Partial Complement Energy of Special Graphs

SWATI NAYAK and K ARATHI BHAT*

***Abstract*—Partial complement of a graph G with respect to a set S is a graph obtained from G by removing edges of $\langle S\rangle$ and adding the edges which are not in $\langle S\rangle$. In this article, we obtain partial complement energy and partial complement characteristic polynomial of few classes of graph.**

***Index Terms*—Partial complement, Partial complement eigenvalue, Partial complement characteristic polynomial.**

I. Introduction

ONE of the fundamental question in graph theory concerns the efficiency of recognition of a graph class Ç. For instance, how quick we can predict whether a graph is planar, 2-connected, triangle free, bipartite, chordal, 3-colorable etc. for modifying an input graph into the desired graph class Ç¸ is one of the main focus of partial complementation. An atomic operation is the change of adjacency required for modifying an input graph into the desired graph class G, and it is one of the main focus of partial complementation. One may be interested to find whether it is possible to add or delete at most k edges to make a graph 2-connected or chordal or triangle free etc. In literature, atomic operation is the change of adjacency i.e., for a pair of vertices x and y, either one can add or delete an edge xy.

Let G be a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges. Let $A = (a_{ij})$ be an adjacency matrix of graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non increasing order, are called eigenvalues of G. The energy of G [2] is defined to be sum of absolute values of the eigenvalues of G. This graph invariant is very closely connected to a chemical quantity known as the total π-electron energy of conjugated hydro carbon molecules. The carbon atoms are represented by the vertices and two vertices are adjacent if and only if there is a carbon-carbon bond. Hydrogen atoms are ignored. The energy level of π-electron in molecules of conjugated hydrocarbons are related to the eigenvalues of a molecular graph. An interesting quantity in Huckel theory is sum of energies of all the electrons in a molecule, so called $\pi-$electron energy of a molecule.
For all terminologies we refer [1].

Recently Fedor V Fomin et al. introduced partial complements of graph [3]. This motivated us to study the energy of partial complement of a graph.

Definition 1.1: Let $G = (V, E)$ be a graph and $S \subseteq V$. The partial complement of a graph G with respect to S, denoted by $G \oplus S$, is a graph (V, E_S), where for any two vertices $x, y \in V$, $xy \in E_S$ if and only if one of the following conditions hold good:

1) $x \notin S$ or $y \notin S$ and $xy \in E$.
2) $x, y \in S$ and $xy \notin E$.

Partial complement of graph G with respect to a set S is also defined as a graph obtained from G by removing edges of $\langle S\rangle$ and adding the edges which are not in $\langle S\rangle$.

Definition 1.2: [4] Let $G \oplus S$ be partial complement of a graph G with respect to S. We define partial complement adjacency matrix of $G\oplus S$ as $n\times n$ matrix defined by $A_p(G\oplus S) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } i \neq j \\ 1, & \text{if } i = j \text{ and } v_i \in S \\ 0, & \text{otherwise.} \end{cases}$$

Adjacency matrix and partial complement adjacency matrix of a graph G are related as follows:
If $A(G) = \left[\begin{array}{c|c} L & M \\ \hline M' & N \end{array}\right]$ then $A_p(G\oplus S) = \left[\begin{array}{c|c} \overline{L} + I & M \\ \hline M' & N \end{array}\right]$
where L is an adjacency matrix of induced subgraph $\langle S\rangle$ and I is an identity matrix.
The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A_p(G \oplus S)$ is called spectrum of $G \oplus S$. Partial complement energy of $G \oplus S$, denoted by $E_p(G \oplus S)$ is defined as $\sum\limits_{i=1}^{n} |\lambda_i|$. In [5], authors have introduced the concept of distance energy of connected partial complements of a graph, and obtained distance energy of connected partial complement of some families of graphs. For more information on energy of graph we refer [5]–[7]. For convenience, throughout this paper $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $G \oplus S$.

II. Partial complement energy of special graphs

In this section, the partial complement energy of some families of graphs is computed.

A. Partial complement energy of Amalgamation of m copies of K_n

A graph amalgamation is a relationship between two graphs (one graph is an amalgamation of another). Amalgamations can provide a way to reduce a graph to a simpler graph while keeping certain structure intact.

Definition 2.1: Let $\{G_1, G_2, G_3, \ldots, G_m\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{0i} called a terminal. The amalgamation $Amal(v_{0i}, G_i)$ is formed by taking all the $G_i's$ and identifying their terminals. In particular, if we take $G_i = K_n$ for $i = 1, 2, \ldots, m$ we get amalgamation of m copies of K_n denoted by $Amal(m, K_n), m \geq 2$. For convenience we denote v_0 as the vertex of amalgamation and $v_{j2}, v_{j3}, \ldots, v_{jn}$ are the remaining vertices of the j^{th} copy of K_n, where $1 \leq j \leq m$.
The amalgamation of 3 copies of K_4 is shown in Figure 1.

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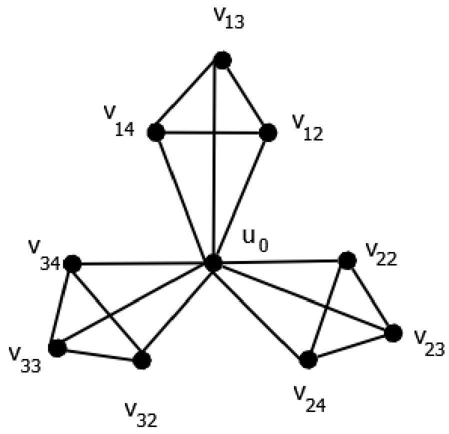

Fig. 1. Amal(3,K_4)

Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$. We obtain partial complement energy of $Amal(m, K_n)$ when $S = \{v_0\}$ and $S = \{v_{ij} | i = 1, 2, \ldots, m, j = 2, 3, \ldots, n\}$ and partial complement characteristic polynomial of $Amal(m, K_n)$ when $S = \{v_{12}, v_{13}, \ldots, v_{1n}\}$ in the following theorems.

Theorem 2.1: Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$ and $S = \{v_0\}$. Then, $E_P(Amal(m, K_n) \oplus S) = (n-2)(2m-1) + \sqrt{(n-3)^2 + 4m(n-1)}$.

Proof: Let

$$A = \begin{pmatrix} J_1 & J_{1\times n-1} & J_{1\times n-1} & \ldots & J_{1\times n-1} \\ J_{n-1\times 1} & B_{n-1} & 0_{n-1} & \ldots & 0_{n-1} \\ J_{n-1\times 1} & 0_{n-1} & B_{n-1} & \ldots & 0_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{n-1\times 1} & 0_{n-1} & 0_{n-1} & \ldots & B_{n-1} \end{pmatrix}_{m(n-1)+1}$$

be a partial complement adjacency matrix of $Amal(m, K_n)$. Here J is matrix of all 1's, 0 is the zero matrix and B is the adjacency matrix of complete subgraph.

Step 1: Consider $|\lambda I - A|$.

Applying row operation $R'_{v_{ij}} \longleftarrow R_{v_{ij}} - R_{v_{ij+1}}, i = 1, 2, \ldots, m, j = 2, 3, \ldots, n-1$ and column operation $C'_{v_{ij}} \longleftarrow C_{v_{ij}} + C_{v_{ij-1}} + C_{v_{ij-2}} + \ldots + C_{v_{i2}}, i = 1, 2, \ldots, m, j = n, n-1, \ldots, 3$ on $|\lambda I - A|$, we get $(\lambda+1)^{m(n-2)} \det(C)$, where $\det(C)$ is of the order $m+1$.

Step 2: On performing row operation $R'_i \longleftarrow R_i - R_{i+1}, i = 2, 3, \ldots, m$ and column operation $C'_i \longleftarrow C_i + C_{i-1} + C_{i-2} + \ldots + C_2, i = m+1, m, \ldots, 3$ on $\det(C)$, we obtain $(\lambda - n + 2)^{m-1} \det(D)$ which is of order 2.

Step 3: Expanding $\det(D)$ leads to the polynomial $\lambda^2 - (n-1)\lambda + m + n - mn - 2$.

Hence partial complement spectrum of $Amal(m, K_n)$ is

$$\begin{pmatrix} -1 & n-2 & \frac{P+Q}{2} & \frac{P-Q}{2} \\ m(n-2) & m-1 & 1 & 1 \end{pmatrix},$$

where $P = (n-1)$ and $Q = \sqrt{(n-1)^2 - 4(m+n-mn-2)}$. Hence, the result follows. ■

Theorem 2.2: Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$ with $S = \{v_{ij} | i = 1, 2, \ldots, m, j = 2, 3, \ldots, n\}$. Then $E_P(Amal(m, K_n) \oplus S) = (n-2)(2m-1) + \sqrt{(m+n-mn-2)^2 + 4m(n-1)}$.

Proof: Let

$$A = \begin{pmatrix} 0_1 & J_{1\times n-1} & J_{1\times n-1} & \ldots & J_{1\times n-1} \\ J_{n-1\times 1} & I_{n-1} & J_{n-1} & \ldots & J_{n-1} \\ J_{n-1\times 1} & J_{n-1} & I_{n-1} & \ldots & J_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{n-1\times 1} & J_{n-1} & J_{n-1} & \ldots & I_{n-1} \end{pmatrix}_{m(n-1)+1}$$

be a partial complement adjacency matrix of $Amal(m, K_n)$. Repeating Step 1 to Step 3 of Theorem 2.1, we get the polynomial

$$(\lambda-1)^{m(n-2)}(\lambda+n-2)^{m-1}[\lambda^2+(m+n-mn-2)\lambda-mn+m].$$

Hence the partial complement spectrum of $Amal(m, k_n)$ is

$$\begin{pmatrix} 1 & 2-n & \frac{P+Q}{2} & \frac{P-Q}{2} \\ m(n-2) & m-1 & 1 & 1 \end{pmatrix},$$

where $P = -(m+n-mn-2)$ and $Q = \sqrt{(m+n-mn-2)^2 + 4m(n-1)}$.

Therefore, $E_P(Amal(m, K_n) \oplus S)$ is $(n-2)(2m-1) + \sqrt{(m+n-mn-2)^2 + 4m(n-1)}$. ■

Theorem 2.3: Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$ and $S = \{v_{12}, v_{13}, \ldots, v_{1n}\}$. Then, partial complement characteristic polynomial of $Amal(m, K_n)$ is given by $\{\lambda^3 + (1-n)\lambda^2 + (m+n-mn-2)\lambda + (mn-m-4n+n^2+3)\}(\lambda-(n-2))^{m-2}(\lambda-1)^{n-2}(\lambda+1)^{(m-1)(n-2)}$.

Proof: Let

$$A = \begin{pmatrix} 0_1 & J_{1\times n-1} & J_{1\times n-1} & \ldots & J_{1\times n-1} \\ J_{n-1\times 1} & I_{n-1} & 0_{n-1} & \ldots & 0_{n-1} \\ J_{n-1\times 1} & 0_{n-1} & B_{n-1} & \ldots & 0_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{n-1\times 1} & 0_{n-1} & 0_{n-1} & \ldots & B_{n-1} \end{pmatrix}_{m(n-1)+1}$$

be a partial complement adjacency matrix of $Amal(m, K_n)$. Repeating row and column operation on $|\lambda I - A|$ as in step 1 of Theorem 2.1, we get

$|\lambda I - A| = (\lambda+1)^{(m-1)(n-2)}(\lambda-1)^{n-2}\det(C)$, where $\det(C)$ is of the order $m+1$ and

$$C = \begin{pmatrix} \lambda & 1-n & (1-n)J_{1,m-1} \\ -1 & \lambda-1 & 0_{1,m-1} \\ -J_{m-1,1} & 0_{m-1,1} & (\lambda-n+2)I_{m-1} \end{pmatrix}_{m+1}.$$

Applying the row operation $R'_i \longleftarrow R_i - R_{i+1}, i = 3, 4, \ldots, m$ on $det(C)$ we get

$|\lambda I - A| = (\lambda+1)^{(m-1)(n-2)}(\lambda-1)^{n-2}(\lambda-(n-2))^{m-2}\det(D)$, where $\det(D)$ is of the order 3 and

$$D = \begin{pmatrix} \lambda & 1-n & (1-n)(m-1) \\ -1 & \lambda-1 & 0 \\ -1 & 0 & \lambda-n+2 \end{pmatrix}.$$

Expanding $\det(D)$ leads to the polynomial $\{\lambda^3 + (1-n)\lambda^2 + (m+n-mn-2)\lambda + (mn-m-4n+n^2+3)\}$.

Hence the result follows. ■

B. Partial complement energy of a Double star

A double star is the graph denoted by $S(l, m)$, consisting of union of two stars $K_{1,l}$ and $K_{1,m}$ together with the line joining their centers. Let $V = \{u_i, v_j | i = 0, 1, \ldots, l, j = 0, 1, \ldots, m\}$ be the vertex set of the double star $S(l, m)$ with u_0 and v_0 as its centers. We obtain the partial complement characteristic polynomial of $S(l, m)$ when $S = \{u_0, v_0\}$, $S = \{u_i, v_j | i = 1, 2, \ldots, l, j = 1, 2, \ldots, m\}$, $S = \{u_i | i =$

$0, 1, 2, \ldots, l\}$ and $S = \{u_0, v_j | j = 1, 2 \ldots, m\}$ in the following theorems.

Theorem 2.4: Partial complement characteristic polynomial of $S(l, m)$ with $S = \{u_0, v_0\}$ is given by $\lambda^{l+m-2}(\lambda^2 - \lambda - l)(\lambda^2 - \lambda - m)$.

Proof: Let $A = \begin{pmatrix} J_1 & 0_1 & J_{1\times l} & 0_{1\times m} \\ 0_1 & J_1 & 0_{1\times l} & J_{1\times m} \\ J_{l\times 1} & 0_{l\times 1} & 0_l & 0_{l\times m} \\ 0_{m\times 1} & J_{m\times 1} & 0_{m\times l} & 0_m \end{pmatrix}_{l+m+2}$ be a partial complement adjacency matrix of $S(l, m)$. Consider $|\lambda I - A|$.

Step 1: Applying row operation $R_i \longleftarrow R_i - R_{i+1}, i = 2, 3, \ldots, l-1, l+1, \ldots, m-1$ and column operations $C_i \longleftarrow C_i + C_{i-1} + \ldots + C_3, i = l, l-1, \ldots, 4$ and $C_j \longleftarrow C_j + C_{j-1} + \ldots + C_{l+1}, j = m, m-1, \ldots, l+2$ on $|\lambda I - A|$, we obtain $\lambda^{l+m-2}\det(C)$.

Step 2: On further simplification of $\det(C)$ which is of order 4, we get $\det(C) = (\lambda^2 - \lambda - l)(\lambda^2 - \lambda - m)$. Hence, the result. ■

From Theorem 2.4, we note the following.

Note 2.1: Partial complement energy of $S(l, m)$ when $S = \{u_0, v_0\}$ is $\sqrt{1 + 4l} + \sqrt{1 + 4m}$.

Theorem 2.5: Partial complement characteristic polynomial of $S(l, m)$ when $S = \{u_i, v_j | i = 1, 2, \ldots, l, j = 1, 2, \ldots, m\}$ is $\lambda^{l+m-2}[\lambda^4 - (l+m)\lambda^3 - (l+m+1)\lambda^2 + (l+m+2lm)\lambda - lm]$.

Proof: Let $A = \begin{pmatrix} 0_1 & J_1 & J_{1\times l} & 0_{1\times m} \\ J_1 & 0_1 & 0_{1\times l} & J_{1\times m} \\ J_{l\times 1} & 0_{l\times 1} & J_l & J_{l\times m} \\ 0_{m\times 1} & J_{m\times 1} & J_{m\times l} & J_m \end{pmatrix}_{l+m+2}$ be a partial complement adjacency matrix of $S(l, m)$. Repeating Step 1 and Step 2 of Theorem 2.4, the result follows. ■

Theorem 2.6: Partial complement characteristic polynomial of $S(l, m)$ with $S = \{u_i | i = 0, 1, 2, \ldots, l\}$ is $\lambda^{l+m-2}(\lambda - l)[\lambda^3 - \lambda^2 - \lambda(m+1) + m]$.

Proof: Let $A = \begin{pmatrix} J_1 & J_1 & 0_{1\times l} & 0_{1\times m} \\ J_1 & 0_1 & 0_{\times l} & J_{1\times m} \\ 0_{l\times 1} & 0_{l\times 1} & J_l & 0_{l\times m} \\ 0_{m\times 1} & J_{m\times 1} & 0_{m\times l} & 0_m \end{pmatrix}_{l+m+2}$ be a partial complement adjacency matrix of $S(l, m)$. Repeating Step 1 and Step 2 of Theorem 2.4, we get the result. ■

Theorem 2.7: Partial complement characteristic polynomial of $S(l, m)$ with $S = \{u_0, v_j | j = 1, 2 \ldots, m\}$ is $\lambda^{l+m-2}[\lambda^4 - (m+1)\lambda^3 - (l+m+1)\lambda^2 + lm\lambda + lm]$.

Proof: Let $A = \begin{pmatrix} J_1 & J_1 & J_{1\times l} & J_{1\times m} \\ J_1 & 0_1 & 0_{1\times l} & J_{1\times m} \\ J_{l\times 1} & 0_{l\times 1} & 0_l & 0_{l\times m} \\ J_{m\times 1} & J_{m\times 1} & 0_{m\times l} & J_m \end{pmatrix}_{l+m+2}$ be a partial complement adjacency matrix of $S(l, m)$. Repeating Step 1 and 2 of Theorem 2.4, the result follows. ■

Theorem 2.8: Let H be a graph which is the union of two star graphs $K_{1,l}$ and $K_{1,m}$. Let $V = \{u_i, v_j | i = 0, 1, \ldots, l, j = 0, 1, \ldots, m\}$ be the vertex set of two star graphs $K_{1,l}$ and $K_{1,m}$ with u_0 and v_0 as its centers. Let G be a graph obtained by adding a new vertex w_0 to the graph H and joining w_0 with u_0 and v_0.

1. If $S = \{u_0, w_0, v_0\}$, then partial complement characteristic polynomial of G is $\lambda^{l+m-2}(\lambda - 1)[\lambda^4 - 2\lambda^3 - (l+m)\lambda^2 + (l+m)\lambda + lm]$.
2. If $S = \{u_i, i = 0, 1, 2, \ldots, l\}$, then partial complement characteristic polynomial of G is $\lambda^{l+m-2}(\lambda - 1)[\lambda^4 - 2\lambda^3 - (2+m)\lambda^2 + (1+m)\lambda + m]$.
3. If $S = \{u_i, v_j | i = 1, 2, \ldots, l, j = 1, 2, \ldots, m\}$, then partial complement characteristic polynomial of G is $\lambda^{l+m-2}[\lambda^5 - (l+m)\lambda^4 - (l+m+2)\lambda^3 + 2(l+m+lm)\lambda^2 + (l+m+lm)\lambda - 4lm]$.
4. If $S = \{u_0, v_i | i = 1, 2, \ldots, m\}$, then partial complement characteristic polynomial of G is $\lambda^{l+m-2}[\lambda^5 - (1+m)\lambda^4 - (l+m+2)\lambda^3 + (1+3m+lm)\lambda^2 + (l-m+lm)\lambda - lm]$.
5. If $S = \{w_0\}$, then partial complement characteristic polynomial of G is $\lambda^{l+m-2}[\lambda^5 - \lambda^4 - (l+m+2)\lambda^3 + (l+m)\lambda^2 + (l+m+lm)\lambda - lm]$.

C. Partial complement energy of Banana tree graph $B_{l,m}$.

A banana tree is a family of stars with a new vertex adjoined to one end vertex of each star. The Banana tree graph $B_{3,5}$ is shown in Figure 2.

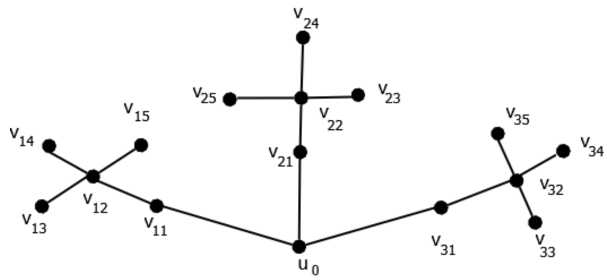

Fig. 2. Banana tree graph $B_{3,5}$

Let $v_0, v_{11}, v_{12}, \ldots, v_{1m}, v_{21}, \ldots, v_{2m}, \ldots, v_{l1}, \ldots, v_{lm}$ be the vertices of $B_{l,m}$. We obtain partial complement characteristic polynomial when $S = \{v_0\}$ and $S = \{v_0, v_{11}, v_{21}, \ldots, v_{l1}\}$ in the following theorems.

Theorem 2.9: Let $v_0, v_{11}, v_{12}, \ldots, v_{1m}, v_{21}, v_{22}, \ldots, v_{2m}, \ldots, v_{l1}, v_{l2}, \ldots, v_{lm}$ be the vertices of $B_{l,m}$ and $S = \{v_0\}$. Then partial complement characteristic polynomial of $B_{l,m}$ is $\lambda^{l(m-2)-1}[\lambda^2 - (m-1)\lambda]^{l-1}[\lambda^4 - \lambda^3 + (1-m-l)\lambda^2 + (m-1)\lambda + l(m-2)]$.

Proof: Let

$$A = \begin{pmatrix} J_1 & C_{1\times m} & C_{1\times m} & \ldots & C_{1\times m} \\ C_{m\times 1} & B_m & 0_m & \ldots & 0_m \\ C_{m\times 1} & 0_m & B_m & \ldots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{m\times 1} & 0_m & 0_m & \ldots & B_m \end{pmatrix}_{lm+1}$$

be a partial complement adjacency matrix of $B_{l,m}$, where $C = [1\ 0\ 0 \ldots 0]$ and

$$B_m = \begin{pmatrix} 0_1 & J_1 & 0_{1\times m-2} \\ J_1 & 0_1 & J_{1\times m-2} \\ 0_{m-2\times 1} & J_{m-2\times 1} & 0_{m-2} \end{pmatrix}.$$

Step 1: Consider $|\lambda I - A|$. Applying block row operation $R_i' \longleftarrow R_i - R_{i+1}, i = 2, 3, \ldots, l$ and block column operation $C_i' \longleftarrow C_i + C_{i-1} + \ldots + C_2, i = l+1, l, l-1, \ldots, 3$ on $|\lambda I - A|$, we obtain $(\det(\lambda I - B))^{l-1}\det D$.

Step 2: On further simplification we get, $\lambda^{l(m-2)-1}[\lambda^2 - (m-1)\lambda]^{l-1}[\lambda^4 - \lambda^3 + (1-m-l)\lambda^2 + (m-1)\lambda + l(m-2)]$. ■

Theorem 2.10: Let $v_0, v_{11}, v_{12}, \ldots, v_{1m}, v_{21}, v_{22}, \ldots, v_{2m}, \ldots, v_{l1}, v_{l2}, \ldots, v_{lm}$ be the vertices of $B_{l,m}$ and

$S = \{v_0, v_{11}, v_{21}, \ldots, v_{l1}\}$. Then partial complement characteristic polynomial of $B_{l,m}$ is $\lambda^{l(m-2)-1}[\lambda^2 - m + 1]^{l-1}(\lambda - 1)[\lambda^3 - l\lambda^2 + (1-m)\lambda + l(m-2)]$.

Proof: Let

$$A = \begin{pmatrix} J_1 & 0_{1\times m} & 0_{1\times m} & \ldots & 0_{1\times m} \\ 0_{m\times 1} & \underset{\sim}{E}_m & \underset{\sim}{D}_m & \ldots & \underset{\sim}{D}_m \\ 0_{m\times 1} & \underset{\sim}{D}_m & \underset{\sim}{E}_m & \ldots & \underset{\sim}{D}_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{m\times 1} & \underset{\sim}{D}_m & \underset{\sim}{D}_m & \ldots & \underset{\sim}{E}_m \end{pmatrix}_{lm+1}$$

be a partial complement adjacency matrix of $B_{l,m}$, where

$$\underset{\sim}{D}_m = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \text{ and } \underset{\sim}{E}_m = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 1 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 1 & 0 & \ldots & 0 \end{pmatrix}.$$

The result follows by repeating Step 1 and Step 2 of Theorem 2.9. ■

D. Partial complement energy of Friendship graph F_n.

The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex. The friendship graph F_4 is shown in Figure 3.

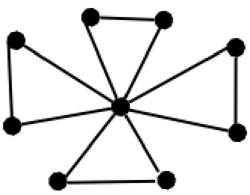

Fig. 3. Friendship graph F_4

In the next theorem we obtain $E_p(F_n \oplus S)$, when $\langle S\rangle = K_3$.

Theorem 2.11: Let F_n be the friendship graph with $\langle S\rangle = K_3$. Then, $E_p(F_n \oplus S) = 2\sqrt{2(n-1)} + 2n - 1$.

Proof: Let

$$A = \begin{pmatrix} J_1 & 0_{1\times 2} & J_{1\times 2} & J_{1\times 2} & \ldots & J_{1\times 2} \\ 0_{2\times 1} & I_2 & 0_2 & 0_2 & \ldots & 0_2 \\ J_{2\times 1} & 0_2 & (J-I)_2 & 0_2 & \ldots & 0_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{2\times 1} & 0_2 & 0_2 & & \ldots & (J-I)_2 \end{pmatrix}_{2n+1}$$

be a partial complement adjacency matrix of F_n. Expanding $|\lambda I - A|$ along second block row gives $|\lambda I - A| = (\lambda - 1)^2 B$. Applying block row operation $R_i' \longleftarrow R_i - R_{i+1}$, $i = 2, 3, \ldots, n-1$ and column operations $C_i' \longleftarrow C_i + C_{i-1} + \ldots + C_2,\ i = n, n-1, \ldots, 3$ on B we get the polynomial

$$(\lambda - 1)^n(\lambda + 1)^{n-1}[\lambda^2 - 2\lambda - 2n + 3].$$

Hence the partial complement spectrum of F_n is

$$\begin{pmatrix} 1 & -1 & 1 + \sqrt{(2(n-1)} & 1 - \sqrt{2(n-1)} \\ n & n-1 & 1 & 1 \end{pmatrix},$$

Therefore $E_P(F_n \oplus S)$ is $\sqrt{2(n-1)} + 2n - 1$. ■

E. Partial complement energy of Cocktail party graph $K_{n\times 2}$

The cocktail party graph, $K_{n\times 2}$ is a regular graph of degree $2n-2$ on $2n$ vertices. It is obtained from K_{2n} by deleting a perfect matching. The cocktail party graph $K_{3\times 2}$

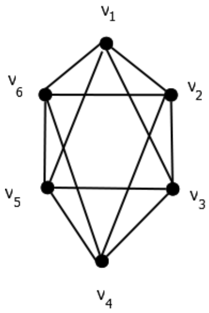

Fig. 4. Cocktail party graph $K_{3\times 2}$

is shown in Figure 4.

We obtain $E_p(K_{n\times 2} \oplus S)$, when $\langle S\rangle = K_n$ and $|S| = 2n$ in the following theorems.

Theorem 2.12: Let $K_{n\times 2}$ be the cocktail party graph with $\langle S\rangle = K_n$. Then, $E_p(K_{n\times 2} \oplus S) = 2\sqrt{2}(n-1) + \sqrt{5n^2 - 12n + 8}$.

Proof: Let $A_p = \begin{bmatrix} I_n & (J-I)_n \\ (J-I)_n & (J-I)_n \end{bmatrix}_{2n\times 2n}$ be the adjacency matrix of $K_{n\times 2} \oplus S$.

Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with A_p.

Consider

$$(A_p - \lambda I)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [(1-\lambda)I]X + (J-I)Y \\ (J-I)X + [J - (\lambda+1)I]Y \end{bmatrix} \quad (1)$$

Case 1: Let $X = X_j, j = 2, 3, \ldots, n$ and $Y = -\dfrac{\lambda - n + 1}{\lambda^2 - (n-2)\lambda - (n-1)}X_j$, where λ is any root of the equation

$$\lambda^2 - 2 = 0.$$

From Equation (1)

$$(1-\lambda)IX_j - (J-I)\left(\frac{\lambda - n + 1}{\lambda^2 - (n-2)\lambda - (n-1)}\right)X_j = \left[\frac{(\lambda - n + 1)(2 - \lambda^2)}{\lambda^2 - (n-2)\lambda - (n-1)}\right]X_j.$$

Hence $\lambda = \sqrt{2}$ and $\lambda = -\sqrt{2}$ are the eigenvalues with multiplicity of at least $n-1$, as there are $n-1$ eigenvectors of the form X_j.

Case 2: Let $X = \mathbf{1}_n$ and $Y = \dfrac{(n-1)(\lambda+1)}{\lambda^2 - (n-2)\lambda - (n-1)}\mathbf{1}_n$, where λ is any root of the equation

$$\lambda^2 - n\lambda + 3n - n^2 - 2 = 0.$$

From Equation (1),

$$(1-\lambda)I\mathbf{1}_n + (J-I)\left(\frac{(n-1)(\lambda+1)}{\lambda^2 - (n-2)\lambda - (n-1)}\right)\mathbf{1}_n = \left[-\frac{(\lambda+1)(\lambda^2 - n\lambda + 3n - n^2 - 2)}{\lambda^2 - (n-2)\lambda - (n-1)}\right]\mathbf{1}_n.$$

Thus $\lambda = \dfrac{n + \sqrt{5n^2 - 12n + 8}}{2}$ and $\lambda = \dfrac{n - \sqrt{5n^2 - 12n + 8}}{2}$ are the eigenvalues with multiplicity of at least one.

Thus, partial complement spectrum of $k_{n\times 2}$ with $|S| = n$ is

$$\begin{pmatrix} \sqrt{2} & -\sqrt{2} & \frac{n+\sqrt{5n^2-12n+8}}{2} & \frac{n-\sqrt{5n^2-12n+8}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$$

and $E_p(K_{n\times 2} \oplus S) = 2\sqrt{2}(n-1) + \sqrt{5n^2-12n+8}$. ■

Theorem 2.13: Let $K_{n\times 2}\oplus S$ be the partial complement of a cocktail party graph with $|S| = 2n$. Then, $E_p(K_{n\times 2}\oplus S) = 2n$.

Proof: Let $A_p = \begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix}_{2n\times 2n}$ be the adjacency matrix of $K_{n\times 2} \oplus S$. Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with A_p.
Consider

$$(A_p - \lambda I)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [(1-\lambda)I]X + IY \\ IX + [(1-\lambda)I]Y \end{bmatrix} \quad (2)$$

Case 1: Let $X = X_j, j = 2, 3, \ldots, n$ and $Y = -\frac{1}{1-\lambda}X_j$, where λ is any root of the equation

$$\lambda^2 - 2\lambda = 0$$

.
From Equation (2),

$$(1-\lambda)IX_j + I\left(\frac{1}{1-\lambda}\right)X_j = \frac{\lambda^2 - 2\lambda}{\lambda - 1}X_j.$$

Hence $\lambda = 0$ and $\lambda = 2$ are the eigenvalues with multiplicity of at least $n-1$, as there are $n-1$ eigenvectors of the form X_j.

Case 2: Let $X = 1_n$ and $Y = -\frac{1}{1-\lambda}1_n$, where λ is any root of the equation

$$\lambda^2 - 2\lambda = 0.$$

From Equation (2),

$$(1-\lambda)I1_n + I\left(\frac{1}{1-\lambda}\right)1_n = \frac{\lambda^2 - 2\lambda}{\lambda - 1}1_n.$$

Hence $\lambda = 0$ and $\lambda = 2$ are the eigenvalues with multiplicity of at least 1.

Thus, partial complement spectrum of $K_{n\times 2}$ with $|S| = 2n$ is $\begin{pmatrix} 0 & 2 \\ n & n \end{pmatrix}$ and $E_p(K_{n\times 2} \oplus S) = 2n$. ■

F. Partial complement energy of Ladder rung graph LR_n

The ladder rung graph LR_n is a regular graph of degree one on $2n$ vertices. Let the vertices of LR_n be $v_1, v_2, \ldots, v_{2n}$ and the vertex v_i is adjacent to $v_{i+1},\ i = 1, 3, \ldots, 2n-1$. We obtain $E_p(LR_n \oplus S)$, when $S = \{v_1, v_3, \ldots, v_{2n-1}\}$ and $S = \{v_1, v_2\}$ in the following theorems.

Theorem 2.14: Let LR_n be the ladder rung graph with $S = \{v_1, v_3, \ldots, v_{2n-1}\}$. Then, $E_p(LR_n \oplus S) = \sqrt{n^2+4} + 2n - 2$.

Proof: Let $A_p = \begin{bmatrix} J_n & I_n \\ I_n & 0_n \end{bmatrix}_{2n\times 2n}$ be the adjacency matrix of $LR_n \oplus S$.
Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with A_p.
Consider

$$(A_p - \lambda I)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [J-\lambda I]X + IY \\ IX - \lambda IY \end{bmatrix} \quad (3)$$

Case 1: Let $X = X_j, j = 2, 3, \ldots, n$. Then,
$(J - \lambda I)X_j + IY = 0$
$-\lambda X_j + Y = 0 \ \Rightarrow Y = \lambda X_j$.
Substituting $Y = \lambda X_j$ in the equation $IX_j - \lambda IY = 0 \Rightarrow (1-\lambda^2)X_j = 0$.
Thus, $\lambda = \pm 1$ occurs $(n-1)$ times as there are $(n-1)$ eigenvectors of the form X_j.

Case 2: Substituting $X = \mathbf{1}_n$ in $IX - \lambda IY = 0$, we get $Y_n = \frac{\mathbf{1}_n}{\lambda}$.
Now substituting $X = \mathbf{1}_n$ and $Y_n = \frac{\mathbf{1}_n}{\lambda}$ in $(J-\lambda I)X_j + IY = 0$, we get,
$(n - \lambda + \frac{1}{\lambda})\mathbf{1}_n = 0 \Rightarrow \lambda^2 - \lambda n - 1 = 0$.
Thus, $\lambda = \frac{n+\sqrt{n^2+4}}{2}$ and $\lambda = \frac{n-\sqrt{n^2+4}}{2}$ are the eigenvalues with multiplicity at least one.
Thus, the partial complement spectrum of LR_n is

$$\begin{pmatrix} 1 & -1 & \frac{n+\sqrt{n^2+4}}{2} & \frac{n-\sqrt{n^2+4}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Hence, $E_p(LR_n \oplus S) = \sqrt{n^2+4} + 2n - 2$. ■

Theorem 2.15: Let LR_n be the ladder rung graph with $S = \{v_1, v_2\}$. Then, $E_p(LR_n \oplus S) = 2n$.

Proof: Let

$$A = \begin{pmatrix} I_2 & 0_2 & 0_2 & 0_2 & \ldots & 0_2 \\ 0_2 & J-I_2 & 0_2 & 0_2 & \ldots & 0_2 \\ 0_2 & 0_2 & (J-I)_2 & 0_2 & \ldots & 0_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0_2 & 0_2 & & \ldots & (J-I)_2 \end{pmatrix}_{2n}$$

be a partial complement adjacency matrix of LR_n.

$$\begin{aligned} |\lambda I - A| &= |\lambda - I|_{2\times 2}\ |\lambda - J + I|_2^{n-1} \\ &= (\lambda-1)^2(\lambda^2-1)^{n-1} \\ &= (\lambda-1)^2(\lambda+1)^{n-1}(\lambda-1)^{n-1} \end{aligned}$$

Hence the partial complement spectrum of LR_n is

$$\begin{pmatrix} 1 & -1 \\ n+1 & n-1 \end{pmatrix},$$

Therefore $E_P(LR_n \oplus S) = 2n$. ■

III. Concluding remarks

An atomic operation is the change of adjacency required for modifying an input graph into the desired graph class G, and it is one of the main focus of partial complementation. Many problems in graph theory come under this graph modification category. Graph energy has so many application in the field of chemistry, physics and mathematics. We have obtained partial complement energy and partial complement characteristic polynomial of a few classes of the graph.

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