Formulated Optimal Solution for EOQ Model with Fuzzy Demand

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Abstract—**This paper aims to prove that when developing an inventory model with fuzzy demand, the order quantity should be identical for each replenishment cycle in order to achieve an optimal replenishment policy. In 2012, Glock, Schwindl and Jaber published a paper in the International Journal Services and Operations Management. They assumed that the order quantities should be identical for each replenishment cycle. We provide a patch work to prove that to attain the minimum solution, then the order quantities must be identical for each replenishment cycle. Moreover, a formulated solution for the optimal replenishment number is derived in this paper. On the other hand, we discuss two related problems in Analytic Hierarchy Process to present some theoretical derivations to help researchers realize the mathematical structure in those problems.**

Index Terms—**Fuzzy demand, Formulated optimal solution, Replenishment number, Economic ordering quantity model, Wash criterion, Rank reversal, Analytic hierarchy process, Decision analysis**

I. Introduction

Glock et al. [1] mentioned their inventory model is limited to order lots of equal sizes. Glock et al. [1] also suggested that in possible future research, practitioners could consider inventory models with unequal order sizes. The purpose of this paper is to further develop their inventory model without the restriction of equal order sizes. Up to now, there are nine papers that have cited Glock et al. [1] in their references. After a careful examination of the related papers, it was found that seven of them: Kumar et al. [2], Al Masud et al. [3], Ghosh et al. [4], Jaaron and Backhouse [5], Shrivastava and Gorantiwar [6], Shekarian et al. [7], and Soni and Joshi [8] only mentioned Glock et al. [1] in their introductions without further discussion with respect to Glock et al. [1]. While the other two papers, Andriolo et al. [9] is a review article analyzing the 219 papers published in the one hundred year development of inventory models. Hence, Andriolo et al. [9] just mentioned Glock et al. [1] in a long publishing list, without providing any improvements for Glock et al. [1]. Kim and Glock [10] only mentioned Glock et al. [1] in the section of possible direction for future study. Recently, there is an important paper of Kazemi et al. [11] to incorporate human learning into a fuzzy EOQ inventory model with backorders. Hence, we can conclude that no researcher provided a solution for the open question proposed by Glock et al. [1] to verify whether or not the optimal solution is found when all order quantities are identical for each replenishment cycle. Therefore the first goal of our paper is to prove that the equal order quantity policy is the best strategy for inventory models under fuzzy environments. Moreover, we provide a formulated optimal solution for their inventory model with fuzzy demand.

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II. Notations and Assumptions

To be compatible with Glock et al. [1], we adopt the same notations and assumptions as follows.

Assumptions

1. Lead time is assumed to be negligible.
2. Shortages are not allowed in this inventory system.
3. The planning horizon is finite. Without loss of generality, we assume that the finite planning horizon as one year.
4. The replenishment is instantaneous, that is, the replenishment rate is infinite.
5. A single manufacturer and a single item are considered in this inventory system.

Notation

$M(n)$ stands for the total cost under fuzzy environment in variable n, with $Q_1 = ... = Q_n = D/n$ assumed by Glock et al. [1]. In this paper, we will prove that $Q_1 = ... = Q_n = D/n$ is the best replenishment strategy.

$M(Q)$ stands for the total cost under fuzzy environment.

$TC(Q)$ stands for the total cost under crisp setting.

Q stands for the order quantity per replenishment cycle.

n stands for the number of orders in the unit time.

h stands for the holding cost per unit item per unit time.

D stands for the demand per unit time.

A stands for the ordering cost per replenishment cycle.

III. Review of the Inventory Model with Fuzzy Demand of Glock et al. [1]

A fuzzy number, $\widetilde{A}$, is denoted by its membership function $\mu_{\widetilde{A}}(x)$ as

$$\mu_{\widetilde{A}}(x) = \begin{cases} A_L(x), & a \le x \le b \\ \omega, & b \le x \le c \\ A_R(x), & c \le x \le d \\ 0, & o.w. \end{cases}, \tag{3.1}$$

where $0 < \omega \le 1$ is a constant, $A_L : [a,b] \to [0,\omega]$ is a non-decreasing and continuous function, and $A_R : [c,d] \to [0,\omega]$ is a non-increasing and continuous function. If $\omega = 1$, then $\widetilde{A}$ is a normal fuzzy number. If $A_L(x)$ and $A_R(x)$ are linear function, then $\widetilde{A}$ is a trapezoidal fuzzy number and is denoted by $\widetilde{A} = (a,b,c,d;\omega)$. In this case,

$$A_L(x) = \omega(x-a)/(b-a), \tag{3.2}$$

and

$$A_R(x) = \omega(d-x)/(d-c). \tag{3.3}$$

In particular, if $b = c$, then the trapezoidal fuzzy number is reduced to a triangular fuzzy number. In this paper, we follow Glock et al. [1] to consider the demand is a normal triangular fuzzy number with $b = c = D$, $b - a = \Delta_1$ and $d - c = \Delta_2$.

For the initial inventory in Glock et al. [1], they considered the classical EOQ model,

$$EOQ(Q) = A\frac{D}{Q} + \frac{Q}{2}h, \tag{3.4}$$

with the crisp demand D.

Remark. In this paper, we use D to express two things: (a) The crisp demand, and (b) The center of a normal triangular fuzzy demand.

Glock et al. [1] extended a crisp demand to a triangular fuzzy number, and denoted as $\widetilde{D}$, where the membership function of $\widetilde{D}$ is expressed as follows,

if $D - \Delta_1 \le x \le D$, then

$$\mu_{\widetilde{D}}(x) = (x - D + \Delta_1)/\Delta_1, \tag{3.5}$$

if $D \le x \le D + \Delta$, then

$$\mu_{\widetilde{D}}(x) = (D + \Delta_2 - x)/\Delta_2, \tag{3.6}$$

otherwise,

$$\mu_{\widetilde{D}}(x) = 0. \tag{3.7}$$

with the center D, the left spread Δ_1 and the right spread Δ_2.

We recall the extension principle proposed by Zadeh [12] and then further discussed by Yager [13], Kaufmann and Gupta [14] and Zimmermann [15].

The mapping f is defined on the powers of $X_1 \times X_2 \times \cdots \times X_n$ to the power sets of Y, as

$$f : P(X_1 \times X_2 \times \cdots \times X_n) \to P(Y), \tag{3.8}$$

where $\widetilde{A}_1$, $\widetilde{A}_2$, …, $\widetilde{A}_n$ are fuzzy sets defined on X_1, X_2, …, X_n, then the image $\widetilde{B} = f(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)$ is a fuzzy set with its membership function, $\mu_{\widetilde{B}}(y)$, as

$$\mu_{\widetilde{B}}(y) = \max_{y = f(x_1, x_2, \ldots, x_n)} \left\{ \min\left[\mu_{\widetilde{A}_1}(x_1), \mu_{\widetilde{A}_2}(x_2), \ldots, \mu_{\widetilde{A}_n}(x_n)\right] \right\}. \tag{3.9}$$

We will apply the extension principle to find the membership function of the fuzzy inventory model proposed by Glock et al. [1], $(A\widetilde{D}/Q) + (Qh/2)$, we denoted it as

$$\widetilde{B} = (A\widetilde{D}/Q) + (Qh/2). \tag{3.10}$$

We have only one membership function, $\mu_{\widetilde{D}}(x)$ and the relation of $y = f(x)$ as

$$f(x) = (AD/x) + (xh/2) \tag{3.11}$$

is one to one such that based on Equation (3.7), we derive the membership function of $\widetilde{B}$ as

$$\mu_{\widetilde{B}}(y) = \mu_{\widetilde{D}}(f^{-1}(y))$$

$$= \mu_{\widetilde{D}}((2y - hQ)Q/2A). \tag{3.12}$$

Based on Equations (3.5-3.7), we obtain the membership function as follows,

if $\Omega - \Delta_3 \le y \le \Omega$, then

$$\mu_{\widetilde{B}}(y) = (y - \Omega + \Delta_3)/\Delta_3, \tag{3.13}$$

if $\Omega \le y \le \Omega + \Delta_4$, then

$$\mu_{\widetilde{B}}(y) = (\Omega + \Delta_4 - y)/\Delta_4, \tag{3.14}$$

otherwise,

$$\mu_{\widetilde{B}}(y) = 0. \tag{3.15}$$

with the center $\Omega = (AD/Q) + (Qh/2)$, left spread $\Delta_3 = A\Delta_1/Q$ and the right spread $\Delta_4 = A\Delta_2/Q$.

We will follow Glock et al. [1] to defuzzy a fuzzy inventory model to a crisp model by centroid method (center of area, center of gravity) proposed by Sugeno [16] and Lee [17] denoted as $M(Q)$, where

$$M(Q) = \int_{-\infty}^{\infty} y\mu_{\widetilde{B}}(y)dy \Big/ \int_{-\infty}^{\infty} \mu_{\widetilde{B}}(y)dy. \tag{3.16}$$

For a normal triangular fuzzy number of Equations (3.13-3.15), we can show that

$$M(Q) = \Omega + \frac{\Delta_4 - \Delta_3}{3}. \tag{3.17}$$

For the numerator, we compute that

$$\int_{\Omega-\Delta_3}^{\Omega} y\frac{y - (\Omega - \Delta_3)}{\Delta_3}dy$$

$$= y\frac{\Delta_3}{2}\left[\frac{y - (\Omega - \Delta_3)}{\Delta_3}\right]^2 - \frac{(\Delta_3)^2}{6}\left[\frac{y - (\Omega - \Delta_3)}{\Delta_3}\right]^3 \Bigg|_{\Omega-\Delta_3}^{\Omega}$$

$$= \Omega\frac{\Delta_3}{2} - \frac{(\Delta_3)^2}{6}, \tag{3.18}$$

and

$$\int_{\Omega}^{\Omega+\Delta_4} y\frac{(\Omega+\Delta_4)-y}{\Delta_4}dy$$
$$= y\frac{-\Delta_4}{2}\left[\frac{(\Omega+\Delta_4)-y}{\Delta_4}\right]^2 - \frac{(\Delta_4)^2}{6}\left[\frac{(\Omega+\Delta_4)-y}{\Delta_4}\right]^3\Bigg|_{\Omega}^{\Omega+\Delta_4}$$
$$= \Omega\frac{\Delta_4}{2} + \frac{(\Delta_4)^2}{6}. \qquad (3.19)$$

For the denominator, we compute that

$$\int_{\Omega-\Delta_3}^{\Omega} \frac{y-(\Omega-\Delta_3)}{\Delta_3}dy$$
$$= \frac{\Delta_3}{2}\left[\frac{y-(\Omega-\Delta_3)}{\Delta_3}\right]^2\Bigg|_{\Omega-\Delta_3}^{\Omega}$$
$$= \frac{\Delta_3}{2}, \qquad (3.20)$$

and

$$\int_{\Omega}^{\Omega+\Delta_4} \frac{(\Omega+\Delta_4)-y}{\Delta_4}dy$$
$$= \frac{-\Delta_4}{2}\left[\frac{(\Omega+\Delta_4)-y}{\Delta_4}\right]^2\Bigg|_{\Omega}^{\Omega+\Delta_4}$$
$$= \frac{\Delta_4}{2}. \qquad (3.21)$$

Hence, we derive that

$$M(Q) = \frac{(\Omega(\Delta_3+\Delta_4)/2) + ((\Delta_4^2-\Delta_3^2)/6)}{(\Delta_3+\Delta_4)/2}$$
$$= \Omega + \left(\frac{\Delta_4-\Delta_3}{3}\right). \qquad (3.22)$$

Based on Equation (3.22), we know that Glock et al. [1] obtained the corrected expression such that for one replenishment cycle, the average cost under a normal triangular fuzzy demand, their centroid is expressed as

$$M(Q) = EOQ(Q) + \frac{A}{3Q}(\Delta_2 - \Delta_1). \qquad (3.23)$$

They considered the finite planning horizon, one year, to convert the above inventory model under equal order sizes with fuzzy demand such that the total cost for one year is expressed as

$$M(n) = An + \frac{Dh}{2n} + \frac{An}{3D}(\Delta_2 - \Delta_1), \qquad (3.24)$$

under the assumption of equal sizes,

$$Q_1 = Q_2 = \ldots = Q_n = \frac{D}{n}. \qquad (3.25)$$

IV. Our proposed inventory model

For the time being, we fixed the number of replenishment n. The order quantity is denoted as Q_1, Q_2, …, Q_n with the restriction

$$Q_1 + Q_2 + \ldots + Q_n = D. \qquad (4.1)$$

Our goal will be to prove

$$Q_1 = Q_2 = \ldots = Q_n, \qquad (4.2)$$

for the optimal replenishment policy.

For the *k*-th replenishment cycle, with order quantity Q_k, the inventory level is

$$I(t) = Q_k - D\left(t - \sum_{i=1}^{k-1}\frac{Q_i}{D}\right), \qquad (4.3)$$

for $\sum_{i=1}^{k-1}\frac{Q_i}{D} \le t \le \sum_{i=1}^{k}\frac{Q_i}{D}$, to imply that

$$\int_{\sum_{i=1}^{k-1}(Q_i/D)}^{\sum_{i=1}^{k}(Q_i/D)} I(t)dt = \frac{1}{2D}Q_k^2. \qquad (4.4)$$

Hence, the holding cost for one year is $\sum_{k=1}^{n}\frac{h}{2D}Q_k^2$.

Motivated by our above derivation of Equation (3.22), the average cost can be rewritten as

$$\frac{A + (hQ^2/2D) + (A(\Delta_2-\Delta_1)/3D)}{Q/D}. \qquad (4.5)$$

Therefore, the total cost of the centroid for the membership function of inventory model with fuzzy demand with order quantity Q_k is expressed as

$$A + \frac{h}{2D}Q_k^2 + \frac{A}{3D}(\Delta_2 - \Delta_1). \qquad (4.6)$$

For the finite planning horizon of one year with n replenishment with order quantity Q_1, Q_2, …, Q_n under the restriction $Q_1 + Q_2 + \ldots + Q_n = D$, the total cost of the centroid for our inventory model with fuzzy demand is expressed as

$$F(Q_1,\ldots,Q_n) = nA + \sum_{k=1}^{n}\frac{h}{2D}Q_k^2 + \frac{nA}{3D}(\Delta_2 - \Delta_1), \qquad (4.7)$$

with $Q_1 + Q_2 + \ldots + Q_n = D$.

In the following, we begin to verify that $Q_1 = \ldots = Q_n$ for the optimal replenishment policy. The Lagrange method is applied to consider $G(Q_1,\ldots,Q_n,\lambda)$ with

$$G(Q_1,\ldots,Q_n,\lambda) = F(Q_1,\ldots,Q_n) + \lambda(Q_1 + Q_2 + \ldots + Q_n - D). \qquad (4.8)$$

We find that

$$\frac{\partial G}{\partial Q_j} = \frac{h}{D}Q_j + \lambda, \qquad (4.9)$$

for $j = 1,\ldots,n$. From Equation (4.9), to solve the system of

$\dfrac{\partial G}{\partial Q_j} = 0$ for $j = 1,\ldots,n$ and $\dfrac{\partial G}{\partial \lambda} = 0$, we know that

$$Q_1 = Q_j, \tag{4.10}$$

for $j = 1,\ldots,n$ and

$$Q_j = \frac{D}{n}, \tag{4.11}$$

for $j = 1,\ldots,n$.

A summary of our results is provided in the following theorem.

Theorem 1. For the finite planning horizon, if the order number is n, we prove that the optimal solution satisfies $Q_1 = Q_2 = \ldots = Q_n = D/n$.

After proving that $Q_1 = Q_2 = \ldots = Q_n = D/n$ is the optimal solution when the replenishment number, n, is given, thus Equation (4.7) can be rewritten as

$$F(n) = nA + \frac{Dh}{2n} + \frac{nA}{3D}(\Delta_2 - \Delta_1). \tag{4.12}$$

In the following, the optimal n^* is solved.

First, we extend the domain from discrete natural numbers $\{1,2,\ldots\}$ to the positive real number $\{x > 0\}$ to convert $F(n)$ to $F(x)$ as

$$F(x) = xA + \frac{Dh}{2x} + \frac{xA}{3D}(\Delta_2 - \Delta_1), \tag{4.13}$$

for $x > 0$.

Based on $F''(x) = \dfrac{Dh}{x^3} > 0$, we know that $F(x)$ is convex up, for $x > 0$ such that the trajectory of $\{(1, F(1)), (2, F(2)), \ldots\}$ is also spotted as a concave up dotted curve. Hence, the optimal replenishment number, n^*, satisfies

$$n^* = \min\{n : F(n) \le F(n+1)\}. \tag{4.14}$$

We compute that

$$F(n+1) - F(n) = \frac{A}{D}\left(D + \frac{\Delta_2 - \Delta_1}{3}\right) - \frac{Dh}{2n(n+1)}. \tag{4.15}$$

Based on Equation (4.15), we obtain that $F(n+1) \ge F(n)$ is equivalent to

$$n(n+1) \ge \Omega, \tag{4.16}$$

where we assume an abbreviation denoted as Ω, with

$$\Omega = \frac{3D^2 h}{2A(3D + \Delta_2 - \Delta_1)}. \tag{4.17}$$

The inequality of (4.16) can be rewritten as

$$n^2 + n + 0.25 \ge \Omega + 0.25 \tag{4.18}$$

to imply that

$$n^* = \left\lceil \sqrt{\Omega + 0.25} - 0.5 \right\rceil \tag{4.19}$$

where $\lceil x \rceil$ is the smallest integer that satisfies $\lceil x \rceil \ge x$. We obtain a theoretical result for the optimal solution.

We summarize our results in the next theorem.

Theorem 2. For inventory models with fuzzy demand, we derive the optimal solution $n^* = \left\lceil \sqrt{\Omega + 0.25} - 0.5 \right\rceil$, where $\Omega = \dfrac{3D^2 h}{2A(3D + \Delta_2 - \Delta_1)}$.

Glock et al. [1] did not offer us any analytical work for the optimal solution for this kind of inventory models. We provide an improvement to find the formulated optimal solution.

V. Numerical Examples

To be compatible with Glock et al. [1], we adopt the same numerical examples with the following data: $A = 75$, $D = 1000$, $h = 5$, $\Delta_1 = 100$ and $\Delta_2 = 500$ for Example 1, and $\Delta_1 = 10$, and $\Delta_2 = 50$ for Example 2.

We list our results in the next table. For comparison with Glock et al. [1], their findings also listed in Table 1.

From Table 1, we know that Glock et al. [1] and our paper both derived the same optimal number of orders. However, for the total cost, we find that there are some typographical errors in their computation.

To support our derivation, we run more detailed computations for Examples 1 and 2 to list them in the next Table 2.

After we know that $TC(n)$ is a convex function in discrete variable n, we observe the tendency of Examples 1 and 2, then the minimum point and minimum value are marked by bold to show that our derivation in Table 1 is the optimal solution.

Moreover, by our Theorem 2, we can directly locate the optimal solution, n*, to illustrate that our formulated optimal solution works for inventory models with fuzzy demand.

VI. A Related Problem

In this section, we will examine a related problem and then provide our improvements. We recall the wash criterion question that have been studied by Lin et al. [18], Saaty and Vargas [19], Liberatore and Nydick [20], and Finan and Hurley [21] to consider whether or not a wash criterion can be removed to simplify the computation manual labor to construct comparison matrices during the analytic hierarchy process. Finan and Hurley [21] pointed out that they provided a strongest challenge for the analytic hierarchy process developed by Saaty [22, 23]. Saaty and Vargas [19] and Liberatore and Nydick [20] have presented responses for this challenge. Jung et al. [24] provided an example to show that working through by the following three various procedures: (i) Lin et al. [18] and Finan and Hurley [21] without wash criterion, (ii) Saaty and Vargas [19] and Liberatore and Nydick [20] without wash criterion, (iii) Finan and Hurley [21] with wash criterion, and then the same relative weights will be derived. Jung et al. [24] claimed that they constructed an invariant phenomenon in the analytic hierarchy process.

VII. Literature Review

Finan and Hurley [21] considered wash criteria to develop several theorems to show that a two-level analytic hierarchy process system with a perfectly consistent corresponding comparison matrix, and then deleting a wash criterion will not influence the ordering for alternatives. On the other hand, Finan and Hurley [21] developed a three-level comparison matrix to show the occurrence of rank reversal after deleting a wash criterion and then they raised a severe question on the well-defined problem of the analytic hierarchy process. Liberatore and Nydick [20] mentioned that the relative weights among alternatives, or alternatives to criteria will be estimated again such that the rank reversal phenomenon will not happen. Saaty and Vargas [19] showed that a wash criterion cannot be arbitrarily canceled. In this section, the debate about deleting a wash criterion will be further examined.

VIII. Our Improvements

Let us recall the solution approach of Finan and Hurley [21]. When deleting the wash criterion, J_0, with weight 0.6, Finan and Hurley [21] considered the weights of J_1 and J_2 as 0.2 and 0.2, then they normalized them to obtain the relative weights of J_1,

$$\frac{0.2}{0.2+0.2}=0.5, \tag{8.1}$$

and of J_2,

$$\frac{0.2}{0.2+0.2}=0.5, \tag{8.2}$$

so after the wash criterion J_0 is deleted, and then the weights of J_1 and J_2 becomes 0.5 and 0.5.

It means that the comparison matrix for J_0, J_1 and J_2 with respect to J is assumed as $\lfloor a_{i,j} \rfloor_{3\times3}$ then the normalized eigenvector will satisfy

$$\begin{bmatrix} 1 & a_{1,2} & a_{1,3} \\ a_{2,1} & 1 & a_{2,3} \\ a_{3,1} & a_{3,2} & 1 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}=\lambda\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}, \tag{8.3}$$

where λ is the maximum eigenvalue corresponding to the matrix of Equation (8.3).

Owing to a_{ij} is defined as the comparison of J_{i-1} with J_{j-1} with respect to criterion J in the upper level, its value will not influence by deleting the wash criterion whether or not. Hence, after deleting the wash criterion J_0, then the first row and the first column of $\lfloor a_{i,j} \rfloor_{3\times3}$ is disappeared. According to every 2 by 2 comparison matrix is consistent, so the eigenvector problem becomes

$$\begin{bmatrix} 1 & a_{2,3} \\ a_{3,2} & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}=2\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \tag{8.4}$$

with $\alpha_1+\alpha_2=1$.

Finan and Hurley [21] did not try to solve Equation (2), instead, they directly based on the eigenvector of Equation (1), to obtain the relative ratio between J_1 and J_2, and then normalized them.

We recall that Finan and Hurley [21] evaluated a comparison matrix $\lfloor b_{i,j} \rfloor_{n+1\times n+1}$ to derive the normalized eigenvector, say $[v_i]_{n+1\times1}$ corresponding to the maximum eigenvalue, and then Finan and Hurley [21] deleted the first row and the first column of $\lfloor b_{i,j} \rfloor_{n+1\times n+1}$ to construct a matrix $\lfloor c_{i,j} \rfloor_{n\times n}$ with $c_{i,j}=b_{i+1,j+1}$ for $1\le i\le n$ and $1\le j\le n$. In their Proposition 1, they claimed that the normalized eigenvector, corresponding to the maximum eigenvalue will be

$$\left[\frac{v_{i-1}}{1-v_1}\right]_{n\times1}. \tag{8.5}$$

From the above discussion, we may say that Finan and Hurley [21] approach is reasonable that is verified by Proposition 1 of Finan and Hurley [21].

However, in Lin et al. [18], they have pointed out that Proposition 1 contained questionable results. When $\lfloor b_{i,j} \rfloor_{n+1\times n+1}$ is a perfect consistent matrix then their claim assertion is corrected. For the general case, their assertion is false.

Based on our previous discussion, we need to consider the following question:

$$\begin{bmatrix} 1 & a_{1,2} & a_{1,3} \\ a_{2,1} & 1 & a_{2,3} \\ a_{3,1} & a_{3,2} & 1 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}=\lambda\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}, \tag{8.6}$$

with $a_{1,2},a_{1,3},a_{2,3}\in\left\{1,2,\ldots,9,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{9}\right\}$ the 1-9 scale proposed by Saaty [22], under the restrictions of $a_{2,1}=\frac{1}{a_{1,2}}$, $a_{3,1}=\frac{1}{a_{1,3}}$, $a_{3,2}=\frac{1}{a_{2,3}}$ and λ is the maximum eigenvalue.

We find that there are three possible solutions as follows and their corresponding reduced 2 by 2 comparison matrix with its maximum eigenvector:

$$\begin{bmatrix} 1 & 1 & 9 \\ 1 & 1 & 1/3 \\ 1/9 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix} = \frac{13}{3}\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}, \quad (8.7)$$

$$\begin{bmatrix} 1 & \frac{1}{3} \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = 2\begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}, \quad (8.8)$$

$$\begin{bmatrix} 1 & 9 & 1 \\ 1/9 & 1 & 3 \\ 1 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix} = \frac{13}{3}\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}, \quad (8.9)$$

$$\begin{bmatrix} 1 & 3 \\ 1/3 & 1 \end{bmatrix}\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = 2\begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}, \quad (8.10)$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 1/3 & 1 & 1 \\ 1/3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix} = 3\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}, \quad (8.11)$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = 2\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}. \quad (8.12)$$

Based on the above computations, we show that there are three possible solutions for the relative weights for J_1 and J_2 as $J_1 : J_2 = 1:3$, $3:1$, or $1:1$, it is actually derived from the 2 by 2 pairwise comparison matrix elements that also satisfy the 1-9 scale proposed by Saaty [22]. We recall that Finan and Hurley [21] approach is directly derived from the relative weights from the 3 by 3 pairwise comparison matrix.

Our findings reveal that the solution procedure of Finan and Hurley [21] to claim the solution must be $1:1$ which is only a partial solution of the all three possible answers.

Therefore, we verify that the solution process proposed by Finan and Hurley [21] is not complete such that their procedure sometimes cannot obtain the corrected relative weights.

IX. A Consistent Problem with Wash Criterion in Analytic Hierarchy Process

In this section, we will discuss a famous problem in Analytic Hierarchy Process of wash criterion. This problem had been studied by Finan and Hurley [21], Liberatore and Nydick [20], Saaty and Vargas [19], Lin et al. [18], Jung et al. [24], and Ke et al. [25].

We will provide a theoretical explanation why their counterexample cannot be constructed.

X. Their Proposed Counterexample

We consider the hierarchy structure studied by Liberatore and Nydick [20] and Saaty and Vargas [19] with the following relative weights where the criterion, "J_{11}" is a wash criterion.

We compute the synthesized weight of alternative A_1 as follows. We know that

$$w(A_1) = w\left(\frac{A_1}{J_1}\right) w(J_1) + w\left(\frac{A_1}{J_2}\right) w(J_2), \quad (10.1)$$

$$w\left(\frac{A_1}{J_1}\right) = w\left(\frac{A_1}{J_{11}}\right) w\left(\frac{J_{11}}{J_1}\right) + w\left(\frac{A_1}{J_{12}}\right) w\left(\frac{J_{12}}{J_1}\right) + w\left(\frac{A_1}{J_{13}}\right) w\left(\frac{J_{13}}{J_1}\right), \quad (10.2)$$

and

$$w\left(\frac{A_1}{J_2}\right) = w\left(\frac{A_1}{J_{21}}\right) w\left(\frac{J_{21}}{J_2}\right) + w\left(\frac{A_1}{J_{22}}\right) w\left(\frac{J_{22}}{J_2}\right). \quad (10.3)$$

Liberatore and Nydick [20] claimed that "criteria J_1" lost J_{11} with 60% weight, then the relative weight between J_1 and J_2 should be changed. However, Liberatore and Nydick [20] did not inform readers how to modify relatives.

Based on Equations (10.1-10.3), we derive that

$$w(A_1) = 0.477. \quad (10.4)$$

Similarly, we also know that

$$w(A_2) = w\left(\frac{A_2}{J_1}\right) w(J_1) + w\left(\frac{A_2}{J_2}\right) w(J_2), \quad (10.5)$$

$$w\left(\frac{A_2}{J_1}\right) = w\left(\frac{A_2}{J_{11}}\right) w\left(\frac{J_{11}}{J_1}\right) + w\left(\frac{A_2}{J_{12}}\right) w\left(\frac{J_{12}}{J_1}\right) + w\left(\frac{A_2}{J_{13}}\right) w\left(\frac{J_{13}}{J_1}\right), \quad (10.6)$$

and

$$w\left(\frac{A_2}{J_2}\right) = w\left(\frac{A_2}{J_{21}}\right) w\left(\frac{J_{21}}{J_2}\right) + w\left(\frac{A_2}{J_{22}}\right) w\left(\frac{J_{22}}{J_2}\right). \quad (10.7)$$

Based on Equations (10.5-10.7), we derive that

$$w(A_2) = 0.523. \quad (10.8)$$

Based on our findings of Equations (10.4) and (10.8), it follows that the restriction of

$$w(A_1) + w(A_2) = 1, \quad (10.9)$$

is satisfied.

Our goal is to find parameters: a, b, c, d, e, f, and g in the unit interval such that the relative ratio between alternatives A_1 and A_2 are altered.

First, with the wash criterion, J_{11}, we compute the synthesized weight of the alternative A_1, and the alternative A_2, then

$$w(A_1) = \frac{ab}{2} + ace + a(1-b-c)f + (1-a)dg + (1-a)(1-d)h, \quad (10.10)$$

and

$$w(A_2) = \frac{ab}{2} + ac(1-e) + a(1-b-c)(1-f) + (1-a)d(1-g) + (1-a)(1-d)(1-h), \quad (10.11)$$

Next, without the wash criterion, J_{11}, we compute the synthesized weight of the alternative A_1, and the alternative A_2, then

$$\overline{w}(A_1) = ace + a(1-c)f + (1-a)dg + (1-a)(1-d)h, \quad (10.12)$$

and

$$\overline{w}(A_2) = ac(1-e) + a(1-c)(1-f) + (1-a)d(1-g) + (1-a)(1-d)(1-h). \quad (10.13)$$

Our purpose is find parameters: a, b, c, d, e, f, and g such that

$$[w(A_1) - w(A_2)][\overline{w}(A_1) - \overline{w}(A_2)] < 0. \quad (10.14)$$

The inequality of Equation (10.14) indicates the rank of A_1 and A_2 are altered, after the wash criterion is removed.

Based on our results of Equations (10.10-10.13), we can simplify the condition of Equation (10.14) as follows,

$$[\Delta - ab(2f-1)]\Delta < 0, \tag{10.15}$$

where Δ is an abbreviation to simplify the expression, with

$$\Delta = ac(2e-1) + a(1-c)(2f-1) + (1-a)d(2g-1) + (1-a)(1-d)(2h-1). \tag{10.16}$$

Based on Equation (10.15), we list the following conditions:

$$2e - 1 > 0, \tag{10.17}$$

$$2f - 1 > 0, \tag{10.18}$$

$$2g - 1 > 0, \tag{10.19}$$

$$2h - 1 > 0, \tag{10.20}$$

and

$$\Delta < ab(2f-1). \tag{10.21}$$

Motivated by conditions of (10.17-10.21), we select

$$e = 0.6 = f = g = h, \tag{10.22}$$

to derive that

$$\Delta = 0.2. \tag{10.23}$$

We know that conditions of (10.17-10.20) are satisfied, and we simplify Equation (10.21) in the following,

$$0.2 < ab(0.2). \tag{10.23}$$

For later discussion, we further simplify Equation (10.23) as follows,

$$1 < ab. \tag{10.24}$$

For a reasonable hierarchy structure, we know that

$$0 \le a \le 1, \tag{10.25}$$

and

$$0 \le b \le 1. \tag{10.26}$$

Based on Equations (10.25) and (10.26), we know that the inequality in Equation (10.24) cannot find example.

Based on our above detailed examination for the example proposed by Liberatore and Nydick [20], we demonstrate the assertion of Liberatore and Nydick [20] may contain questionable results that deserves further investigation.

XI. Directions for Future Research

We reference several related articles to assist researchers in identifying potential directions for future research. Belton and Gear [26] and Saaty [27, 28] are mentioned. Belton and Gear [26] present a significant challenge to the analytic hierarchy process by constructing an example that results in rank reversal among alternatives when an additional alternative is introduced. Saaty [27] is a well-known textbook that introduces the analytic hierarchy process to researchers. Saaty [28] provides explanations for some of the challenges associated with the rank reversal problem that occurs when a new alternative is added during the decision-making process. Additionally, there are several recently published papers that are also of great importance. We cite four papers: Yen [29, 30], Yang and Chen [31], and Wang and Chen [32]. Yen [29] explores intuitive algebraic methods for solving inventory models with two back-ordered costs and poses an open question regarding the solution procedure of Chang and Schonfeld [30]. Yen [31] first points out questionable results in Çalışkan [32, 33] and subsequently demonstrates that the challenge proposed by Çalışkan [34] to Wee et al. [35] is invalid. Finally, Yen [31] provides a patchwork solution for Minner [36]. Yang and Chen [37] address the open question raised by Yen [29] and Osler [38] independently. Furthermore, Yang and Chen [37] identify questionable findings in Çalışkan [39, 40]. Wang and Chen [41] make improvements to Aguaron and Moreno-Jimenez [42], rendering their stability interval inadequate for solving the inconsistency problems in the analytic hierarchy process. Wang and Chen [41] also demonstrate that the intuitive approach developed by Yen [29] represents only one possible selection among various choices. We also refer to the works of Nurcahayani et al. [43], Em [44], Ren et al. [45], and Che et al. [46]. Nurcahayani et al. [43] study combined estimators with nonparametric regression bi-response functions. Em [44] derives a sufficient criterion using multiple reaction and diffusion models. Ren et al. [45] develop a new iterative decoding algorithm for two-dimensional problems, and Che et al. [46] examine a novel entropy system for estimating wind velocity within short time periods. Finally, we quote Thangkhenpau and Panday [47], Hwang et al. [48], Zhao et al. [49], and Chanmanee et al. [50]. Thangkhenpau and Panday [47] obtain the optimal solution for nonlinear systems with an eight-dimensional family without derivatives. Hwang et al. [48] discover analytical findings for multi-condition operational systems. Zhao et al. [49] apply a discrete approach with naked mole rats to derive a locally optimal solution for the traveling salesman problem. Chanmanee et al. [50] study UP algebra to uncover new findings for internal direct multiplications. The articles mentioned above serve to inspire researchers and shed light on the current trends in hop research.

XII. Conclusion

For inventory models under fuzzy environment, it is proved that the equal order quantity for each replenishment cycle is the optimal strategy. Meanwhile, we derived the formulated optimal solution for the number of order that will help researchers locate the optimal solution. Moreover, we examine two problems in Analytic Hierarchy Process. First, we provide a reason why Finan and Hurley [21] only consider one-third of possible results. Second, we provide a theoretical development to explain why Liberatore and Nydick [20] cannot construct a counterexample to demonstrate that after deleting a wash criterion, the ranking of alternatives will be altered. Our findings will help researchers to deal operational research problems with analytical approach under a solid mathematical background.

TABLE I

THE COMPARISONS BETWEEN OUR DERIVATIONS WITH THEIRS OF GLOCK ET AL. [1]

	Findings of Glock et al. [1]		Our results	
	Ex. 1	Ex. 2	Ex. 1	Ex. 2
n_{opt}	5	6	5	6
Q_{opt}	200	166.67	200	166.67
$TC(n^*)$	885.00	867.67	925	872.67

Table II
Detailed computations for Examples 1 and 2

Ex. 1									
n	1	2	3	4	**5**	6	7	8	9
$TC(n)$	2585	1420	1088	965	**925**	926.7	952.1	992.5	1043
Ex. 2									
n	1	2	3	4	5	**6**	7	8	9
$TC(n)$	2576	1402	1061	929	880	**872.7**	889.1	920.5	961.8

Table III.
Our proposed hierarchy with relative weights among criteria and alternatives

	J_1				J_2	
	a				1-a	
	J_{11}	J_{12}	J_{13}	J_{21}	J_{22}	
J_1	b	c	1-b-c	d	1-d	J_2
	J_{11}	J_{12}	J_{13}	J_{21}	J_{22}	
A_1	0.5	e	f	g	h	A_1
A_2	0.5	1-e	1-f	1-g	1-h	A_2

REFERENCES

[1] C. H. Glock, K. Schwindl, M. Y. Jaber, "An EOQ model with fuzzy demand and learning in fuzziness," *International Journal Services and Operations Management,* vol. 12, no. 1, 2012, pp. 90-100.
[2] N. Kumar, S. R. Singh, R. Kumari, "Learning effect on an inventory model with two-level storage and partial backlogging under inflation," *International Journal of Services and Operations Management,* vol. 16, no. 1, 2013, pp. 105-122.
[3] M. A. Al. Masud, S. K. Paul, A. Azeem, "Optimization of a production inventory model with reliability considerations," *International Journal of Logistics Systems and Management,* vol. 17, no. 1, 2014, pp. 22-45.
[4] S. Ghosh, R. Piplani, S. Viswanathan, "Optimal inventory rationing policy under service differentiation for multiple customer classes with deterministic demand," *International Journal of Services and Operations Management,* vol. 18, no. 1, 2014, pp. 38-56.
[5] A. Jaaron, C. J. Backhouse, "Building antifragility in service organizations: Going beyond resilience," *International Journal of Services and Operations Management,* vol. 19, no. 4, 2014, pp. 491-513.
[6] R. L. Shrivastava, V. S. Gorantiwar, "An instrument for identification of critical factors for quality and productivity improvement in sponge iron industry," *International Journal of Services and Operations Management,* vol. 19, no. 3, 2014, pp. 377-412.
[7] E. Shekarian, C. H. Glock, S. M. P. Amiri, K. Schwindl, "Optimal manufacturing lot size for a single-stage production system with rework in a fuzzy environment," *Journal of Intelligent and Fuzzy Systems,* vol. 27, no. 6, 2014, pp. 3067-3080.
[8] H. N. Soni, M. Joshi, "EOQ model with backorders under linear combination of possibility and necessity measure," *International Journal of Mathematics in Operational Research,* vol. 7, no. 2, 2015, pp. 194-214.
[9] A. Andriolo, D. Battini, R. Grubbström, W. A. Persona, F. Sgarbossa, "A century of evolution from Harris's basic lot size model: Survey and research agenda," *International Journal of Production Economics,* vol. 155, 2014, pp. 16-38.
[10] T. Kim, C. H. Glock, "On the use of RFID in the management of reusable containers in closed-loop supply chains under stochastic container return quantities," *Transportation Research Part E: Logistics and Transportation Review,* vol. 64, 2014, pp. 12-27.
[11] N. Kazemi, E. Shekarian, L. E. Cárdenas-Barrón, E. U. Olugu, "Incorporating human learning into a fuzzy EOQ inventory model with backorders," *Computers and Industrial Engineering,* vol. 87, 2015, pp. 540-542.
[12] L. Zadeh, ‘‘The concept of a linguistic variable and its application to approximate reasoning, Part I,’’ *Information Sciences*, vol. 8, 1975, pp. 199–249.
[13] R. R. Yager, ‘‘A characterization of the extension principle,’’ *Fuzzy Sets Systems,* vol. 18, 1986, pp. 205–217.
[14] A. Kaufmann, M. M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Applications.* New York: Van Nostrand Reinhold, 1991.
[15] H. J. Zimmermann, *Fuzzy Set Theory and its Application.* 3rd edition, Dordrecht, The Netherlands: Kluwer Academic Publishers, 1996.
[16] M. Sugeno, ‘‘An introductory survey of fuzzy control,’’ *Information Sciences*, vol. 36, 1985, pp. 59–83.
[17] C. Lee, ‘‘Fuzzy logic in control systems: fuzzy logic controller, Parts I and II,’’ *IEEE Transactions on Systems, Man, and Cybernetics. Part A, Systems and Humans,* vol. 20, 1990, pp. 404–435.
[18] J. S. J. Lin, S. Y. Chou, W. T. Chouhuang, C. P. Hsu, "Note on “wash criterion in analytic hierarchy process”," *European Journal of Operations Research*, vol. 185, 2008, pp. 444-447.
[19] T. L. Saaty, L. G. Vargas, "The analytic hierarchy process: wash criteria should not be ignored," *International Journal of Management and Decision Making*, vol. 7, 2006, pp. 180-188.
[20] M. J. Liberatore, R. L. Nydick, "Wash criteria and the analytic hierarchy process," *Computers and Operations Research*, vol. 31, 2004, pp. 889-892.
[21] J. S. Finan, W. J. Hurley, "The analytic hierarchy process: can wash criteria be ignored?," *Computers and Operations Research*, vol. 29, 2002, pp. 1025-1030.
[22] T. L. Saaty, "A scaling method for priorities in hierarchical structures," *Journal of Mathematical Psychology*, vol. 15, 1977, pp. 234–281.
[23] T. L. Saaty, *The Analytic Hierarchy Process*, McGraw-Hill, New York, 1980.
[24] S. T. Jung, Y. W. Wou, S. P. Li, P. Julian, "A Revisit to Wash Criteria in Analytic Hierarchy Process," *Far East Journal of Mathematical Sciences*, vol. 34, no. 1, 2009, pp. 31-36.
[25] H. J. Ke, C. T. Tung, P. Deng, "Rank reversal problem related to wash criterion in analytic hierarchy process (AHP)," *African Journal of Business Management*, vol. 5, no. 20, 2011, pp. 8301-8306.
[26] V. Belton, A. E. Gear, "On a shortcoming of Saaty’s method of analytic hierarchy," *Omega*, vol. 11, 1983, pp. 227–230.
[27] T. L. Saaty, *Decision Making for Leaders: The Analytical Hierarchy Process for Decisions in a Complex World*, third ed. RWS Publications, Pittsburgh, PA. 1995.
[28] T. L. Saaty, "Rank generation preservation and reversal in the analytic hierarchy decision process," *Decision Sciences*, vol. 18, 1987, pp. 157–177.
[29] C. P. Yen, "Further Study for Inventory Models with Compounding," *IAENG International Journal of Applied Mathematics,* vol. 52, no. 3, 2022, pp. 684-691.
[30] S. K. Chang, P. M. Schonfeld, “Multiple period optimization of bus systems,” *Transportation Research Part B*, vol. 25, no. 6, 1991, pp. 453-478.
[31] C. P. Yen, "Solving Inventory Models by the Intuitive Algebraic Method," *IAENG International Journal of Applied Mathematics,* vol. 51, no. 2, 2021, pp. 341-345.
[32] C. Çalışkan, "On the Economic Order Quantity Model with Compounding," *American Journal of Mathematical and Management Sciences*, 2020, DOI: 10.1080/01966324.2020.1847224.
[33] C. Çalışkan, "A Simple Derivation of the Optimal Solution for the EOQ Model for Deteriorating Items with Planned Backorders," *Applied Mathematical Modelling*, vol. 89,2021, pp. 1373-1381.

[34] C. Çalışkan, "A note on "A Modified Method to Compute Economic Order Quantities without Derivatives by Cost-Difference Comparisons"", *Journal of Statistics and Management Systems*, 2021, DOI: 10.1080/09720510.2020.1859809.
[35] H. M. Wee, W. T. Wang, C. J. Chung, "A Modified Method to Compute Economic Order Quantities without Derivatives by Cost-Difference Comparisons," *European Journal of Operational Research*, vol. 194, 2009, pp. 336-338.
[36] S. Minner, "A Note on how to Compute Economic Order Quantities without Derivatives by Cost Comparisons," *International Journal of Production Economics*, vol. 105, 2007, 293–296.
[37] T. C. Yang, Y. C. Chen, "Solution for Open Questions in Yen (2021) Osler (2001) and Caliskan (2020, 2022)," *IAENG International Journal of Applied Mathematics*, vol. 53, no.1, 2023, pp48-57.
[38] T.J. Osler, "Cardan polynomials and the reduction of radicals," *Mathematics Magazine*, vol. 47, no. 1, 2001, pp. 26–32.
[39] C. Çalışkan, "A derivation of the optimal solution for exponentially deteriorating items without derivatives," *Computers & Industrial Engineering*, vol. 148,2020, 106675.
[40] C. Çalışkan, "Derivation of the optimal solution for the economic production quantity model with planned shortages without derivatives," *Modelling*, vol. 2022, no. 3, 2022, pp. 54–69.
[41] Y. L. Wang, M. L. Chen, "Study for Local Stability Intervals in Analytic Hierarchy Process," *IAENG International Journal of Applied Mathematics*, vol. 53, no.1, 2023, pp194-201.
[42] J. Aguaron, J. M. Moreno-Jimenez, "Local stability intervals in the analytic hierarchy process," *European Journal of Operational Research*, vol. 125, 2000, pp. 113-132.
[43] H. Nurcahayani, I. N. Budiantara, I. Zain, "The New Estimation of Bi-response Nonparametric Regression Curve with Combined Estimators," *Engineering Letters*, vol. 31, no.1, 2023, pp. 435-443.
[44] P. V. L. Em, "Sufficient Condition for Synchronization in Complete Networks of n Reaction-Diffusion Systems of Hindmarsh-Rose Type with Nonlinear Coupling," Engineering Letters, vol. 31, no.1, , 2023, pp. 413-418.
[45] X. Ren, J. Ma, V. Y. Tsviatkou, V. K. Kanapelka, "Improved Hard-Decision Iterative Decoding Method for 2D SEC-DED Codes," Engineering Letters, vol. 31, no.1, , 2023, pp. 364-372.
[46] J. Che, Y. Ye, H. Wang, W. Huang, "A Sample Entropy Parsimonious Model Using Decomposition-ensemble with SSA and CEEMDAN for Short-term Wind Speed Prediction," Engineering Letters, vol. 31, no.1, , 2023, pp. 320-327.
[47] G. Thangkhenpau, S. Panday, "Optimal Eight Order Derivative-Free Family of Iterative Methods for Solving Nonlinear Equations," *IAENG International Journal of Computer Science*, vol. 50, no.1, 2023, pp. 335-341.
[48] Y. A. Hwang, Y. H. Liao, B. Y. Wang, L. S. Chou, "Outcomes and Relative Axiomatic Results under Multicriteria Management Models," *IAENG International Journal of Computer Science*, vol. 50, no.1, 2023, pp. 328-334.
[49] Z. L. Zhao, J. S. Wang, S. S. Guo, Z. F. Li, J. S. Yu, J. Sun, "A Discrete Naked Mole Algorithm to Solve Traveling Salesman Problem Based on Multiple Local Dynamic Searching Strategies," *IAENG International Journal of Computer Science*, vol. 50, no.1, 2023, pp. 315-327.
[50] C. Chanmanee, R. Chinram, R. Prasertpong, P. Julatha, A. Iampan, "Two New Concepts of Internal Direct Products of UP (BCC)-Algebras," *IAENG International Journal of Computer Science*, vol. 50, no.1, 2023, pp. 306-314.

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