + \\
&\quad \sum_{i=1}^{m}\sum_{k=i+1}^{2m}\sum_{h=1}^{n} d_G(x_i y_{n+1}, x_k y_h)) \\
&= [4mn^3 + (6m^2 + 6m)n^2 + (28m^3 + 6m^2 - \\
&\quad 2m)n + m^4 + 14m^3 - m^2 - 2m]/12.
\end{aligned} \tag{22}
$$

Situation 2: The proof process of this situation is same as the Situation 2 in Theorem 2. Therefore, we will not repeat the explanations here but give the final result directly.

$$
\begin{aligned}
2\sum_{i=1}^{m}\sum_{j=1}^{n} D_{(i,j)} &= 2\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{h=1}^{2n+1} d_G(x_i y_j, x_i y_h) \\
&\quad + 2\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{i-1}\sum_{h=1}^{2n+1} d_G(x_i y_j, x_i y_h) \\
&\quad + 2\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=i+1}^{2m}\sum_{h=1}^{2n+1} d_G(x_i y_j, x_i y_h) \\
&= [5mn^4 + 290m^2n^3 + (70m^3 + \\
&\quad 360m^2 - 15m)n^2 + (85m^4 + \\
&\quad 45m^2)n - m^5 + 5m^3 - 4m]/60.
\end{aligned} \tag{23}
$$

In summary, from Eqn (21-23), when $1 < m \le n$, we obtain the Wiener index of G which is expressed as follows:

$$
\begin{aligned}
W(G) &= \sum_{i=1}^{m} D_{(i,n+1)} + 2\sum_{i=1}^{m}\sum_{j=1}^{n} D_{(i,j)} \\
&= [5mn^4 + (290m^2 + 20m)n^3 + (70m^3 + 390m^2 + \\
&\quad 15m)n^2 + (85m^4 + 140m^3 + 75m^2 - 10m)n - \\
&\quad m^5 + 5m^4 + 75m^3 - 5m^2 - 14m]/60.
\end{aligned}
$$

Finally, when $m = 1$, the structure of G is relatively simple, and we give the Wiener index of G separately,

$$
\begin{aligned}
W(G) &= D_{(1,n+1)} + 2\sum_{j=1}^{n} D_{(1,j)} \\
&= (16n^3 + 24n^2 + 14n + 3)/3.
\end{aligned}
$$

□

IV. Application

By using the shortest path selection of Section III, we determine a closed upper bound of the edge forwarding index of the strong product network $G = P_{2n+1} \boxtimes P_{2m} (m > n)$. The edge forwarding index is proposed for routing, defined as follows: For a certain edge e of a routing β in network G, the edge forwarding index β is defined as the maximum number of times that the routing is determined by β passing through e, i.e., $\varepsilon(G) = \max\{\varepsilon_e(G, \beta) : e \in E(G)\}$. For a network G, its edge forwarding index is defined as the minimum value of the edge forwarding index of all routing in G, i.e., $\varepsilon(G) = \min\{\varepsilon(G, \beta) : e \in E(G)\beta\}$.

Considering comprehensively of the above definitions, in order to prove that it is meaningful to determine the upper bound of the edge forwarding index of the strong product networks, we transform the problem of maximizing network capacity into the problem of minimizing the edge forwarding index. Suppose that G is a certain communication network with T edges, and the constant n is the data transmission efficiency of the routing determined by β. Because we are aiming at undirected network, so the data can be transmitted both forward and backward on each edge in the network. Therefore, the transmission rate e is

$$2n + 2(T - 1)n = 2Tn,$$

the total transmission rate of all edges in the entire network is

$$2Tn + T(T - 1)n = T(T + 1)n.$$

Since the capacity C_e of an edge limits the maximum amount of data forwarding of the edge, the data or signal processed by an edge should not exceed its capacity, i.e.,

$$2Tn + \varepsilon_e n \leq C_e$$

where $\varepsilon_e = \varepsilon_e(G, \beta)$. Assuming that the maximum capacity of all edges in the network is C, it is obvious that there is

$$2Tn + \varepsilon_e n \leq C_e \leq C.$$

From the above formula, we can conclude that the transmission rate n of G must meet the requirement

$$n \leq C/(2T + \varepsilon),$$

where $\varepsilon = \varepsilon(G, \beta)$. Therefore, the total transmission rate of the edge must meet the requirement

$$T(T + 1)n \leq T(T + 1)C/(\varepsilon + 2T).$$

Since n, T, C in the above formula are constants, while the number of edges, the transmission efficiency and the capacity of the edge are determined, the capacity of the network is inversely proportional to the edge forwarding index of the network. So far, we have successfully transform the problem of maximizing network capacity into the problem of minimum edge forwarding index. Therefore, it is highly significance to determine the upper bound of the edge forwarding index of strong product networks. Next, the upper bound of edge forwarding index of strong product network $G = P_{2m} \boxtimes P_{2m}$ is given.

Theorem 3: Let P_{2n+1} and P_{2m} be two paths with order $2n + 1$ and $2m, m > n$, respectively. A new upper bound of the edge forwarding index of the strong product network $G = P_{2n+1} \boxtimes P_{2m}$ constructed by these two paths is:

$$\varepsilon(G) \leq (4m^2 - 2m)n + 2m^2 - 2mn^2.$$

Proof : As illustrated in Fig. 7, for the convenience of observation, we list the cases of $m = 6, n = 3,\ 4$. In the shortest path selection of Theorem 1, the edge with the largest forwarding index is the dotted line in Fig. 7, and the forwarding index of edge $(x_{n+1}y_m, x_{n+1}y_{m+1})$ is determined by the number of black vertices. In strong product network G, the number of the black vertices is

$$2m(2n + 1) - 2n(n + 1) = -2n^2 + (4m - 2)n + 2m. \tag{24}$$

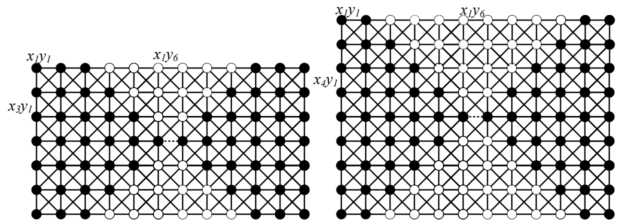

Fig. 7. Symmetry of $P_{2n+1} \boxtimes P_{2m}, m = 6, n = 3,\ 4$

In Fig. 7, we find that the position and number of black vertices on the left and right sides of edge $(x_{n+1}y_m, x_{n+1}y_{m+1})$ are symmetrical, and the number of black vertices on the left and right sides is

$$(2m - 1)n + m - n^2.$$

Take the $(2m - 1)n + m - n^2$ vertices on the left side of edge $(x_{n+1}y_m, x_{n+1}y_{m+1})$ as the sending-points, the last m vertices of X_{n+1}, i.e., $x_{n+1}y_{m+1}, x_{n+1}y_{m+2} \ldots, x_{n+1}y_{2m}$, as the receiving-points. According to the shortest path selection in Theorem 1, when the sending-points send data or signal to the receiving-points, it must pass through $(x_{n+1}y_m, x_{n+1}y_{m+1})$. At this time, the forwarding index of this edge is

$$(2m^2 - m)n + m^2 - mn^2.$$

By using the symmetry of G, the forwarding index of edge $(x_{n+1}y_m, x_{n+1}y_{m+1})$ is

$$(4m^2 - 2m)n + 2m^2 - 2mn^2,$$

which is an upper bound of edge forwarding index of strong product network $G = P_{2n+1} \boxtimes P_{2m} (m > n)$. □

V. Conclusion

In network planning and design, the strong product method is one of the simple, reasonable and efficient methods. This paper analyze the structural characteristics of strong product graphs, by using symmetry and classification methods to determine the exact Wiener index of two paths. The obtained Wiener index is only determined by the orders of the factor graphs. In the process of proving the conclusion, we also determine a closed upper bound of the edge-forwarding index, which also depends on the orders of the factor graphs. In the next step, we will attempt to deal with the Wiener indices of different graphs.

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