Abelian Subgroups Based on $(3,2)$-Fuzzy Sets

A. Iampan, N. Rajesh and S. Shanthi

***Abstract*—In this article, we apply the concept of $(3,2)$-fuzzy sets to introduce and study the concepts of $(3,2)$-fuzzy abelian subgroups ($(3,2)$-FASs), $(3,2)$-fuzzy normal subgroups ($(3,2)$-FNSs) and $(3,2)$-fuzzy cyclic subgroups ($(3,2)$-FCSs). We study those concepts in terms of the Cartesian product of $(3,2)$-fuzzy sets. Finally, homomorphic images and preimages of $(3,2)$-fuzzy sets are established.**

***Index Terms*—$(3,2)$-fuzzy set, $(3,2)$-fuzzy group, $(3,2)$-fuzzy abelian subgroup, $(3,2)$-fuzzy cyclic subgroup.**

I. Introduction

ZADEH [15] first developed the notion of fuzzy sets. The concept of fuzzy sets has numerous real-world applications, and many scholars have researched it. Following the introduction of the concept of fuzzy sets, several research works on the generalizations of fuzzy sets were completed. In [1], [3], [4], [12], [16], [17], [18], the merging of fuzzy sets with various uncertainty techniques, like as soft sets and rough sets, has been studied. One of the more relevant expansions of fuzzy sets is the idea of intuitionistic fuzzy sets given by Atanassov [2]. Intuitionistic fuzzy sets are employed in a range of disciplines, such as medical diagnostics, optimization problems, and multi-criteria decision making [5], [6], [7], [10], [14]. Yager [13] presented a new fuzzy set known as a Pythagorean fuzzy set, which is an extension of intuitionistic fuzzy sets. Senapati and Yager [11] introduced Fermatean fuzzy sets and gave essential approaches on the Fermatean fuzzy sets. In [9], the notion of $(3,2)$-fuzzy sets is presented and researched.

In this paper, the notions of $(3,2)$-FASs, $(3,2)$-FNSs, and $(3,2)$-FCSs are proposed, and then their properties and relations are discussed. We study those concepts in terms of the Cartesian product of $(3,2)$-fuzzy sets. Finally, homomorphic images and preimages of $(3,2)$-fuzzy sets are established.

Definition I.1. [9] A $(3,2)$-*fuzzy set* on a nonempty set X is defined as a structure

$$\mathcal{C} := \{\langle x, f(x), g(x)\rangle \mid x \in X\}, \tag{1}$$

where $f : X \to [0,1]$ and $g : X \to [0,1]$ such that $0 \le f(x)^3 + g(x)^2 \le 1$.

Manuscript received January 21, 2022; revised June 12, 2022.

A. Iampan is an Associate Professor of Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand (corresponding author to provide phone: +6654466666 ext. 1792; fax: +6654466664; e-mail: aiyared.ia@up.ac.th).

N. Rajesh is a Lecturer of Department of Mathematics, Rajah Serfoji Government College, Thanjavur-613005, Tamilnadu, India (e-mail: nrajesh_topology@yahoo.co.in).

S. Shanthi is a Lecturer of Department of Mathematics, Arignar Anna Government Arts College, Namakkal-637002, Tamilnadu, India (e-mail: shanthiwini2005@yahoo.co.in).

The notations $f^3(x)$ and $g^2(x)$ are used instead of $f(x)^3$ and $g(x)^2$, respectively, and the $(3,2)$-fuzzy set in (1) is simply indicated by $\mathcal{C} := (X, f, g)$.

II. Main Results

Definition II.1. Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-fuzzy set of a group $\mathcal{G}$. Then A is called a $(3,2)$-fuzzy subgroup ($(3,2)$-FS) of $\mathcal{G}$ if the following conditions hold:

(1) $f_A^3(mn) \ge f_A^3(m) \wedge f_A^3(n)$ and $g_A^2(mn) \le g_A^2(m) \vee g_A^2(n)\ \forall m,n \in \mathcal{G}$,

(2) $f_A^3(m^{-1}) \ge f_A^3(m)$ and $g_A^2(m^{-1}) \le g_A^2(m)\ \forall m \in \mathcal{G}$.

Equivalently, a $(3,2)$-fuzzy set $A = (\mathcal{G}, f_A, g_A)$ of $\mathcal{G}$ is a $(3,2)$-FS of $\mathcal{G}$ if and only if $f_A^3(mn^{-1}) \ge f_A^3(m) \wedge f_A^3(n)$ and $g_A^2(mn^{-1}) \le g_A^2(m) \vee g_A^2(n)\ \forall m,n \in \mathcal{G}$.

Definition II.2. Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FS of a group $\mathcal{G}$. Let $N(A) = \{a \in \mathcal{G} : f_A^3(a^{-1}xa) = f_A^3(x), g_A^2(a^{-1}xa) = g_A^2(x) \forall x \in \mathcal{G}\}$. Then $N(A)$ is called the $(3,2)$-fuzzy normalizer of A in $\mathcal{G}$.

Definition II.3. A $(3,2)$-FS $A = (\mathcal{G}, f_A, g_A)$ of a group $\mathcal{G}$ is called a $(3,2)$-fuzzy normal subgroup ($(3,2)$-FNS) of $\mathcal{G}$ if $f_A^3(mn) = f_A^3(nm)$ and $g_A^2(mn) = g_A^2(nm)$ $\forall m,n \in \mathcal{G}$. Equivalently, a $(3,2)$-FS $A = (\mathcal{G}, f_A, g_A)$ of $\mathcal{G}$ is a $(3,2)$-FNS of $\mathcal{G}$ if and only if $f_A^3(n^{-1}mn) = f_A^3(m)$ and $g_A^2(n^{-1}mn) = g_A^2(m)\ \forall m,n \in \mathcal{G}$.

Theorem II.4. *Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FS of a group $\mathcal{G}$. Then we have the followings.*

(1) $N(A)$ is a subgroup of $\mathcal{G}$.

(2) A is a $(3,2)$-FNS of $\mathcal{G}$ if and only if $N(A) = \mathcal{G}$.

(3) A is a $(3,2)$-FNS of $N(A)$.

Proof: (1) Let $a, b \in N(A)$. Then

(1) $f_A^3(a^{-1}xa) = f_A^3(x)$ and $g_A^2(a^{-1}xa) = f_A^3(x)\ \forall x \in \mathcal{G}$,

(2) $f_A^3(b^{-1}xb) = f_A^3(y)$ and $g_A^2(b^{-1}yb) = g_A^2(y)\ \forall y \in \mathcal{G}$.

Put $y = a^{-1}xa$ in (2) and using (1), we get $f_A^3(b^{-1}a^{-1}xab) = f_A^3(a^{-1}xa) = f_A^3(x)$ and $g_A^2(b^{-1}a^{-1}xab) = g_A^2(a^{-1}xa) = g_A^2(x)$, that is, $f_A^3((ab)^{-1}x(ab)) = f_A^3(x)$ and $g_A^2((ab)^{-1}x(ab)) = g_A^2(x)$. Thus $ab \in N(A)$. Next, change x to x^{-1} in (1), we get $f_A^3(a^{-1}x^{-1}a) = f_A^3(x^{-1}) = f_A^3(x)$ and $g_A^2(a^{-1}x^{-1}a) = g_A^2(x^{-1}) = g_A^2(x)$, that is, $f_A^3((axa^{-1})^{-1}) = f_A^3(axa^{-1}) = f_A^3(x)$ and $g_A^2((axa^{-1})^{-1}) = g_A^2(axa^{-1}) = g_A^2(x)$, that is, $f_A^3((a^{-1})^{-1}x(a^{-1})) = f_A^3(x)$ and $g_A^2((a^{-1})^{-1}x(a^{-1})) = g_A^2(x) \Rightarrow a^{-1} \in N(A)$. Hence $N(A)$ is a subgroup of $\mathcal{G}$.

(2) Obviously, when $N(A) = \mathcal{G}$, then $f_A^3(a^{-1}xa) = f_A^3(x)$ and $g_A^2(a^{-1}xa) = g_A^2(x)\ \forall x, a \in \mathcal{G}$. Hence A is a $(3,2)$-FNS of $\mathcal{G}$.

Conversely, when A is a $(3,2)$-FNS of $\mathcal{G}$, then $f_A^3(a^{-1}xa) = f_A^3(x)$ and $g_A^2(a^{-1}xa) = g_A^2(x)\ \forall x, a \in \mathcal{G}$, that is, the set $\{a \in \mathcal{G} : f_A^3(a^{-1}xa) = f_A^3(x)$ and $g_A^2(a^{-1}xa) = g_A^2(x) \forall x \in \mathcal{G}\} = \mathcal{G}$, that is, $N(A) = \mathcal{G}$.

(3) Let $a, b \in N(A)$. Then $f_A^3(a^{-1}xa) = f_A^3(x)$ and $g_A^2(a^{-1}xa) = g_A^2(x)\ \forall x \in \mathcal{G}$. Put $x = ab$, we get $f_A^3(ab) = f_A^3(a^{-1}aba) = f_A^3(ba)$ and $g_A^2(ab) = g_A^2(a^{-1}aba) = g_A^2(ba)$. Hence A is a $(3,2)$-FNS of $N(A)$. ∎

Definition II.5. Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FS of a group $\mathcal{G}$. Let $C(A) = \{a \in \mathcal{G} : f_A^3([a,x]) = f_A^3(e)$ and $g_A^2([a,x]) = g_A^2(e) \forall x \in \mathcal{G}\}$. Then $C(A)$ is called the $(3,2)$-fuzzy centralizer of A in $\mathcal{G}$, where $[x,y]$ is the commutator of x and y in $\mathcal{G}$, that is, $[x,y] = x^{-1}y^{-1}xy$.

Theorem II.6. *Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FS of a group $\mathcal{G}$. Then we have the followings.*
(1) $C(A)$ is a subgroup of $\mathcal{G}$.
(2) $C(A)$ is a normal subgroup of $N(A)$.

Proof: (1) Clearly, $C(A) \neq \emptyset$, for $e \in C(A)$. Let $a, b \in C(A)$. Then $f_A^3([a,x]) = f_A^3(e)$ and $g_A^2([a,x]) = g_A^2(e)$ and $f_A^3([b,y]) = f_A^3(e)$ and $g_A^2([b,y]) = g_A^2(e)\ \forall x, y \in \mathcal{G}$, that is,
(1) $f_A^3(a^{-1}x^{-1}ax) = f_A^3(e)$ and $g_A^2(a^{-1}x^{-1}ax) = g_A^2(e)$,
(2) $f_A^3(b^{-1}y^{-1}by) = f_A^3(e)$ and $g_A^2(b^{-1}y^{-1}by) = g_A^2(e)$.
Put $y = a^{-1}za$ in (b), we get $f_A^3(b^{-1}a^{-1}z^{-1}aba^{-1}za) = f_A^3(e)$ and $g_A^2(b^{-1}a^{-1}z^{-1}aba^{-1}za) = g_A^2(e) \Rightarrow f_A^3((ab)^{-1}z^{-1}(ab)z)(z^{-1}a^{-1}za) = f_A^3(e)$ and $g_A^2((ab)^{-1}z^{-1}(ab)z)(z^{-1}a^{-1}za)) = g_A^2(e) \Rightarrow f_A^3((ab)^{-1}z^{-1}(ab)z) = f_A^3(e)$ and $g_A^2((ab)^{-1}z^{-1}(ab)z) = g_A^2(e)$ (using (a)) $\Rightarrow ab \in C(A)$. Also, from (a), we have $f_A^3(e) = f_A^3(a^{-1}x^{-1}ax) = f_A^3((a^{-1}x^{-1}ax)) = f_A^3(x^{-1}a^{-1}xa)$, that is, $f_A^3(x^{-1}a^{-1}xa) = f_A^3(e)$. Similarly, we have $g_A^2(x^{-1}a^{-1}xa) = g_A^2(e)$. Put $x = ta^{-1}$, we get $f_A^3(at^{-1}a^{-1}ta^{-1}a) = f_A^3(at^{-1}a^{-1}t) = f_A^3(e)$ and $g_A^2(at^{-1}a^{-1}ta^{-1}a) = g_A^2(at^{-1}a^{-1}t) = g_A^2(e)$. Hence $a^{-1} \in C(A)$. Hence $C(A)$ is a subgroup of $\mathcal{G}$.

(2) Let $a \in C(A)$ and $b \in N(A)$. Now,
(1) $f_A^3(a^{-1}x^{-1}ax) = f_A^3(e)$ and $g_A^2(a^{-1}x^{-1}ax) = g_A^2(e)$ $\forall x \in \mathcal{G}$,
(2) $f_A^3(b^{-1}yb) = f_A^3(y)$ and $g_A^2(b^{-1}yb) = g_A^2(y)\ \forall y \in \mathcal{G}$.
Put $y = a^{-1}x^{-1}ax$ in (b) and using (a), $f_A^3(b^{-1}a^{-1}x^{-1}axb) = f_A^3(a^{-1}x^{-1}ax) = f_A^3(e)$ and $g_A^2(b^{-1}a^{-1}x^{-1}axb) = g_A^2(a^{-1}x^{-1}ax) = g_A^2(e)$. Again put $x = bzb^{-1}$ above, $f_A^3(b^{-1}a^{-1}bz^{-1}b^{-1}abzb^{-1}b) = f_A^3(e)$ and $g_A^2(b^{-1}a^{-1}bz^{-1}b^{-1}abzb^{-1}b) = g_A^2(e)$, that is, $f_A^3(b^{-1}a^{-1}bz^{-1}b^{-1}abz) = f_A^3(e)$ and $g_A^2(b^{-1}a^{-1}bz^{-1}b^{-1}abz) = g_A^2(e)$, that is, $f_A^3((b^{-1}ab)^{-1}z^{-1}(b^{-1}ab)z) = f_A^3(e)$ and $g_A^2((b^{-1}ab)^{-1}z^{-1}(b^{-1}ab)z) = g_A^2(e)$, that is, $b^{-1}ab \in C(A)$. Hence $C(A)$ is a normal subgroup of $N(A)$. ∎

Proposition II.7. *Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FNS of a group $\mathcal{G}$. Let $N = \{a \in \mathcal{G} : f_A^3(a) = f_A^3(e)$ and $g_A^2(a) = g_A^2(e)\}$. Then $N \subseteq C(A)$.*

Proof: Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FNS of $\mathcal{G}$. Therefore, $f_A^3(y^{-1}xy) = f_A^3(x)$ and $g_A^2(y^{-1}xy) = g_A^2(x)$ $\forall x, y \in \mathcal{G}$. Let $a \in N$. Then $f_A^3(a) = f_A^3(e)$ and $g_A^2(a) = g_A^2(e)$. Now, $f_A^3([ax]) = f_A^3(a^{-1}x^{-1}ax) \geq f_A^3(a^{-1}) \wedge f_A^3(x^{-1}ax) = f_A^3(a) \wedge f_A^3(a) = f_A^3(e) \wedge f_A^3(e) = f_A^3(e)$. Thus $f_A^3([a,x]) = f_A^3(e)$. Similarly, $g_A^2([a,x]) = g_A^2(e)$, that is, $a \in C(A)$. Hence $N \subseteq C(A)$. ∎

Definition II.8. Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FS of a group $\mathcal{G}$. Then $A = (\mathcal{G}, f_A, g_A)$ is called a $(3,2)$-fuzzy abelian subgroup ($(3,2)$-FAS) of $\mathcal{G}$ if $C_{\alpha,\beta}(A)$ is an abelian subgroup of $\mathcal{G}\ \forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1$.

Remark II.9. If $\mathcal{G}$ is an abelian group, then every $(3,2)$-FS of $\mathcal{G}$ is a $(3,2)$-FAS of $\mathcal{G}$.

Theorem II.10. *Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FAS of a group $\mathcal{G}$. Then the set $H = \{a \in \mathcal{G} : f_A^3(ab) = f_A^3(ba)$ and $g_A^2(ab) = g_A^2(ba)\ \forall\ b \in \mathcal{G}\}$ is an abelian subgroup of $\mathcal{G}$.*

Proof: Since $A = (\mathcal{G}, f_A, g_A)$ is a $(3,2)$-FAS of $\mathcal{G}$, $C_{\alpha,\beta}(A)$ is an abelian subgroup of $\mathcal{G}\ \forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1$. Clearly, $H \neq \emptyset$ for $e \in H$. Let $a, b \in H$. Then $f_A^3(ax) = f_A^3(xa)$, $g_A^2(ax) = g_A^2(xa)$ and $f_A^3(ax) = f_A^3(xa)$, $g_A^2(ax) = g_A^2(xa)\ \forall x \in \mathcal{G}$. Now $f_A^3((ab)x) = f_A^3(a(bx)) = f_A^3((bx)a) = f_A^3(b(xa)) = f_A^3((xa)b) = f_A^3(x(ab))$ and $g_A^2((ab)x) = g_A^2(a(bx)) = g_A^2((bx)a) = g_A^2(b(xa)) = g_A^2((xa)b) = g_A^2(x(ab))$ $\forall x \in \mathcal{G}$. Therefore, $ab \in H$. Also, let $a \in H$. Now, $a \in H \Rightarrow f_A^3(ax) = f_A^3(xa)$, $g_A^2(ax) = g_A^2(xa)\ \forall x \in \mathcal{G}$ $(\star)$. Put $x = y^{-1}$ in $(\star)$, we get $f_A^3(ay^{-1}) = f_A^3(y^{-1}a)$, $g_A^2(ay^{-1}) = g_A^2(y^{-1}a)$. Now $f_A^3(a^{-1}y) = f_A^3((a^{-1}y)^{-1}) = f_A^3(y^{-1}a) = f_A^3(ay^{-1}) = f_A^3((ay^{-1})^{-1}) = f_A^3(ya^{-1})$. Similarly, $g_A^2(a^{-1}y) = g_A^2(ya^{-1})\ \forall y \in \mathcal{G}$. Thus $a^{-1} \in H$. So H is a subgroup of $\mathcal{G}$. Let $a, b \in H$. Without loss of generality, let $f_A^3(a) = \alpha, g_A^2(a) \leq 1 - \alpha$ and $f_A^3(b) = \alpha_1, g_A^2(b) \leq 1 - \alpha_1$. Then $a \in C_{\alpha,1-\alpha}(A), b \in C\alpha_1, 1 - \alpha_1(A)$. Let $\alpha < \alpha_1$. Then $f_A^3(b) = \alpha_1 > \alpha$ and $g_A^2(b) \leq 1 - \alpha_1 < 1 - \alpha$ $\Rightarrow b \in C_{\alpha,1-\alpha}(A)$. Thus $a, b \in C_{\alpha,1-\alpha}(A)$ and so $ab = ba$. Hence H is an abelian subgroup of $\mathcal{G}$. ∎

Proposition II.11. *(1) If $A = (\mathcal{G}, f_A, g_A)$ is a $(3,2)$-FAS of a group $\mathcal{G}$, then A is a $(3,2)$-FNS of $\mathcal{G}$.*
(2) The sets H and $C(A)$ are same, that is, $C(A) = H$.

Proof: $C(A) = \{a \in \mathcal{G} : f_A^3([a,x]) = f_A^3(e)$ and $g_A^2([a,x]) = g_A^2(e) \forall\ x \in \mathcal{G}\} = \{a \in \mathcal{G} : f_A^3(a^{-1}x^{-1}ax) = f_A^3(e)$ and $g_A^2(a^{-1}x^{-1}ax) = g_A^2(e),\ \forall\ x \in \mathcal{G}\} = \{a \in \mathcal{G} : f_A^3((xa)^{-1}ax) = f_A^3(e)$ and $g_A^2((xa)^{-1}ax) = g_A^2(e),\ \forall\ x \in \mathcal{G}\} = \{a \in \mathcal{G} : f_A^3(xa) = f_A^3(ax)$ and $g_A^2(xa) = g_A^2(ax) \forall\ x \in \mathcal{G}\} = H$. ∎

Theorem II.12. *Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FAS of $\mathcal{G}$. Then $C(A)$ is an abelian subgroup of $\mathcal{G}$.*

Theorem II.13. *Let $A = (\mathcal{G}_1, f_A, g_A)$ and $B = (\mathcal{G}_2, f_B, g_B)$ be $(3,2)$-FSs of groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then $A \times B$ is a $(3,2)$-FAS of $\mathcal{G}_1 \times \mathcal{G}_2$ if and only if both A and B are $(3,2)$-FASs of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.*

Proof: First, let A and B be $(3,2)$-FASs of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then $C_{\alpha,\beta}(A)$ and $C_{\alpha,\beta}(B)$ are abelian subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively $\forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1 \Rightarrow C_{\alpha,\beta}(A) \times C_{\alpha,\beta}(B)$ is an abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$. But $C_{\alpha,\beta}(A \times B) = C_{\alpha,\beta}(A) \times C_{\alpha,\beta}(B)$ (by Proposition 2.8)). Therefore, $C_{\alpha,\beta}(A \times B)$ is an abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2\ \forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1 \Rightarrow A \times B$ is a $(3,2)$-FAS of $\mathcal{G}_1 \times \mathcal{G}_2$.

Conversely, let $A \times B$ is a $(3,2)$-FAS of $\mathcal{G}_1 \times \mathcal{G}_2$. Then $C_{\alpha,\beta}(A \times B)$ is an abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$, that is, $C_{\alpha,\beta}(A) \times C_{\alpha,\beta}(B)$ is an abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2 \Rightarrow$ $C_{\alpha,\beta}(A)$ and $C_{\alpha,\beta}(B)$ are abelian subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, A and B are $(3,2)$-FASs of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. ∎

Definition II.14. Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FS of a group $\mathcal{G}$. Then A is called a $(3,2)$-fuzzy cyclic subgroup ($(3,2)$-FCS) of $\mathcal{G}$ if $C_{\alpha,\beta}(A)$ is a cyclic subgroup of $\mathcal{G}$ $\forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1$.

Proposition II.15. *If $\mathcal{G}$ is a cyclic group, then every $(3,2)$-FS of $\mathcal{G}$ is a $(3,2)$-FCS of $\mathcal{G}$.*

Proof: Let $\mathcal{G} = \langle x \rangle$ be a cyclic group and let A be any $(3,2)$-FS of $\mathcal{G}$. Then $f_A^3(x^n) \geq f_A^3(x^{n-1}) \geq f_A^3(x^{n-2}) \geq \ldots \geq f_A^3(x^2)$ and $g_A^2(x^n) \leq g_A^2(x^{n-1}) \leq g_A^2(x^{n-2}) \leq \ldots \leq g_A^2(x^2)$ $\forall n \in \mathbb{N}$. Therefore, if $x^m \in C_{\alpha,\beta}(A)$, for some $m \in N$, then $x^m, x^{m+1}, x^{m+2} \ldots \in C_{\alpha,\beta}(A)$, that is, $C_{\alpha,\beta}(A) = \langle x^{-1} \rangle$, which is a cyclic subgroup of $\mathcal{G}$ $\forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1$. Hence A is a $(3,2)$-FCS of $\mathcal{G}$. ∎

Theorem II.16. *Let $h : \mathcal{G}_1 \to \mathcal{G}_2$ be a homomorphism of a group $\mathcal{G}_1$ into a group $\mathcal{G}_2$. Let B be a $(3,2)$-FAS of $\mathcal{G}_2$. Then $h^{-1}(B)$ is a $(3,2)$-FAS of $\mathcal{G}_1$.*

Proof: Let B be a $(3,2)$-FAS of $\mathcal{G}_2$. Therefore, $C_{\alpha,\beta}(B)$ is an abelian subgroup of $\mathcal{G}_2$ $\forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1$. It follows that $C_{\alpha,\beta}(h^{-1}(B)) = h^{-1}(C_{\alpha,\beta}(B)) = \{x \in \mathcal{G}_1 : h(x) \in C_{\alpha,\beta}(B)\}$. Let $x_1, x_2 \in C_{\alpha,\beta}(h^{-1}(B))$. Then $h(x_1), h(x_2) \in C_{\alpha,\beta}(B)$ as $C_{\alpha,\beta}(B)$ is an abelian subgroup of $\mathcal{G}_2$. Therefore, $h(x_1)h(x_2) = h(x_2)h(x_1) \Rightarrow h(x_1x_2) = h(x_2x_1)$ and so $f_B^3(h(x_1x_2)) = f_B^3(hx_1x_2))$ and $g_B^2(h(x_1x_2)) = g_B^2(h(x_2x_1)) \Rightarrow f_{h^{-1}(B)}^3(x_1x_2) = f_{h^{-1}(B)}^3(x_2x_1)$ and $g_{h^{-1}(B)}^2(x_1x_2) = g_{h^{-1}(B)}^2(x_2x_1) \Rightarrow x_1x_2 = x_2x_1$. Thus $C_{\alpha,\beta}(h^{-1}(B))$ is an abelian subgroup of $\mathcal{G}_1$ $\forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1$. Hence $h^{-1}(B)$ is a $(3,2)$-FAS of $\mathcal{G}_1$. ∎

Theorem II.17. *Let $h : \mathcal{G}_1 \to \mathcal{G}_2$ be surjective homomorphism of a group $\mathcal{G}_1$ into a group $\mathcal{G}_2$ and A a $(3,2)$-FAS of $\mathcal{G}_1$. Then $h(A)$ is a $(3,2)$-FAS of $\mathcal{G}_2$.*

Proof: Since A is a $(3,2)$-FAS of $\mathcal{G}_1$, $C_{\alpha,\beta}(A)$ is an abelian subgroup of $\mathcal{G}_1$ $\forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1$. Let $y_1, y_2 \in C_{\alpha,\beta}(h(A))$. Then there exist $x_1, x_2 \in \mathcal{G}_1$ such that $h(x_1) = y_1$, $h(x_2) = y_2$. Then $h(x_1), h(x_2) \in C_{\alpha,\beta}(h(A))$ as $C_{\alpha,\beta}(A)$ is an abelian subgroup of $\mathcal{G}_1$. Therefore, there exists $C_{\delta,\theta}(A)$ such that $x_1, x_2 \in C_{\delta,\theta}(A)$, where $\delta, \theta \in (0,1]$ and $0 < \delta + \theta \leq 1$. But $C_{\alpha,\beta}(A)$ is an abelian group. Therefore, $x_1x_2 = x_2x_1 \Rightarrow h(x_1x_2) = h(x_2x_1) \Rightarrow h(x_1)h(x_2) = h(x_2)h(x_1)$, that is, $y_1y_2 = y_2y_1$. Thus $C_{\alpha,\beta}(h(A))$ is an abelian subgroup of $\mathcal{G}_2$. Hence $h(A)$ is an a $(3,2)$-FAS of $\mathcal{G}_2$. ∎

Theorem II.18. *Let $h : \mathcal{G}_1 \to \mathcal{G}_2$ be a homomorphism of a group $\mathcal{G}_1$ into a group $\mathcal{G}_2$. Let B be a $(3,2)$-FCS of $\mathcal{G}_2$. Then $h^{-1}(B)$ is $(3,2)$-FCS of $\mathcal{G}_1$.*

Proof: Since B is a $(3,2)$-FCS of $\mathcal{G}_2$, $C_{\alpha,\beta}(B)$ is a cyclic subgroup of $\mathcal{G}_2$ $\forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1$. Let $C_{\alpha,\beta}(B) = \langle g_2 \rangle$ for some $g_2 \in \mathcal{G}_2$. Now for $g_2 \in \mathcal{G}_2$, $\exists g_1 \in \mathcal{G}_1$ such that $h(g_1) = g_2$. Thus $C_{\alpha,\beta}(B) = \langle f(g_1) \rangle$. And so $h^{-1}(C_{\alpha,\beta}) = C_{\alpha,\beta}(h^{-1}(B)) = \langle g_1 \rangle$. Hence $h^{-1}(B)$ is a $(3,2)$-FCS of $\mathcal{G}_1$. ∎

Theorem II.19. *Let $h : \mathcal{G}_1 \to \mathcal{G}_2$ be a surjective homomorphism a group $\mathcal{G}_1$ into a group $\mathcal{G}_2$ and A a $(3,2)$-FCS of $\mathcal{G}_1$. Then $h(A)$ is a $(3,2)$-FCS of $\mathcal{G}_2$.*

Proof: Let A be a $(3,2)$-FCS of $\mathcal{G}_1$. Therefore, $C_{\alpha,\beta}(A)$ is a cyclic subgroup of $\mathcal{G}_1$ $\forall \alpha, \beta \in (0,1]$ with $0 < \alpha + \beta \leq 1$. Let $g \in C_{\alpha,\beta}(f(A))$. As h is surjective, therefore let $g = h(g_1)$ for some $g_1 \in \mathcal{G}_1$. As $g_1 \in \mathcal{G}_1$, we can find one $C_{\alpha,\beta}(A)$ which exists $\forall g_1 \in \mathcal{G}_1$ (and hence $\forall g \in C_{\alpha,\beta}(h(A))$ such that $g_1 \in C_{\alpha,\beta}(A)$. But $C_{\alpha,\beta}(A)$ is a cyclic subgroup of $\mathcal{G}_1$. Let $C_{\alpha,\beta}(A) = \langle g_1 \rangle$. So $g_1 = (g)^n$. Thus $g = h(g_1)h((g_1)^n) = (h(g_1))^n$, that is, $C_{\alpha,\beta}(h(A))$ is a cyclic subgroup of $\mathcal{G}_2$. Hence $h(A)$ is a $(3,2)$-FCS of $\mathcal{G}_2$. ∎

Definition II.20. The support of a $(3,2)$-fuzzy set A of X is defined to be

$$\text{supp}_X(A) = \{x \in X : f_A^3(x) > 0 \text{ and } g_A^2(x) < 1\}.$$

Clearly, $\text{supp}_X(A)$ is $\bigcup\{C_{\alpha,\beta}(A) : \forall \alpha, \beta \in (0,1]$ such that $\alpha + \beta \leq 1\}$.

Proposition II.21. *For $f : X \to Y$ and $(3,2)$-fuzzy sets A, B of X and Y, respectively, we have*

(1) $f(\text{supp}_X(A)) \subseteq \text{supp}_Y(f(A))$, equality holds if f is bijective,
(2) $f^{-1}(\text{supp}_Y(B)) = \text{supp}_X(f^{-1}(B))$.

Proposition II.22. *If A is a non-zero $(3,2)$-FS of a group $\mathcal{G}$, the $\text{supp}_G(A)$ is a subgroup of $\mathcal{G}$.*

The following example shows that the converse of Proposition II.22 is not true.

Example II.23. Let $\mathcal{G} = (\mathbb{R}, +)$ be a group of real numbers under addition. Define the $(3,2)$-fuzzy set A on $\mathcal{G}$ by

$$f_A(x) = \begin{cases} 0.31 & \text{if } x = 0 \\ 0.72 & \text{if } x \in \mathbb{Q} - \{0\} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

$$g_A(x) = \begin{cases} 0.51 & \text{if } x = 0 \\ 0.22 & \text{if } x \in \mathbb{Q} - \{0\} \\ 1 & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Clearly, A is not a $(3,2)$-FS of $\mathcal{G}$, but $\text{supp}_G(A) = \mathbb{Q}$ is a subgroup of $\mathcal{G}$.

Proposition II.24. *If A is a $(3,2)$-FNS of a group $\mathcal{G}$, the $\text{supp}_G(A)$ is a normal subgroup of $\mathcal{G}$.*

The following example shows that the converse of Proposition II.24 is not true.

Example II.25. Let $G = S_3 = \{e, a, a^2, b, ab, a^2b\}$, where $b^2 = c = a^3$ be the symmetric group on 3 symbols. Define the $(3,2)$-fuzzy set A on $\mathcal{G}$ by

$$f_A(x) = \begin{cases} 0 & \text{if } x = e \\ \frac{1}{2} & \text{if } x = b \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

$$g_A(x) = \begin{cases} 1 & \text{if } x = e \\ \frac{1}{3} & \text{if } x = b \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Clearly, A is a $(3,2)$-FS of $\mathcal{G}$ and $\text{supp}_G(A) = S_3$ is normal in $\mathcal{G}$. But A is not $(3,2)$-FNS of $\mathcal{G}$, for $C_{(\frac{1}{2},1)} = \{x \in \mathcal{G} : f_A^3(x) \geq \frac{1}{2}$ and $g_A^2(x) \leq 1\} = \{e, b\}$ is not normal in $\mathcal{G}$.

Theorem II.26. *Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-FS of a group $\mathcal{G}$. The A is $(3,2)$-FAS of $\mathcal{G}$ if and only if $\text{supp}_G(A)$ ia an abelian (cyclic) subgroup of $\mathcal{G}$.*

Proof: If $\text{supp}_G(A)$ is an abelian subgroup of $\mathcal{G}$, then the result follows as $C_{\alpha,\beta} \subseteq \text{supp}_G(A)$ for $\alpha, \beta \in (0,1]$ such that $\alpha + \beta < 1$.

Conversely, let A be a $(3,2)$-FAS of $\mathcal{G}$. let $a, b \in \text{supp}_G(A)$. Then $a \in C_{\alpha_1,\beta_1}(A)$ and $b \in C_{\alpha_2,\beta_2}(A)$ for some $\alpha_i, \beta_i \in (0,1]$ such that $\alpha_i + \beta_i < 1$, where $i = 1, 2$.

Case i: When $\alpha_1\alpha_2$ and $\beta_1 > \beta_2$, $a, b \in C_{\alpha_1,\beta_1}(A)$ and $ab = ba$.

Case ii: When $\alpha_1\alpha_2$ and $\beta_1 < \beta_2$, $a, b \in C_{\alpha_1,\beta_2}(A)$ and $ab = ba$ for $\alpha_1 < \alpha_2$ implies $0 < \alpha_1 + \beta_1 \leq 1$.

Other cases can similarly be dealt with. That is, when A is a $(3,2)$-FCS of $\mathcal{G}$, $\text{supp}_G(A)$ is cyclic can be proved on the same lines. ■

Definition II.27. If $A = (\mathcal{G}, f_A, g_A)$ is a $(3,2)$-fuzzy set of $\mathcal{G}$ and H is a subgroup of $\mathcal{G}$, then the restriction of A on H is denoted by $A|H$ is a $(3,2)$-fuzzy set on H defined as $(A|H)(x) = (f_{A|H}(x), g_{A|H}(x))$, where $f_{A|H}(x) = f_A(x)$ and $g_{A|H}(x) = g_A(x)$.

The proof of the following propositions are easy and hence omitted.

Proposition II.28. *Let $A = (\mathcal{G}, f_A, g_A)$ be a $(3,2)$-fuzzy set of $\mathcal{G}$. Then we have the followings.*

(1) If A is a $(3,2)$-FS of $\mathcal{G}$ and H is a subgroup of $\mathcal{G}$, then $A|H$ is also a $(3,2)$-FS of H.
(2) If $A|H$ is the restriction of the $(3,2)$-fuzzy set A of a group $\mathcal{G}$ on the subgroup H of $\mathcal{G}$, then $\text{supp}_H(A|H) = \text{supp}_G(A) \cap H$.
(3) If A is a cyclic $(3,2)$-FS of $\mathcal{G}$ and H is a subgroup of $\mathcal{G}$, then $A|H$ is also a cyclic $(3,2)$-FS of H if and only if H is a cyclic subgroup of $\mathcal{G}$.

Proposition II.29. *If A and B are cyclic $(3,2)$-FSs of $\mathcal{G}$, then $A \times B$ is a cyclic $(3,2)$-FS of $\mathcal{G}$ if and only if both A and B are abelian subgroups of $\mathcal{G}$.*

Proposition II.30. *If A and B are cyclic $(3,2)$-FSs of a group $\mathcal{G}$, then $A \times B$ is a cyclic $(3,2)$-FS of $\mathcal{G}$ if and only if both A and B are cyclic subgroups of $\mathcal{G}$ if and only if A and B are cyclic $(3,2)$-FSs of $\mathcal{G}$ such that $gcd\{|\text{supp}_G(A)|, |\text{supp}_G(B)|\} = 1$.*

Theorem II.31. *Let $h : \mathcal{G} \to \mathcal{G}'$ be a homomorphism of a group $\mathcal{G}$ into a group $\mathcal{G}'$. Then we have the followings.*

(1) If A is a $(3,2)$-FAS of $\mathcal{G}$, then $h(A)$ is a $(3,2)$-FAS of $\mathcal{G}'$.
(2) If A is a $(3,2)$-FCS of $\mathcal{G}$, then $h(A)$ is a $(3,2)$-FCS of $\mathcal{G}'$.
(3) If A' is a $(3,2)$-FAS on $\mathcal{G}'$, then $h^{-1}(A')$ is a $(3,2)$-FAS on $\mathcal{G}$.
(4) If A' is a $(3,2)$-FCS on $\mathcal{G}'$, then $h^{-1}(A')$ is a $(3,2)$-FCS on $\mathcal{G}$.

Definition II.32. Let $A = (f_A, g_A)$ and $B = (f_B, g_B)$ be $(3,2)$-fuzzy sets of X and Y, respectively,. Then A and B are said to be $(3,2)$-fuzzy equipotent sets written $A \approx B$, if there is a bijective map $f : \text{supp}_X(A) \to \text{supp}_Y(B)$ and constants $k_1, k_2 \in R^+$ such that $f_A^3(x) = k_1 f_B^3(f(x))$ and $g_A^2(x) = k_2 g_B^2(f(x))$ $\forall x \in \text{supp}_X(A) \backslash \text{core}_X(A)$.

Definition II.33. Let A and B be $(3,2)$-FSs of groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. An isomorphism $f : \text{supp}_{\mathcal{G}_1}(A) \to \text{supp}_{\mathcal{G}_2}(B)$ is a $(3,2)$-fuzzy isomorphism ($(3,2)$-FI) of A onto B, if there exist constants $k_1, k_2 \in R^+$ such that $f_A^3(x) = k_1 f_B^3(f(x))$ and $g_A^2(x) = k_2 g_B^2(f(x))$ $\forall x \in \text{supp}_X(A) \backslash \text{core}_X(A)$.

We then say that A is isomorphic to B and write it as $A \approx B$.

Proposition II.34. *Let A and B be $(3,2)$-FSs of groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Suppose that $A \approx B$.The for $\alpha_1, \beta_1 \in (0,1]$ with $o < \alpha_1 + \beta_1 < 1$, there exist $\alpha_2, \beta_2 \in (0,1]$ with $0 < \alpha_2 + \beta_2 < 1$ such that $C_{(\alpha_1,\beta_1)}(A) \approx C_{(\alpha_2,\beta_2)}(B)$.*

Proof: Let $f : \text{supp}_{\mathcal{G}_1}(A) \to \text{supp}_{\mathcal{G}_2}(B)$ be a $(3,2)$-FI defined by $f_A^3(x) = k_1 f_B^3(f(x))$ and $g_A^2(x) = k_2 g_B^2(f(x))$ $\forall x \in \text{supp}_X(A) \backslash \text{core}_X(A)$, where $k_1, k_2 \in R^+$ are fixed real numbers such that $k_1 > 1$ and $k_2 < 1$. Define $g : C_{(\alpha_1,\beta_1)}(A) \to C_{(\frac{\alpha_1}{k_1}, \frac{\beta_1}{k_1})}(B)$ by $g = f|C_{(\alpha_1,\beta_1)}(A)$. Then it is easy to verify that g is an isomorphism with required conditions. ■

The proof of the following propositions are easy and hence omitted.

Proposition II.35. *Let A and B be $(3,2)$-FSs of groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Suppose that $A \approx B$. Then A is a $(3,2)$-FAS (resp., $(3,2)$-FCS) of $\mathcal{G}_1$ if and only if B ia a $(3,2)$-FAS (resp., $(3,2)$-FCS) of $\mathcal{G}_2$.*

Proposition II.36. *Let A and B be $(3,2)$-FSs of groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Suppose that $A \approx B$. If A is a $(3,2)$-FNS in $\text{supp}_{\mathcal{G}_1}(A)$, then B ia a $(3,2)$-FNS in $\text{supp}_{\mathcal{G}_1}(B)$.*

III. Conclusion

In this work, we have introduced the notion of $(3,2)$-fuzzy sets and established their properties. We also present certain counterexamples to prove some properties of them. As interesting kinds, we have introduced and studied the concepts of $(3,2)$-FASs and $(3,2)$-FCSs. Finally, homomorphic images and preimages of $(3,2)$-fuzzy sets are established.

References

[1] B. Ahmad and A. Kharal, *On fuzzy soft sets*, Advances in Fuzzy Systems, **2009** (2009), Article ID 586507, 6 pages.
[2] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1) (1986), 87-96.
[3] M. Atef, M. I. Ali and T. M. Al-shami, *Fuzzy soft covering based multi-granulation fuzzy rough sets and their applications*, Computational and Applied Mathematics, **40** (4) (2021), 115.
[4] N. Cağman, S. Enginoğlu and F. Citak, *Fuzzy soft set theory and its application*, Iranian Journal of Fuzzy Systems, **8** (3) (2011), 137-147.
[5] H. Garg and S. Singh, *A novel triangular interval type-2 intuitionistic fuzzy set and their aggregation operators*, Iranian Journal of Fuzzy Systems, **15** (2018), 69-93.
[6] H. Garg and K. Kumar, *An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making*, Soft Computing, **22** (15) (2018), 4959-4970.
[7] H. Garg and K. Kumar, *Distance measures for connection number sets based on set pair analysis and its applications to decision-making process*, Applied Intelligence, **48** (10) (2018), 3346-3359.
[8] A. Hussain, T. Mahmood and M. I. Ali, *Rough Pythagorean fuzzy ideals in semigroups*, Computational and Applied Mathematics, **67** (2019), 1-15.
[9] H. Z. Ibrahim, T. M. Al-shami and O. G. Elbarbary, *$(3,2)$-Fuzzy sets and their applications to topology and optimal choices*, Computational Intelligence and Neuroscience, **2021** (2021), Article ID 1272266, 14 pages.
[10] C. Rotjanasom, C. Inbunleu and P. Suebsan, *Applications of fuzzy parameterized relative soft sets in decision-making problems*, IAENG International Journal of Applied Mathematics, **51** (3) (2021), 607-612.

[11] T. Senapati and R. R. Yager, *Fermatean fuzzy sets*, Journal of Ambient Intelligence and Humanized Computing, **11** (2) (2020), 663-674.
[12] P. Suebsan and M. Siripitukdet, *Applications of fuzzy soft sets over some semigroups based on extended averages*, IAENG International Journal of Computer Science, **47** (4) (2020), 816-822.
[13] R. R. Yager, *Pythagorean fuzzy subsets*, Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), IEEE, Edmonton, Canada, (2013), 57-61.
[14] W. Ye, J. Geng, X. Cui and D. Xu, *A new method for multi-attribute decision-making based on single-valued neutrosophic sets*, Engineering Letters, **28** (4) (2020), 1302-1309.
[15] L. A. Zadeh, *Fuzzy sets*, Information and Control, **18** (3) (1965), 338-353.
[16] K. Zhu, J. Wang and Y. Yang, *A new approach to rough lattices and rough fuzzy lattices based on fuzzy ideals*, IAENG International Journal of Applied Mathematics, **49** (4) (2019), 408-414.
[17] K. Zhu, J. Wang and Y. Yang, *A study on Z-soft fuzzy rough sets in BCI-algebras*, IAENG International Journal of Applied Mathematics, **50** (3) (2020), 577-583.
[18] K. Zhu, J. Wang and Y. Yang, *Lattices of (generalized) fuzzy ideals in residuated lattices*, IAENG International Journal of Applied Mathematics, **50** (3) (2020), 505-511.