Inner Local Exponent of Two-coloured Digraphs with Two Cycles of Length n and $4n+1$

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***Abstract*—A two-coloured digraph $\mathcal{D}^{(2)}$ is a digraph in which each arc is coloured with one of two colours – for example, red or black. A two-coloured digraph $\mathcal{D}^{(2)}$ is said to be primitive if there are positive integers a and i such that for each pair of points x and y in $\mathcal{D}^{(2)}$ there is an (a, i)-walk from x to y. The inner local exponent of a point p_v in $\mathcal{D}^{(2)}$ denoted by $\exp\text{in}(p_v, \mathcal{D}^{(2)})$ is the smallest positive integer $a+i$ over all non-negative integers a and i such that there is a walk from each vertex in $\mathcal{D}^{(2)}$ to p_v consisting of a red arcs and i black arcs. In a two-coloured primitive digraph, two cycles of length n and $4n+1$ result in four or five red arcs. For the two-coloured digraphs, primitivity and inner local exponent are discussed at each point.**

***Index Terms*—primitive-digraph, two-coloured-digraph, digraph-with-two-cycles, inner-local-exponent.**

I. Introduction

A digraph $\mathcal{D}$ consists of a non-empty finite set $P(D)$ and a set $A(D)$ which is a sequential pair of different elements which are still members of $P(D)$. Set $P(D)$ is a set of points on digraph $\mathcal{D}$ and set $A(D)$ is a directed side called the set of arcs on digraph $\mathcal{D}$. A digraph in which the arc is coloured with only two colours, namely red or black, is called a two-coloured digraph. An (a, i)-walk on a digraph whose arcs are given two colours is a walk consisting of a combination of the number of red arcs (a) and black arcs (i). For a walk K in two-coloured digraph $\mathcal{D}^{(2)}$, $r(K)$ and $b(K)$ denote the number of red arcs and the number of black arcs contained in walk K, respectively. The column matrix $\begin{bmatrix} r(K) \\ b(K) \end{bmatrix}$ is the composition of the walk K, and $\ell(K) = r(K) + b(K)$ is the length of the walk K.

Let a and i be non-negative integers. The primitivity of a digraph is determined by the presence of a non-negative integer representing the number of red and black arcs contained in the (a, i)-walk. The exponent of a two-coloured digraph $\mathcal{D}^{(2)}$ denoted by $\exp(\mathcal{D}^{(2)})$ is the smallest positive integer $a+i$ such that for each pair of points x and y in $\mathcal{D}^{(2)}$ there is a (a, i)-walk from x to y. As in the digraph, local exponents on the digraph are divided into two, namely inner local exponents and outer local exponents. The smallest positive integer $a+i$ such that there is a path (a, i) from each point at $\mathcal{D}^{(2)}$ to p_v is called the inner local exponent from a point p_v at $\mathcal{D}^{(2)}$ and denoted by $\exp\text{in}(\mathcal{D}^{(2)})$.

Digraph motivation is coloured with two colours found in computer science, namely in automata theory. In automata theory, there is an on and off button. Red represents on, and black represents off. The term synchronizing words in automata theory is a sequence (0,1) with the same length, and the sequence (0,1) is the same. So the related problem is how to make colouring so that it can find local exponents from points with the same length and colour sequence. Another motivation is the Road Colouring Problem, namely determining whether we can find a specific point from each point so that we move from each point to a certain point using the same number of red and black colours and the same colour sequence.

The study of exponent numbers in the two-cycle two-coloured digraph in terms of the length of each cycle is classified into several types. The first type is two-cycle two-coloured digraph exponent number research with a difference t as in the study by Gao and Shao [1]. Included in the first type are Suwilo [2], Suwilo [3] with a difference of 1, Shao et al. [4], Syahmarani and Suwilo [5] with a difference of 2 and Mardiningsih et al. [6] with a difference of 3. The second type is research on exponent numbers of two-cycle two-coloured digraphs with a difference of $(k-1)n+1$. The second type of research has been conducted by Luo [7] and Sumardi and Suwilo [8] with a difference of $n+1$ and Prasetyo et al. [12] with a difference of $2n+1$. The third type, apart from the first and second types, were studied by Mardiningsih et al. [9] with a difference of $n-1$. This study discusses the inner local exponent in a two-cycle two-coloured digraph with a length of n and $4n+1$. In other words, this study is a study of the inner local exponent of two-cycle two-coloured digraphs with a difference of $3n+1$.

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II. Method

The primitivity requirements of the two-coloured digraph have been discussed by Fornasini and Valcher [10]. Iff the content of the cycle matrix is equal to 1, then the two-coloured digraph is said to be primitive. The content of the cycle matrix is defined as the greatest common divisor of the submatrix determinant 2×2. The cycle matrix for a two-cycle two-colored digraph is $M = \begin{bmatrix} r(C_1) & r(C_2) \\ b(C_1) & b(C_2) \end{bmatrix}$, with C_1 and C_2 representing the first and second cycles.

Corollary II.1. *Given a strongly connected two-coloured digraph $\mathcal{D}^{(2)}$ consisting of two cycles, namely cycle n and* *cycle* $4n+1$. *If* $\mathcal{D}^{(2)}$ *is primitive then the matrix cycle is equal to* $M = \begin{bmatrix} 1 & 4 \\ n-1 & 4n-3 \end{bmatrix}$ *or* $M = \begin{bmatrix} n-1 & 4n-3 \\ 1 & 4 \end{bmatrix}$.

Proof: The cycle matrix form of $\mathcal{D}^{(2)}$ is $M = \begin{bmatrix} r_1 & r_2 \\ n-r_1 & 4n+1-r_2 \end{bmatrix}$ where $0 \leq r_1 \leq n$ and $0 \leq r_2 \leq 4n+1$. Clearly $\mathcal{D}^{(2)}$ is a primitive two-coloured digraph. Therefore, the determinant of the cycle matrix is equal to ± 1. If $\det(M) = 1$, then $(4r_1 - r_2)\,n + r_1 = 1$. As $0 \leq r_2 \leq 4n+1$, we obtain $4r_1 - r_2 = 0$. Consequently $r_1 = 1$ and $r_2 = 4$. Thus, $M = \begin{bmatrix} 1 & 4 \\ n-1 & 4n-3 \end{bmatrix}$. If $\det(M) = -1$, then $(r_2 - 4r_1)\,n - r_1 = 1$. Since $0 \leq r_2 \leq 4n+1$, we obtain $r_2 - 4r_1 = 1$. Hence, $r_1 = n-1$ and $r_2 = 4n-3$. Thus, $M = \begin{bmatrix} n-1 & 4n-3 \\ 1 & 4 \end{bmatrix}$. ■

The reversal of arc colours from red to black or from black to red does not affect the yield of the local exponent. Therefore, we can conclude that $M = \begin{bmatrix} 1 & 4 \\ n-1 & 4n-3 \end{bmatrix}$. For a Hamiltonian two-coloured digraph, the number of red arcs formed from the cycle matrix is four or five red arcs.

The upper and lower bounds of the inner local exponent in the two-coloured digraph are proved by the proposition and lemma stated by Suwilo [6].

Proposition II.1. [2] *Given a two-cycle two-coloured digraph* $\mathcal{D}^{(2)}$ *and any point* p_v *located on both cycles in* $\mathcal{D}^{(2)}$. *If for some nonnegative integers* a *and* i, *there is a path* P_{p_j,p_v} *from point* p_j *to* p_v *such that system*

$$M\mathbf{z} + \begin{bmatrix} r(P_{p_j,p_v}) \\ b(P_{p_j,d_v}) \end{bmatrix} = \begin{bmatrix} a \\ i \end{bmatrix}$$

has a non-negative integer solution, then $\operatorname{expin}(p_v, D^{(2)}) \leq a + i$.

Lemma II.1. [2] *Given a primitive two-coloured digraph* $\mathcal{D}^{(2)}$ *and* p_j *is any point in* $\mathcal{D}^{(2)}$ *with the inner local exponent* $\operatorname{expin}(p_j, D^{(2)})$. *For every* $v = 1, 2, \ldots, 4n+1$ *it follows that* $\operatorname{expin}(p_v, D^{(2)}) \leq \operatorname{expin}(p_j, D^{(2)}) + d(p_j, p_v)$.

Lemma II.2. [11] *Given a primitive two-coloured digraph* $\mathcal{D}^{(2)}$ *which has two cycles, namely* C_1 *and* C_2 *with cycle matrix* $M = \begin{bmatrix} r(C_1) & r(C_2) \\ b(C_1) & b(C_2) \end{bmatrix}$ *and that* $\det(M) = 1$. *If* $\operatorname{expin}(p_v, D^{(2)})$ *is obtained using the* (a_v, i_v)*-walk, then*

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$= M \begin{bmatrix} b(C_2)r(P_{p_j,p_v}) - r(C_2)b(P_{p_j,p_v}) \\ r(C_1)b(P_{p_m,p_v}) - b(C_1)r(P_{p_m,p_v}) \end{bmatrix}$$

for the paths P_{p_j,p_v} *and* P_{p_m,p_v}.

III. Results and Discussion

A. Hamiltonian Two-coloured Digraphs with Two Cycles of Length n and $4n+1$

The two-coloured digraph discussed in this subsection is Hamiltonian two-coloured digraphs with two cycles of length n and $4n+1$ (see Fig.1). Let the first cycle with length n be $C_1 : p_1 \to p_2 \to \cdots \to p_{n-1} \to p_n \to p_1$ and the second cycle with length $4n+1$ be $C_2 : p_1 \to p_2 \to \cdots \to p_{n-1} \to p_n \to p_{n+1} \cdots \to p_{4n} \to p_{4n+1} \to p_1$.

Let the four red arcs in $\mathcal{D}^{(2)}$ be the first arc $p_e \to p_{e+1}$ where $1 \leq e \leq n-1$ and let the second, third and fourth arcs be $p_f \to p_{f+1}, p_g \to p_{g+1}$ and arcs $p_h \to p_{h+1}$, respectively, where $n \leq f < g < h \leq 4n+1$. Let the five red arcs in $\mathcal{D}^{(2)}$ be arc $p_n \to p_1$, arc $p_e \to p_{e+1}$, arc $p_f \to p_{f+1}$, arc $p_g \to p_{g+1}$ and arc $p_h \to p_{h+1}$, for $n \leq e < f < g < h \leq 4n+1$. In Theorem III.1, the red arcs are placed consecutively in C_2, while in Theorem III.2, the red arcs are placed alternately in C_2. Let d_{11} represent the distance from p_{e+1} to p_1 in C_1, d_{12} represent the distance from p_{e+1} to p_1 in C_2, d_2 represent the distance from p_{f+1} to p_1, d_3 represent the distance from p_{g+1} to p_1 and d_4 represent the distance from p_{h+1} to p_1.

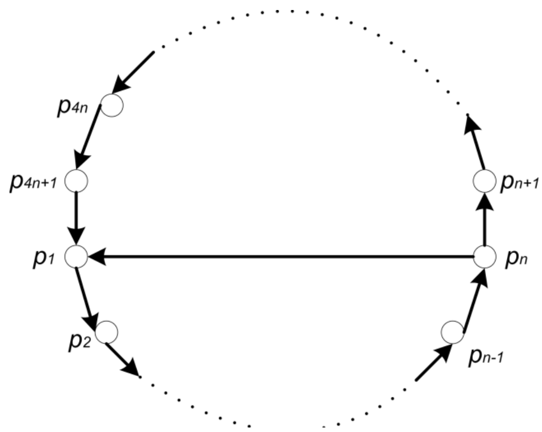

Fig. 1. Hamiltonian digraph with two cycles of length n and $4n+1$

Theorem III.1. *Given* $\mathcal{D}^{(2)}$, *a Hamiltonian two-cycle primitive two-coloured digraph with length* n *and* $4n+1$. *If* $\mathcal{D}^{(2)}$ *has three or four consecutive red arcs at* C_2, *then for every* $v = 1, 2, \ldots, 4n+1$ *it follows*
$\operatorname{expin}(p_v, D^{(2)}) =$

$$\begin{cases} 16n^2 + 4n\,(d_4 - d_{12}) + d_4 + d\,(p_1, p_v)\,, \\ \quad \text{for } d_{12} - d_2 \leq n \\ 12n^2 - 9n + d_4 + d\,(p_1, p_v)\,, \\ \quad \text{for } n < d_{12} - d_2 < 3n \\ 12n^2 - 9n + 4n\,(d_{11} - d_4) + d_{11} + d\,(p_1, p_v)\,, \\ \quad \text{for } d_{12} - d_2 \geq 3n. \end{cases}$$

Proof: Assume that $\operatorname{expin}(p_v, \mathcal{D}^{(2)})$ for every $v = 1, 2, \ldots, 4n+1$ is obtained using path (a_v, i_v). The proof will be divided into three cases as follows.

Case 1.1 : $d_{12} - d_2 \leq n$.

The first step is to show that $\operatorname{expin}(p_v, D^{(2)}) \geq 16n^2 + 4n\,(d_4 - d_{12}) + d_4 + d\,(p_1, p_v)$. Look at the P_{p_e,p_v} and P_{p_{h+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_e,p_v}) - r(C_2)b(P_{p_e,p_v})$ and $q_2 = r(C_1)b(P_{p_{h+1},p_v}) - b(C_1)r(P_{p_{h+1},p_v})$. The following five subcases are taken into consideration.

Subcase 1.1.1.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_e,p_v} is obtained, namely the path $(4, d_{12} - 3 + d(p_1, p_v))$. Using this path, we get $q_1 = 16n - 4d_{12} - 4d(p_1, p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(0, d_4 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 + d(p_1, p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 16n+4d_4 \\ 16n^2-4nd_{12}+4nd_4-16n-3d_4+d(p_1,p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 16n^2+4n(d_4-d_{12})+d_4+d(p_1,p_v) \quad (1)$$

for every point p_v on the path $p_1 \to p_e$.

Subcase 1.1.2.

The point p_v is on the path $p_{e+1} \to p_f$. The path P_{p_e,p_v} is obtained, namely the path $(1, d_{12}-4n-1+d(p_1,p_v))$. Using this path, we get $q_1 = 20n+1-4d_{12}-4d(p_1,p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(1, d_4-1+d(p_1,p_v))$. Using this path, we get $q_2 = d_4-n+d(p_1,p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 16n+1+4d_4 \\ 16n^2-4nd_{12}+4nd_4-16n-1-3d_4+d(p_1,p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 16n^2+4n(d_4-d_{12})+d_4+d(p_1,p_v) \quad (2)$$

for every point p_v on the path $p_{e+1} \to p_f$.

Subcase 1.1.3.

The point p_v is on the path $p_{f+1} \to p_g$. The path P_{p_e,p_v} is obtained, namely the path $(2, d_{12}-4n-2+d(p_1,p_v))$. Using this path, we get $q_1 = 24n+2-4d_{12}-4d(p_1,p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(2, d_4-2+d(p_1,p_v))$. Using this path, we get $q_2 = d_4-2n+d(p_1,p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 16n+2+4d_4 \\ 16n^2-4nd_{12}+4nd_4-16n-2-3d_4+d(p_1,p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 16n^2+4n(d_4-d_{12})+d_4+d(p_1,p_v) \quad (3)$$

for every point p_v on the path $p_{f+1} \to p_g$.

Subcase 1.1.4.

The point p_v is on the path $p_{g+1} \to p_h$. The path P_{p_e,p_v} is obtained, namely the path $(3, d_{12}-4n-3+d(p_1,p_v))$. Using this path, we get $q_1 = 28n+3-4d_{12}-4d(p_1,p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(3, d_4-3+d(p_1,p_v))$. Using this path, we get $q_2 = d_4-3n+d(p_1,p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 16n+3+4d_4 \\ 16n^2-4nd_{12}+4nd_4-16n-3-3d_4+d(p_1,p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 16n^2+4n(d_4-d_{12})+d_4+d(p_1,p_v) \quad (4)$$

for every point p_v on the path $p_{g+1} \to p_h$.

Subcase 1.1.5.

The point p_v is on the path $p_{h+1} \to p_{4n+1}$. The path P_{p_e,p_v} is obtained, namely the path $(4, d_{12}-4n-4+d(p_1,p_v))$. Using this path, we get $q_1 = 32n+4-4d_{12}-4d(p_1,p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(0, d_4-4n-1+d(p_1,p_v))$. Using this path, we get $q_2 = d_4-4n-1+d(p_1,p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 16n+4d_4 \\ 16n^2-4nd_{12}+4nd_4-20n-1-3d_4+d(p_1,p_v) \end{bmatrix}.$$

Let $a_1 = 16n-4d_{12}+4d_4$ and $a_2 = 16n^2-4nd_{12}+4nd_4-20n-1+4d_{12}-3d_4+d(p_1,p_v)$. Considering the path (a_1,a_2) from p_{h+1} to p_v, note that the path P_{p_{h+1},p_v} is $(0, d_4-4n-1+d(p_1,p_v))$ and the solution to the system $M\mathbf{z} + \begin{bmatrix} r(P_{p_{h+1},p_v}) \\ b(P_{p_{h+1},p_v}) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is $z_1 = 16n-4d_{12}+4d_4$ and $z_2 = 0$. The path P_{p_{h+1},p_v} lies on cycle C_2 and there is no walk (a_1,a_2) from p_{h+1} to p_v. Therefore, $\text{expin}(p_v, D^{(2)}) > a_1+a_2$. Note that the shortest walk from p_{h+1} to p_v containing at least a_1 red arc and least a_2 black arc is $(a_1+r(C_2), a_2+b(C_2))$-walk. Since $r(C_2)+b(C_2) = 4n+1$, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} r(C_2) \\ b(C_2) \end{bmatrix} =$$

$$\begin{bmatrix} 16n+4d_4+4 \\ 16n^2-4nd_{12}+4nd_4-16n-4-3d_4+d(p_1,p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 16n^2+4n(d_4-d_{12})+d_4+d(p_1,p_v) \quad (5)$$

for every point p_v on the path $p_{h+1} \to p_{4n+1}$.

The conclusion of (1), (2), (3), (4) and (5) is $\text{expin}(p_v, D^{(2)}) \geq 16n^2+4n(d_4-d_{12})+d_4+d(p_1,p_v)$ for every $v = 1,2,\ldots,4n+1$.

Next, we prove $\text{expin}(p_v, D^{(2)}) \leq 16n^2+4n(d_4-d_{12})+d_4+d(p_1,p_v)$ for every $v = 1,2,...,4n+1$. First, we show that $\text{expin}(p_1, D^{(2)}) = 16n^2+4n(d_4-d_{12}+d_4$ and then by Lemma II.1 to guarantee that $\text{expin}(p_v, D^{(2)}) \leq 16n^2+4n(d_4-d_{12})+d_4+d(p_1,p_v)$ for every $v = 1,2,...,4n+1$.

From (1) we get $\text{expin}(p_1, D^{(2)}) \geq 16n^2+4n(d_4-d_{12})+d_4$. Next simply show that $\text{expin}(p_1, D^{(2)}) \leq 16n^2+4n(d_4-d_{12})+d_4$ for every $p_u = 1,2,...,4n+1$, the system of equations

$$M\mathbf{z} + \begin{bmatrix} r(P_{p_u},p_1) \\ b(P_{p_u,p_1}) \end{bmatrix} = \begin{bmatrix} 16n+4d_4 \\ 16n^2-4nd_{12}+4nd_4-16n-3d_4 \end{bmatrix} \quad (6)$$

has a non-negative integer solution for the path P_{p_u,p_1}. From (6) we get $z_1 = 16n-4d_{12}-(4n-3)r(P_{p_u,p_1})+4b(P_{p_u,p_1})$ and $z_2 = d_4-(1-n)r(P_{p_u,p_1})-b(P_{p_u,p_1})$.

If p_u is on $p_1 \to p_e$, then there is path $(4, 4n-3-d(p_1,p_u))$. Using this path, we get $z_1 = 16n-4(d_{12}+d(p_1,p_u)) \geq 0$ since $d_{12}+d(p_1,p_u) \leq 4n$ and $z_2 = d_4+d(p_1,p_u)-1 \geq 7$ since $d_4+d(p_1,p_u) \geq 2n+2$ with $n \geq 3$. If p_u is on $p_{e+1} \to p_f$, then there is a path $(3, 4n-2-d(p_1,p_u))$. Using this path, we get $z_1 = 20n+1-4(d_{12}+d(p_1,p_u)) \geq 17$ since $d_{12}+d(p_1,p_u) \leq 4n-1$ with $n \geq 3$ and $z_2 = d_4+d(p_1,p_u)-n-1 \geq 5$ since $d_4+d(p_1,p_u) \geq 3n$ with $n \geq 3$. If p_u is on $p_{f+1} \to p_g$, then there is a path $(2, 4n-1-d(p_1,p_u))$. Using this path, we get $z_1 = 24n+2-4(d_{12}+d(p_1,p_u)) \geq 14$ since $d_{12}+d(p_1,p_u) \leq 5n$ with $n \geq 3$ and $z_2 = d_4+d(p_1,p_u)-2n-1 \geq 4$ since

$d_4 + d(p_1, p_u) \geq 4n - 1$ with $n \geq 3$. If p_u is on $p_{g+1} \to p_h$, then there is a path $(1, 4n - d(p_1, p_u))$. Using this path, we get $z_1 = 28n + 3 - 4(d_{12} + d(p_1, p_u)) \geq 23$ since $d_{12} + d(p_1, p_u) \leq 5n + 1$ with $n \geq 3$ and $z_2 = d_4 + d(p_1, p_u) - 3n - 1 \geq 2$ since $d_4 + d(p_1, p_u) \geq 4n$ with $n \geq 3$. If p_u is on $p_{h+1} \to p_{4n+1}$, then there is a path $(0, 4n + 1 - d(p_1, p_u))$. Using this path, we get $z_1 = 32n + 4 - 4(d_{12} + d(p_1, p_u)) \geq 4$ since $d_{12} + d(p_1, p_u) \leq 8n$ and $z_2 = d_4 + d(p_1, p_u) - 4n - 1 \geq 0$ since $d_4 + d(p_1, p_u) \geq 4n + 1$.

Therefore, for every $u = 1, 2, ..., 4n + 1$, the system of equations (6) has a non-negative integer solution. Proposition II.1 guarantees for every $u = 1, 2, ..., 4n + 1$, there is a path P_{p_u, p_1} with $a = 16n - 4d_{12} + 4d_4$ and $i = 16n^2 - 4nd_{12} + 4nd_4 - 16n + 4d_{12} - 3d_4$. Therefore, $\text{expin}(p_1, D^{(2)}) = 16n^2 + 4n(d_4 - d_{12}) + d_4$ and by Lemma II.1 we get the conclusion that $\text{expin}(p_v, D^{(2)}) \leq 16n^2 + 4n(d_4 - d_{12}) + d_4 + d(p_1, p_v)$ for every $v = 1, 2, ..., 4n + 1$.

Case 2.1 : $n < d_{12} - d_2 < 3n$.

The first step is to show that $\text{expin}(p_v, D^{(2)}) \geq 12n^2 - 9n + d_4 + d(p_1, p_v)$. Look at the P_{p_f, p_v} and P_{p_{h+1}, p_v} paths and define $q_1 = b(C_2)r(P_{p_f, p_v}) - r(C_2)b(P_{p_f, p_v})$ and $q_2 = r(C_1)b(P_{p_{h+1}, p_v}) - b(C_1)r(P_{p_{h+1}, p_v})$. The following five subcases are taken into consideration.

Subcase 2.1.1.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_f, p_v} is obtained, namely the path $(3, d_4 + d(p_1, p_v))$. Using this path, we get $q_1 = 12n - 9 - 4d_4 - 4d(p_1, p_v)$. The path P_{p_{h+1}, p_v} is obtained, namely the path $(0, d_4 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 12n - 9 \\ 12n^2 - 21n + 9 + d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - 9n + d_4 + d(p_1, p_v) \quad (7)$$

for every point p_v on the path $p_1 \to p_e$.

Subcase 2.1.2.

The point p_v is on the path $p_{e+1} \to p_f$. The path P_{p_f, p_v} is obtained, namely the path $(4, d_4 - 1 + d(p_1, p_v))$. Using this path, we get $q_1 = 16n - 8 - 4d_4 - 4d(p_1, p_v)$. The path P_{p_{h+1}, p_v} is obtained, namely the path $(1, d_4 - 1 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 - n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 12n - 8 \\ 12n^2 - 21n + 8 + d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - 9n + d_4 + d(p_1, p_v) \quad (8)$$

for every point p_v on the path $p_{e+1} \to p_f$.

Subcase 2.1.3.

The point p_v is on the path $p_{f+1} \to p_g$. The path P_{p_f, p_v} is obtained, namely the path $(1, d_4 - 4n + 1 + d(p_1, p_v))$. Using this path, we get $q_1 = 20n - 7 - 4d_4 - 4d(p_1, p_v)$. The path P_{p_{h+1}, p_v} is obtained, namely the path $(2, d_4 - 2 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 - 2n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 12n - 7 \\ 12n^2 - 21n + 7 + d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - 9n + d_4 + d(p_1, p_v) \quad (9)$$

for every point p_v on the path $p_{f+1} \to p_g$.

Subcase 2.1.4.

The point p_v is on the path $p_{g+1} \to p_h$. The path P_{p_f, p_v} is obtained, namely the path $(2, d_4 - 4n + d(p_1, p_v))$. Using this path, we get $q_1 = 24n - 6 - 4d_4 - 4d(p_1, p_v)$. The path P_{p_{h+1}, p_v} is obtained, namely the path $(3, d_4 - 3 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 - 3n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 12n - 6 \\ 12n^2 - 21n + 6 + d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - 9n + d_4 + d(p_1, p_v) \quad (10)$$

for every point p_v on the path $p_{g+1} \to p_h$.

Subcase 2.1.5.

The point p_v is on the path $p_{h+1} \to p_{4n+1}$. The path P_{p_f, p_v} is obtained, namely the path $(3, d_4 - 4n - 1 + d(p_1, p_v))$. Using this path, we get $q_1 = 28n - 5 - 4d_4 - 4d(p_1, p_v)$. The path P_{p_{h+1}, p_v} is obtained, namely the path $(0, d_4 - 4n - 1 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 - 4n - 1 + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 12n - 9 \\ 12n^2 - 25n + 8 + d_4 + d(p_1, p_v) \end{bmatrix}.$$

Let $a_1 = 12n - 9$ and $a_2 = 12n^2 - 25n + 8 + d_4 + d(p_1, p_v)$. Considering the path (a_1, a_2) from p_{h+1} to p_v, note that the path P_{p_{h+1}, p_v} is $(0, d_4 - 4n - 1 + d(p_1, p_v))$ and the solution to the system $M\mathbf{z} + \begin{bmatrix} r(P_{p_{h+1}, p_v}) \\ b(P_{p_{h+1}, p_v}) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is $z_1 = 12n - 9$ and $z_2 = 0$. The path P_{p_{h+1}, p_v} lies on cycle C_2 and there is no walk (a_1, a_2) from p_{h+1} to p_v. Therefore, $\text{expin}(p_v, D^{(2)}) > a_1 + a_2$. Note that the shortest walk from $p_{h+1} \to p_v$ containing at least a_1 red arc and least a_2 black arc is $(a_1 + r(C_2), a_2 + b(C_2))$-walk. Since $r(C_2) + b(C_2) = 4n + 1$, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} r(C_2) \\ b(C_2) \end{bmatrix} = \begin{bmatrix} 12n - 5 \\ 12n^2 - 21n + 5 + d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - 9n + d_4 + d(p_1, p_v) \quad (11)$$

for every point p_v on the path $p_{h+1} \to p_{4n+1}$.

The conclusion of (7), (8), (9), (10) and (11) is $\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - 9n + d_4 + d(p_1, p_v)$ for every $v = 1, 2, \ldots, 4n+1$.

Next, we prove that $\exp\text{in}(p_v, D^{(2)}) \leq 12n^2 - 9n + d_4 + d(p_1, p_v)$ for every $v = 1, 2, \ldots, 4n+1$. First we show that $\exp\text{in}(p_1, D^{(2)}) = 12n^2 - 9n + d_4$ and then by Lemma II.1 to guarantee that $\exp\text{in}(p_v, D^{(2)}) \leq 12n^2 - 9n + d_4 + d(p_1, p_v)$ for every $v = 1, 2, \ldots, 4n+1$.

From (7) we get $\exp\text{in}(p_1, D^{(2)}) \geq 12n^2 - 9n + d_4$. Next simply show that $\exp\text{in}(p_1, D^{(2)}) \leq 12n^2 - 9n + d_4$ for every p_u, $u = 1, 2, \ldots, 4n+1$, the system of equations

$$M\mathbf{z} + \begin{bmatrix} r(P_{p_u,p_1}) \\ b(P_{p_u,p_1}) \end{bmatrix} = \begin{bmatrix} 12n - 9 \\ 12n^2 - 21n + 9 + d_4 \end{bmatrix} \quad (12)$$

has a non-negative integer solution for the path P_{p_u,p_1}. From (12) we get $z_1 = 12n - 9 - 4d_4 - (4n-3)r(P_{p_u,p_1}) + 4b(P_{p_u,p_1})$ and $z_2 = d_4 - (1-n)r(P_{p_u,p_1}) - b(P_{p_u,p_1})$.

If p_u is on $p_1 \to p_e$, then there is a path $(4, 4n - 3 - d(p_1, p_u))$. Using this path, we get $z_1 = 12n - 9 - 4(d_4 + d(p_1, p_u)) \geq 3$ since $d_4 + d(p_1, p_u) \leq 2n$ with $n \geq 3$ and $z_2 = d_4 + d(p_1, p_u) - 1 \geq 1$ since $d_4 + d(p_1, p_u) \geq n - 1$ with $n \geq 3$. If p_u is on $p_{e+1} \to p_f$, then there is a path $(3, 4n - 2 - d(p_1, p_u))$. Using this path, we get $z_1 = 16n - 8 - 4(d_4 + d(p_1, p_u)) \geq 0$ since $d_4 + d(p_1, p_u) \leq 4n - 2$ and $z_2 = d_4 + d(p_1, p_u) - n - 1 \geq 3$ since $d_4 + d(p_1, p_u) \geq 2n + 1$ with $n \geq 3$. If p_u is on $p_{f+1} \to p_g$, then there is a path $(2, 4n - 1 - d(p_1, p_u))$. Using this path, we get $z_1 = 20n - 7 - 4(d_4 + d(p_1, p_u)) \geq 9$ since $d_4 + d(p_1, p_u) \leq 4n - 1$ and $z_2 = d_4 + d(p_1, p_u) - 2n - 1 \geq 4$ since $d_4 + d(p_1, p_u) \geq 4n - 1$ with $n \geq 3$. If p_u is on $p_{g+1} \to p_h$, then there is a path $(1, 4n - d(p_1, p_u))$. Using this path, we get $z_1 = 24n - 6 - 4(d_4 + d(p_1, p_u)) \geq 14$ since $d_4 + d(p_1, p_u) \leq 4n + 1$ and $z_2 = d_4 + d(p_1, p_u) - 3n - 1 \geq 3$ since $d_4 + d(p_1, p_u) \geq 4n + 1$ with $n \geq 3$. If p_u is on $p_{h+1} \to p_{4n+1}$, then there is a path $(0, 4n + 1 - d(p_1, p_u))$. Using this path, we get $z_1 = 28n - 5 - 4(d_4 + d(p_1, p_u)) \geq 7$ since $d_4 + d(p_1, p_u) \leq 6n$ with $n \geq 3$ and $z_2 = d_4 + d(p_1, p_u) - 4n - 1 \geq 0$ since $d_4 + d(p_1, p_u) \geq 4n + 1$.

Therefore, for every $u = 1, 2, ..., 4n + 1$, the system of equations (12) has a non-negative integer solution. Proposition II.1 guarantees for every $u = 1, 2, ..., 4n + 1$, there is a path P_{p_u,p_1} with $a = 12n - 9$ and $i = 12n^2 - 21n + 9 + d_4$. Therefore, $\exp\text{in}(p_1, D^{(2)}) = 12n^2 - 9n + d_4$ and by Lemma II.1 we get the conclusion that $\exp\text{in}(p_v, D^{(2)}) \leq 12n^2 - 9n + d_4 + d(p_1, p_v)$ for every $v = 1, 2, ..., 4n + 1$.

Case 3.1 : $d_{12} - d_2 \geq 3n$**.**

The first step is to show that $\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11} + d(p_1, p_v)$. Look at the P_{p_f,p_v} and P_{p_{e+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_f,p_v}) - r(C_2)b(P_{p_f,p_v})$ and $q_2 = r(C_1)b(P_{p_{e+1},p_v}) - b(C_1)r(P_{p_{e+1},p_v})$. The following five subcases are taken into consideration.

Subcase 3.1.1.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_f,p_v} is obtained, namely the path $(3, d_4 + d(p_1, p_v))$. Using this path, we get $q_1 = 12n - 9 - 4d_4 - 4d(p_1, p_v)$. The path P_{p_{e+1},p_v} is obtained, namely the path $(0, d_{11} + d(p_1, p_v))$. Using this path, we get $q_2 = d_{11} + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 9 + 4d_{11} \\ 12n^2 - 21n + 9 + 4n(d_{11} - d_4) - 3d_{11} + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11} + d(p_1, p_v) \quad (13)$$

for every point p_v on the path $p_1 \to p_e$.

Subcase 3.1.2.

The point p_v is on the path $p_{e+1} \to p_f$. The path P_{p_f,p_v} is obtained, namely the path $(4, d_4 - 1 + d(p_1, p_v))$. Using this path, we get $q_1 = 16n - 8 - 4d_4 - 4d(p_1, p_v)$. The path P_{p_{e+1},p_v} is obtained, namely the path $(0, d_{11} - n + d(p_1, p_v))$. Using this path, we get $q_2 = d_{11} - n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 8 + 4d_{11} \\ 12n^2 - 21n + 8 + 4n(d_{11} - d_4) - 3d_{11} + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11} + d(p_1, p_v) \quad (14)$$

for every point p_v on the path $p_{e+1} \to p_f$.

Subcase 3.1.3.

The point p_v is on the path $p_{f+1} \to p_g$. The path P_{p_f,p_v} is obtained, namely the path $(1, d_4 - 4n + 1 + d(p_1, p_v))$. Using this path, we get $q_1 = 20n - 7 - 4d_4 - 4d(p_1, p_v)$. The path P_{p_{e+1},p_v} is obtained, namely the path $(0, d_{11} - 2n - 4 + d(p_1, p_v))$. Using this path, we get $q_2 = d_{11} - 2n - 4 + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 23 + 4d_{11} \\ 12n^2 - 37n + 19 + 4n(d_{11} - d_4) - 3d_{11} + d(p_1, p_v) \end{bmatrix}.$$

Let $a_1 = 12n - 21 - 4d_4 + 4d_{11}$ and $a_2 = 12n^2 - 37n + 19 + 4n(d_{11} - d_4) + 4d_4 - 3d_{11} + d(p_1, p_v)$. Considering the path (a_1, a_2) from p_{e+1} to p_v, note that the path P_{p_{e+1},p_v} is $(0, d_{11} - 2n - 4 + d(p_1, p_v))$ and the solution to the system $M\mathbf{z} + \begin{bmatrix} r(P_{p_{e+1},p_v}) \\ b(P_{p_{e+1},p_v}) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is $z_1 = 12n + 23 - 4d_4 + 4d_{11}$ and $z_2 = 0$. The path P_{p_{e+1},p_v} lies on cycle C_2 and there is no walk (a_1, a_2) from p_{e+1} to p_v. Therefore, $\exp\text{in}(p_v, D^{(2)}) > a_1 + a_2$. Note that the shortest walk from p_{e+1} to p_v containing at least a_1 red arc and at least a_2 black arc is $(a_1 + r(C_2), a_2 + b(C_2))$-walk. Since $r(C_2) + b(C_2) = 4n + 1$, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + 4 \begin{bmatrix} r(C_2) \\ b(C_2) \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 7 + 4d_{11} \\ 12n^2 - 21n + 7 + 4n(d_{11} - d_4) - 3d_{11} + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11} + d(p_1, p_v) \quad (15)$$

for every point p_v on the path $p_{f+1} \to p_g$.

Subcase 3.1.4.

The point p_v is on the path $p_{g+1} \to p_h$. The path P_{p_f,p_v} is obtained, namely the path $(2, d_4 - 4n + d(p_1,p_v))$. Using this path, we get $q_1 = 24n - 6 - 4d_4 - 4d(p_1,p_v)$. The path P_{p_{e+1},p_v} is obtained, namely the path $(0, d_{11} - 3n - 2 + d(p_1,p_v))$. Using this path, we get $q_2 = d_{11} - 3n - 2 + d(p_1,p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 14 + 4d_{11} \\ 12n^2 - 29n + 12 + 4n(d_{11} - d_4) - 3d_{11} + d(p_1,p_v) \end{bmatrix}.$$

Let $a_1 = 12n - 14 - 4d_4 + 4d_{11}$ and $a_2 = 12n^2 - 29n + 12 + 4n(d_{11} - d_4) + 4d_4 - 3d_{11} + d(p_1,p_v)$. Considering the path (a_1,a_2) from p_{e+1} to p_v, note that the path P_{p_{e+1},p_v} is $(0, d_{11} - 3n - 2 + d(p_1,p_v))$ and the solution to the system $M\mathbf{z} + \begin{bmatrix} r(P_{p_{e+1},p_v}) \\ b(P_{p_{e+1},p_v}) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is $z_1 = 12n + 14 - 4d_4 + 4d_{11}$ and $z_2 = 0$. The path P_{p_{e+1},p_v} lies on cycle C_2 and there is no walk (a_1,a_2) from p_{e+1} to p_v. Therefore, $\text{expin}(p_v, D^{(2)}) > a_1 + a_2$. Note that the shortest walk from p_{e+1} to p_v containing at least a_1 red arc and at least a_2 black arc is $(a_1 + r(C_2), a_2 + b(C_2))$-walk. Since $r(C_2) + b(C_2) = 4n + 1$, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + 2\begin{bmatrix} r(C_2) \\ b(C_2) \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 6 - 4d_4 + 4d_{11} \\ 12n^2 - 21n + 6 + 4n(d_{11} - d_4) + 4d_4 - 3d_{11} + d(p_1,p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11} + d(p_1,p_v) \tag{16}$$

for every point p_v on the path $p_{g+1} \to p_h$.

Subcase 3.1.5.

The point p_v is on the path $p_{h+1} \to p_{4n+1}$. The path P_{p_f,p_v} is obtained, namely the path $(3, d_4 - 4n - 1 + d(p_1,p_v))$. Using this path, we get $q_1 = 28n - 5 - 4d_4 - 4d(p_1,p_v)$. The path P_{p_{e+1},p_v} is obtained, namely the path $(0, d_{11} - 4n + d(p_1,p_v))$. Using this path, we get $q_2 = d_{11} - 4n + d(p_1,p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 5 + 4d_{11} \\ 12n^2 - 21n + 5 + 4n(d_{11} - d_4) - 3d_{11} + d(p_1,p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11} + d(p_1,p_v) \tag{17}$$

for every point p_v on the path $p_{h+1} \to p_{4n+1}$.

The conclusion of (13), (14), (15), (16) and (17) is $\text{expin}(p_v, D^{(2)}) \geq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11} + d(p_1,p_v)$ for every $v = 1, 2, \ldots, 4n + 1$.

Next, we prove $\text{expin}(p_v, D^{(2)}) \leq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11} + d(p_1,p_v)$ for every $v = 1, 2, \ldots, 4n+1$. First we show that $\text{expin}(p_1, D^{(2)}) = 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11}$ and then by Lemma II.1 to guarantee that $\text{expin}(p_v, D^{(2)}) \leq 12n^2 - 9n + d_4 + d(p_1,p_v)$ for every $v = 1, 2, \ldots, 4n + 1$. From (13) we get $\text{expin}(p_1, D^{(2)}) \geq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11}$. Next simply show that $\text{expin}(p_1, D^{(2)}) \leq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11}$ for every $p_u, u = 1, 2, \ldots, 4n + 1$, the system of equations

$$M\mathbf{z} + \begin{bmatrix} r(P_{p_u,p_1}) \\ b(P_{p_u,p_1}) \end{bmatrix} = \begin{bmatrix} 12n - 9 - 4d_4 + 4d_{11} \\ 12n^2 - 21n + 9 + 4n(d_{11} - d_4) + 4d_4 + 3d_{11} \end{bmatrix} \tag{18}$$

has a non-negative integer solution for the path P_{p_u,p_1}. From (18) we get $z_1 = 12n - 9 - 4d_4 - (4n - 3)r(P_{p_u,p_1}) + 4b(P_{p_u,p_1})$ and $z_2 = d_{11} - (1 - n)r(P_{p_u,p_1}) - b(P_{p_u,p_1})$.

If p_u is on $p_1 \to p_e$, then there is a path $(4, 4n - 3 - d(p_1,p_u))$. Using this path, we get $z_1 = 12n - 9 - 4(d_4 + d(p_1,p_u)) \geq 23$ since $d_4 + d(p_1,p_u) \leq n - 2$ with $n \geq 3$ and $z_2 = d_{11} + d(p_1,p_u) - 1 \geq 1$ since $d_{11} + d(p_1,p_u) \geq n - 1$ with $n \geq 3$. If p_u is on $p_{e+1} \to p_f$, then there is a path $(3, 4n - 2 - d(p_1,p_u))$. Using this path, we get $z_1 = 16n - 8 - 4(d_4 + d(p_1,p_u)) \geq 0$ since $d_4 + d(p_1,p_u) \leq 4n - 2$ and $z_2 = d_{11} + d(p_1,p_u) - n - 1 \geq 0$ since $d_{11} + d(p_1,p_u) \geq n + 1$. If p_u is on $p_{f+1} \to p_g$, then there is a path $(2, 4n - 1 - d(p_1,p_u))$. Using this path, we get $z_1 = 20n - 7 - 4(d_4 + d(p_1,p_u)) \geq 9$ since $d_4 + d(p_1,p_u) \leq 4n - 1$ with $n \geq 3$ and $z_2 = d_{11} + d(p_1,p_u) - 2n - 1 \geq 5$ since $d_{11} + d(p_1,p_u) \geq 4n$ with $n \geq 3$. If p_u is on $p_{g+1} \to p_h$, then there is a path $(1, 4n - d(p_1,p_u))$. Using this path, we get $z_1 = 24n - 6 - 4(d_4 + d(p_1,p_u)) \geq 18$ since $d_4 + d(p_1,p_u) \leq 4n$ with $n \geq 3$ and $z_2 = d_{11} + d(p_1,p_u) - 3n - 1 \geq 3$ since $d_{11} + d(p_1,p_u) \geq 4n + 1$ with $n \geq 3$. If p_u is on $p_{h+1} \to p_{4n+1}$, then there is a path $(0, 4n + 1 - d(p_1,p_u))$. Using this path, we get $z_1 = 28n - 5 - 4(d_4 + d(p_1,p_u)) \geq 31$ since $d_4 + d(p_1,p_u) \leq 4n$ with $n \geq 3$ and $z_2 = d_{11} + d(p_1,p_u) - 4n - 1 \geq 0$ since $d_{11} + d(p_1,p_u) \geq 4n + 1$.

Therefore, for every $u = 1, 2, \ldots, 4n + 1$, the system of equations (18) has a non-negative integer solution. Proposition II.1 guarantees for every $u = 1, 2, \ldots, 4n + 1$, there is a path P_{p_u,p_1} with $a = 12n - 9 - 4d_4 + 4d_{11}$ and $i = 12n^2 - 21n + 9 + 4n(d_{11} - d_4) + 4d_4 + 3d_{11}$. Therefore, $\text{expin}(p_1, D^{(2)}) = 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11}$ and by Lemma II.1 we get the conclusion that $\text{expin}(p_v, D^{(2)}) \leq 12n^2 - 9n + 4n(d_{11} - d_4) + d_{11} + d(p_1,p_v)$ for every $v = 1, 2, ..., 4n + 1$.

■

Theorem III.2. *Given $\mathcal{D}^{(2)}$, a Hamiltonian two-cycle primitive two-coloured digraph with cycle C_1 and C_2 of length n and $4n+1$. If $\mathcal{D}^{(2)}$ has three or four red arcs alternating with a difference of 1 at C_2, then for every $v = 1, 2, \ldots, 4n + 1$ we have*

$$\text{expin}(p_v, D^{(2)}) = \begin{cases} 16n^2 + 4n(d_4 - d_{12}) + d_4 + d(p_1,p_v), & \text{for } d_{12} - d_2 \leq n \\ 12n^2 - n + 4n(d_4 - d_2) + d_4 + d(p_1,p_v), & \text{for } n < d_{12} - d_2 < 3n - 2 \\ 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1,p_v), & \text{for } d_{12} - d_2 \geq 3n - 2 \end{cases}$$

Proof: Assume that $\text{expin}(p_v, \mathcal{D}^{(2)})$ for every $v = 1, 2, \ldots, 4n + 1$ is obtained using path (a_v, i_v). The proof will be divided into three cases as follows.

Case 1.2 : $d_{12} - d_2 \leq n$**.**

The proof for Case 1.2 of Theorem III.2 is the same as Case 1.1 in Theorem III.1.

Case 2.2 : $n < d_{12} - d_2 < 3n - 2$**.**

The first step is to show that $\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_4 - d_2) + d_4 + d(p_1, p_v)$. Look at the P_{p_f,p_v} and P_{p_{h+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_f,p_v}) - r(C_2)b(P_{p_f,p_v})$ and $q_2 = r(C_1)b(P_{p_{h+1},p_v}) - b(C_1)r(P_{p_{h+1},p_v})$. The following five subcases are taken into consideration.

Subcase 2.2.1.

The point p_v is on the path $p_1 \rightarrow p_e$. The path P_{p_f,p_v} is obtained, namely the path $(3, d_2 - 2 + d(p_1, p_v))$. Using this path, we get $q_1 = 12n - 1 - 4d_2 - 4d(p_1, p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(0, d_4 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 1 + 4d_4 \\ 12n^2 + 4n(d_4 - d_2) - 13n + 1 - 3d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_4 - d_2) + d_4 + d\,(p_1, p_v) \tag{19}$$

for every point p_v on the path $p_1 \rightarrow p_e$.

Subcase 2.2.2.

The point p_v is on the path $p_{e+1} \rightarrow p_f$. The path P_{p_f,p_v} is obtained, namely the path $(4, d_2 - 3 + d(p_1, p_v))$. Using this path, we get $q_1 = 16n - 4d_2 - 4d(p_1, p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(1, d_4 - 1 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 - n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n + 4d_4 \\ 12n^2 + 4n(d_4 - d_2) - 13n - 3d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_4 - d_2) + d_4 + d\,(p_1, p_v) \tag{20}$$

for every point p_v on the path $p_{e+1} \rightarrow p_f$.

Subcase 2.2.3.

The point p_v is on the path $p_{f+1} \rightarrow p_g$. The path P_{p_f,p_v} is obtained, namely the path $(1, d_2 - 4n - 1 + d(p_1, p_v))$. Using this path, we get $q_1 = 20n + 1 - 4d_2 - 4d(p_1, p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(2, d_4 - 2 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 - 2n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n + 1 + 4d_4 \\ 12n^2 + 4n(d_4 - d_2) - 13n - 1 - 3d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_4 - d_2) + d_4 + d\,(p_1, p_v) \tag{21}$$

for every point p_v on the path $p_{f+1} \rightarrow p_g$.

Subcase 2.2.4.

The point p_v is on the path $p_{g+1} \rightarrow p_h$. The path P_{p_f,p_v} is obtained, namely the path $(2, d_2 - 4n - 2 + d(p_1, p_v))$. Using this path, we get $q_1 = 24n + 2 - 4d_2 - 4d(p_1, p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(3, d_4 - 3 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 - 3n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n + 2 + 4d_4 \\ 12n^2 + 4n(d_4 - d_2) - 13n - 2 - 3d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_4 - d_2) + d_4 + d\,(p_1, p_v) \tag{22}$$

for every point p_v on the path $p_{g+1} \rightarrow p_h$.

Subcase 2.2.5.

The point p_v is on the path $p_{h+1} \rightarrow p_{4n+1}$. The path P_{p_f,p_v} is obtained, namely the path $(3, d_2 - 4n - 3 + d(p_1, p_v))$. Using this path, we get $q_1 = 28n + 3 - 4d_2 - 4d(p_1, p_v)$. The path P_{p_{h+1},p_v} is obtained, namely the path $(0, d_4 - 4n - 1 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 - 4n - 1 + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 1 + 4d_4 \\ 12n^2 + 4n(d_4 - d_2) - 17n - 3d_4 + d(p_1, p_v) \end{bmatrix}.$$

Let $a_1 = 12n - 1 - 4d_2 + 4d_4$ and $a_2 = 12n^2 + 4n(d_4 - d_2) - 17n + 4d_2 - 3d_4 + d(p_1, p_v)$. Considering the path (a_1, a_2) from p_{h+1} to p_v, note that the path P_{p_{h+1},p_v} is $(0, d_4 - 4n - 1 + d(p_1, p_v))$ and the solution to the system $M\mathbf{z} + \begin{bmatrix} r(P_{p_{h+1},p_v}) \\ b(P_{p_{h+1},p_v}) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is $z_1 = 12n - 1 - 4d_2 + 4d_4$ and $z_2 = 0$. The path P_{p_{h+1},p_v} lies on cycle C_2 and there is no walk (a_1, a_2) from p_{h+1} to p_v. Therefore, $\exp\text{in}(p_v, D^{(2)}) > a_1 + a_2$. Note that the shortest walk from p_{h+1} to p_v containing at least a_1 red arc and least a_2 black arc is $(a_1 + r(C_2), a_2 + b(C_2))$-walk. Since $r(C_2) + b(C_2) = 4n + 1$, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} r(C_2) \\ b(C_2) \end{bmatrix} =$$

$$\begin{bmatrix} 12n + 3 + 4d_4 \\ 12n^2 + 4n(d_4 - d_2) - 13n - 3 - 3d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_4 - d_2) + d_4 + d\,(p_1, p_v) \tag{23}$$

for every point p_v on the path $p_{h+1} \rightarrow p_{4n+1}$.

The conclusion of (19), (20), (21), (22) and (23) is $\exp\text{in}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_4 - d_2) + d_4 + d\,(p_1, p_v)$ for every $v = 1, 2, \ldots, 4n + 1$.

Next, we will prove that $\exp\text{in}(p_v, D^{(2)}) \leq 12n^2 - n + 4n(d_4 - d_2) + d_4 + d\,(p_1, p_v)$ for every $v = 1, 2, \ldots, 4n + 1$. First we show that $\exp\text{in}(p_1, D^{(2)}) = 12n^2 - n + 4n(d_4 - d_2) + d_4$ and then by Lemma II.1 to guarantee that $\exp\text{in}(p_v, D^{(2)}) \leq 12n^2 - n + 4n(d_4 - d_2) + d_4$ for every $v = 1, 2, \ldots, 4n + 1$.

From (19) we get $\text{expin}(p_1, D^{(2)}) \geq 12n^2 - n + 4n(d_4 - d_2) + d_4$. Next simply show that $\text{expin}(p_1, D^{(2)}) \leq 12n^2 - n + 4n(d_4 - d_2) + d_4$ for every p_u, $u = 1, 2, \ldots, 4n+1$, the system of equations

$$M\mathbf{z} + \begin{bmatrix} r(P_{p_u,p_1}) \\ b(P_{p_u,p_1}) \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 1 - 4d_2 + 4d_4 \\ 12n^2 + 4n(d_4 - d_2) - 13n + 1 + 4d_2 - 3d_4 \end{bmatrix} \tag{24}$$

has a non-negative integer solution for the path P_{p_u,p_1}. From (24) we get $z_1 = 12n - 1 - 4d_2 - (4n-3)r(P_{p_u,p_1}) + 4b(P_{p_u,p_1})$ and $z_2 = d_4 - (1-n)r(P_{p_u,p_1}) - b(P_{p_u,p_1})$.

If p_u is on $p_1 \to p_e$, then there is a path $(4, 4n - 3 - d(p_1, p_u))$. Using this path, we get $z_1 = 12n - 1 - 4(d_2 + d(p_1, p_u)) \geq 3$ since $d_2 + d(p_1, p_u) \leq 2n+2$ with $n \geq 3$ and $z_2 = d_4 + d(p_1, p_u) - 1 \geq 1$ since $d_4 + d(p_1, p_u) \geq n - 1$ with $n \geq 3$. If p_u is on $p_{e+1} \to p_f$, then there is a path $(3, 4n - 2 - d(p_1, p_u))$. Using this path, we get $z_1 = 16n - 4(d_2 + d(p_1, p_u)) \geq 0$ since $d_2 + d(p_1, p_u) \leq 4n$ and $z_2 = d_4 + d(p_1, p_u) - n - 1 \geq 0$ since $d_4 + d(p_1, p_u) \geq n+1$. If p_u is on $p_{f+1} \to p_g$, then there is a path $(2, 4n - 1 - d(p_1, p_u))$. Using this path, we get $z_1 = 20n + 1 - 4(d_2 + d(p_1, p_u)) \geq 5$ since $d_2 + d(p_1, p_u) \leq 5n - 1$ with $n \geq 3$ and $z_2 = d_4 + d(p_1, p_u) - 2n - 1 \geq 2$ since $d_4 + d(p_1, p_u) \geq 3n$ with $n \geq 3$. If p_u is on $p_{g+1} \to p_h$, then there is a path $(1, 4n - d(p_1, p_u))$. Using this path, we get $z_1 = 24n + 2 - 4(d_2 + d(p_1, p_u)) \geq 10$ since $d_2 + d(p_1, p_u) \leq 6n - 2$ and $z_2 = d_4 + d(p_1, p_u) - 3n - 1 \geq 1$ since $d_4 + d(p_1, p_u) \geq 4n - 1$ with $n \geq 3$. If p_u is on $p_{h+1} \to p_{4n+1}$, then there is a path $(0, 4n + 1 - d(p_1, p_u))$. Using this path, we get $z_1 = 28n + 3 - 4(d_2 + d(p_1, p_u)) \geq 7$ since $d_2 + d(p_1, p_u) \leq 6n + 2$ with $n \geq 3$ and $z_2 = d_4 + d(p_1, p_u) - 4n - 1 \geq 0$ since $d_4 + d(p_1, p_u) \geq 4n + 1$.

Therefore, for every $u = 1, 2, ..., 4n+1$, the system of equations (24) has a non-negative integer solution. Proposition II.1 guarantees for every $u = 1, 2, ..., 4n+1$, there is a path P_{p_u,p_1} with $a = 12n - 1 - 4d_2 + 4d_4$ and $i = 12n^2 + 4n(d_4 - d_2) - 13n + 1 + 4d_2 - 3d_4$.

Therefore, $\text{expin}(p_1, D^{(2)}) = 12n^2 - n + 4n(d_4 - d_2) + d_4$ and by Lemma II.1 we get the conclusion that $\text{expin}(p_v, D^{(2)}) \leq 12n^2 - n + 4n(d_4 - d_2) + d_4 + d(p_1, p_v)$ for every $v = 1, 2, ..., 4n+1$.

Case 3.2 : $d_{12} - d_2 \geq 3n - 2$**.**

The first step is to show that $\text{expin}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1, p_v)$. Look at the P_{p_f,p_v} and P_{p_{e+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_f,p_v}) - r(C_2)b(P_{p_f,p_v})$ and $q_2 = r(C_1)b(P_{p_{e+1},p_v}) - b(C_1)r(P_{p_{e+1},p_v})$. The following four subcases are taken into consideration.

Subcase 3.2.1.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_f,p_v} is obtained, namely the path $(3, d_2 - 2 + d(p_1, p_v))$. Using this path, we get $q_1 = 12n - 1 - 4d_2 - 4d(p_1, p_v)$. The path P_{p_{e+1},p_v} is obtained, namely the path $(0, d_{11} + d(p_1, p_v))$. Using this path, we get $q_2 = d_{11} + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n - 1 + 4d_{11} \\ 12n^2 - 13n + 1 + 4n(d_{11} - d_2) - 3d_{11} + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1, p_v) \tag{25}$$

for every point p_v on the path $p_1 \to p_e$.

Subcase 3.2.2.

The point p_v is on the path $p_{e+1} \to p_f$. The path P_{p_f,p_v} is obtained, namely the path $(4, d_2 - 3 + d(p_1, p_v))$. Using this path, we get $q_1 = 16n - 4d_2 - 4d(p_1, p_v)$. The path P_{p_{e+1},p_v} is obtained, namely the path $(0, d_{11} - n + d(p_1, p_v))$. Using this path, we get $q_2 = d_{11} - n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n + 4d_{11} \\ 12n^2 - 13n + 4n(d_{11} - d_2) - 3d_{11} + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1, p_v) \tag{26}$$

for every point p_v on the path $p_{e+1} \to p_f$.

Subcase 3.2.3.

The point p_v is on the path $p_{f+1} \to p_g$. The path P_{p_f,p_v} is obtained, namely the path $(1, d_2 - 4n - 1 + d(p_1, p_v))$. Using this path, we get $q_1 = 20n + 1 - 4d_2 - 4d(p_1, p_v)$. The path P_{p_{e+1},p_v} is obtained, namely the path $(1, d_{11} - n - 1 + d(p_1, p_v))$. Using this path, we get $q_2 = d_{11} - 2n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n + 1 + 4d_{11} \\ 12n^2 - 13n - 1 + 4n(d_{11} - d_2) - 3d_{11} + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1, p_v) \tag{27}$$

for every point p_v on the path $p_{f+1} \to p_g$.

Subcase 3.2.4.

The point p_v is on the path $p_{g+1} \to p_h$. The path P_{p_f,p_v} is obtained, namely the path $(2, d_2 - 4n - 2 + d(p_1, p_v))$. Using this path, we get $q_1 = 24n + 2 - 4d_2 - 4d(p_1, p_v)$. The path P_{p_{e+1},p_v} is obtained, namely the path $(2, d_{11} - n - 2 + d(p_1, p_v))$. Using this path, we get $q_2 = d_{11} - 3n + d(p_1, p_v)$. Based on Lemma II.2 we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 12n + 2 + 4d_{11} \\ 12n^2 - 13n - 2 + 4n(d_{11} - d_2) - 3d_{11} + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1, p_v) \tag{28}$$

for every point p_v on the path $p_{g+1} \to p_h$.

The conclusion of (25), (26), (27) and (28) is $\text{expin}(p_v, D^{(2)}) \geq 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1, p_v)$ for every $v = 1, 2, \ldots, 4n+1$.

Next, we prove $\operatorname{expin}(p_v, D^{(2)}) \le 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1, p_v)$ for every $v = 1, 2, \ldots, 4n+1$. First we show that $\operatorname{expin}(p_v, D^{(2)}) = 12n^2 - n + 4n(d_{11} - d_2) + d_{11}$ and then by Lemma II.1 to guarantee that $\operatorname{expin}(p_v, D^{(2)}) \le 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1, p_v)$ for every $v = 1, 2, \ldots, 4n + 1$.

From (25) we get $\operatorname{expin}(p_1, D^{(2)}) \ge 12n^2 - n + 4n(d_{11} - d_2) + d_{11}$. Next simply show that $\operatorname{expin}(p_1, D^{(2)}) \le 12n^2 - n + 4n(d_{11} - d_2) + d_{11}$ for every $p_u, u = 1, 2, \ldots, 4n + 1$, the system of equations

$$M\mathbf{z} + \begin{bmatrix} r(P_{p_u,p_1}) \\ b(P_{p_u,p_1}) \end{bmatrix} = \begin{bmatrix} 12n - 1 + 4d_{11} \\ 12n^2 - 13n + 1 + 4n(d_{11} - d_2) - 3d_{11} + d(p_1, p_v) \end{bmatrix} \quad (29)$$

has a non-negative integer solution for the path P_{p_u,p_1}. From (29) we get $z_1 = 12n - 1 - 4d_2 - (4n - 3)r(P_{p_u,p_1}) + 4b(P_{p_u,p_1})$ and $z_2 = d_{11} - (1 - n)r(P_{p_u,p_1}) - b(P_{p_u,p_1})$.

If p_u is on $p_1 \to p_e$, then there is a path $(4, 4n - 3 - d(p_1, p_u))$. Using this path, we get $z_1 = 12n - 1 - 4(d_2 + d(p_1, p_u)) \ge 15$ since $d_2 + d(p_1, p_u) \le 2n - 1$ with $n \ge 3$ and $z_2 = d_{11} + d(p_1, p_u) - 1 \ge 1$ since $d_{11} + d(p_1, p_u) \ge n - 1$ with $n \ge 3$. If p_u is on $p_{e+1} \to p_f$, then there is a path $(3, 4n - 2 - d(p_1, p_u))$. Using this path, we get $z_1 = 16n - 4(d_2 + d(p_1, p_u)) \ge 0$ since $d_2 + d(p_1, p_u) \le 4n$ and $z_2 = d_{11} + d(p_1, p_u) - n - 1 \ge 0$ since $d_{11} + d(p_1, p_u) \ge n + 1$. If p_u is on $p_{f+1} \to p_g$, then there is a path $(2, 4n - 1 - d(p_1, p_u))$. Using this path, we get $z_1 = 20n + 1 - 4(d_2 + d(p_1, p_u)) \ge 5$ since $d_2 + d(p_1, p_u) \le 5n - 1$ with $n \ge 3$ and $z_2 = d_{11} + d(p_1, p_u) - 2n - 1 \ge 3$ since $d_{11} + d(p_1, p_u) \ge 3n + 1$ with $n \ge 3$. If p_u is on $p_{g+1} \to p_h$, then there is a path $(1, 4n - d(p_1, p_u))$. Using this path, we get $z_1 = 24n + 2 - 4(d_2 + d(p_1, p_u)) \ge 10$ since $d_2 + d(p_1, p_u) \le 5n + 1$ with $n \ge 3$ and $z_2 = d_{11} + d(p_1, p_u) - 3n - 1 \ge 2$ since $d_{11} + d(p_1, p_u) \ge 4n$ with $n \ge 3$. If p_u is on $p_{h+1} \to p_{4n+1}$, then there is a path $(0, 4n + 1 - d(p_1, p_u))$. Using this path, we get $z_1 = 28n - 3 - 4(d_2 + d(p_1, p_u)) \ge 13$ since $d_2 + d(p_1, p_u) \le 5n + 2$ with $n \ge 3$ and $z_2 = d_{11} + d(p_1, p_u) - 4n - 1 \ge 0$ since $d_{11} + d(p_1, p_u) \ge 4n + 1$.

Therefore, for every $u = 1, 2, \ldots, 4n + 1$, the system of equations (29) has a non-negative integer solution. Proposition II.1 guarantees for every $u = 1, 2, \ldots, 4n + 1$, there is a path P_{p_u,p_1} with $a = 12n - 1 - 4d_2 + 4d_{11}$ and $i = 12n^2 - 13n + 1 + 4n(d_{11} - d_2) + 4d_2 - 3d_{11} + d(p_1, p_v)$.

Therefore, $\operatorname{expin}(p_1, D^{(2)}) = 12n^2 - n + 4n(d_{11} - d_2) + d_{11}$ and by Lemma II.1 we get the conclusion that $\operatorname{expin}(p_v, D^{(2)}) \le 12n^2 - n + 4n(d_{11} - d_2) + d_{11} + d(p_1, p_v)$ for every $v = 1, 2, ..., 4n + 1$.

■

B. Non-Hamiltonian Two-coloured Digraphs with Two Cycles of Length n and $4n + 1$

Next, the two-coloured digraph discussed in this article is non-Hamiltonian two-coloured digraphs with two cycles of length n and $4n + 1$ (see Fig.2). Let the first cycle with length n be $C_1 : p_1 \to p_2 \to \cdots \to p_{n-1} \to p_n = p_1$ and the second cycle with length $4n + 1$ be $C_2 : p_1 \to p_{n+1} \to p_{n+2} \to \cdots \to p_{4n} \to p_{4n+1} \to p_1$.

Let the five red arcs in $\mathcal{D}^{(2)}$ be the first arc $p_h \to p_{h+1}$ where $1 \le h \le n - 1$ and let the second, third, fourth and fifth arcs be $p_e \to p_{e+1=f}, p_f \to p_{f+1}, p_{f+1} \to p_{f+2=g}$ and arcs $p_g \to p_{g+1}$, respectively, where $1 \le e < f < f + 1 < g \le 4n + 1$. The second, third, fourth, and fifth red arcs are laid consecutively in the second cycle (C_2). Let d_1 represent the distance from p_{e+1} to p_1, d_2 represent the distance from p_{f+1} to p_1, d_3 represent the distance from p_{g+1} to p_1, and d_4 represent the distance from p_{h+1} to p_1.

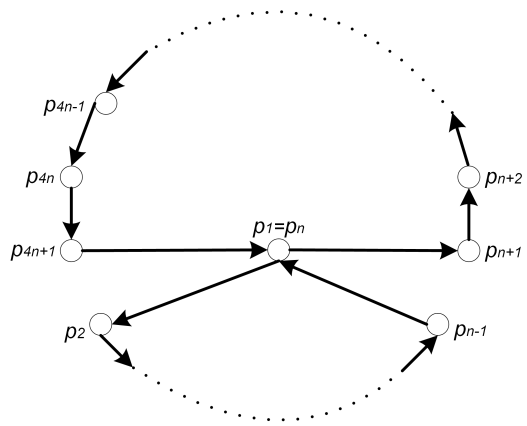

Fig. 2. Non-Hamiltonian digraph with two cycles of length n and $4n + 1$

Conjecture III.1. *Given $\mathcal{D}^{(2)}$, a non-Hamiltonian two-cycle primitive two-coloured digraph with length n and $4n + 1$. If $\mathcal{D}^{(2)}$ has four consecutive red arcs at C_2, then for every $v = 1, 2, \ldots, 4n + 1$ it follows*
$\operatorname{expin}(p_v, D^{(2)}) =$

$$\begin{cases} 16n^2 - 12n + d_3 + d(p_1, p_v), \\ \quad \text{for } d_3 \ge d_4,\ d_3 - d_4 \le 2n + 1,\ d_4 \le n - 1 \\ 16n^2 - 12n + 4n(d_4 - d_3) + d_4 + d(p_1, p_v), \\ \quad \text{for } d_3 < d_4 \\ 4n^2 - 3n + 4n(d_3 - d_4) + d_3 + d(p_1, p_v), \\ \quad \text{for } d_3 > d_4,\ d_3 - d_4 \ge 2n + 1,\ d_4 \le n - 2. \end{cases}$$

The following are the steps in making conjecture III.1. Assume that $\operatorname{expin}(p_v, \mathcal{D}^{(2)})$ for every $v = 1, 2, \ldots, 4n + 1$ is obtained using path (a_v, i_v). The step will be divided into three cases as follows.

Case 1.3 : $d_3 \ge d_4,\ d_3 - d_4 \le 2n + 1,\ d_4 \le n - 1$**.**

The first step is to show that $\operatorname{expin}(p_v, D^{(2)}) \ge 16n^2 - 12n + d_3 + d(p_1, p_v)$. Look at the P_{p_e,p_v} and P_{p_{g+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_e,p_v}) - r(C_2)b(P_{p_e,p_v})$ and $q_2 = r(C_1)b(P_{p_{g+1},p_v}) - b(C_1)r(P_{p_{g+1},p_v})$.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_e,p_v} is obtained, namely the path $(4, d_3 + d(p_1, p_v))$. Using this path, we get $q_1 = 16n - 12 - 4(d_3 + d(p_1, p_v))$. The path P_{p_{g+1},p_v} is obtained, namely the path $(0, d_3 + d(p_1, p_v))$. Using this path, we get $q_2 = d_3 + d(p_1, p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \ge M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 16n - 12 \\ 16n^2 - 28n + 12 + d_3 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\operatorname{expin}(p_v, D^{(2)}) \ge 16n^2 - 12n + d_3 + d(p_1, p_v) \quad (30)$$

for every point p_v on the path $p_1 \to p_e$.

Case 2.3 : $d_3 < d_4$.
The first step is to show that $\text{expin}(p_v, D^{(2)}) \geq 16n^2 - 12n + 4n(d_4 - d_3) + d_4 + d(p_1, p_v)$. Look at the P_{p_e,p_v} and P_{p_{h+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_e,p_v}) - r(C_2)b(P_{p_e,p_v})$ and $q_2 = r(C_1)b(P_{p_{h+1},p_v}) - b(C_1)r(P_{p_{h+1},p_v})$.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_e,p_v} is obtained, namely the path $(4, d_3 + d(p_1, p_v))$. Using this path, we get $q_1 = 16n - 12 - 4(d_3 + d(p_1, p_v))$. The path P_{p_{h+1},p_v} is obtained, namely the path $(0, d_4 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 + d(p_1, p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 16n + 4(d_4 - d_3) \\ 16n^2 - 28n + 4n(d_4 - d_3) + 4d_3 - 3d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 16n^2 - 12n + 4n(d_4 - d_3) + d_4 + d(p_1, p_v) \quad (31)$$

for every point p_v on the path $p_1 \to p_e$.

Case 3.3 : $d_3 > d_4,\ d_3 - d_4 \geq 2n + 1,\ d_4 \leq n - 2$.
The first step is to show that $\text{expin}(p_v, D^{(2)}) \geq 4n^2 - 3n + 4n(d_3 - d_4) + d_3 + d(p_1, p_v)$. Look at the P_{p_h,p_v} and P_{p_{g+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_h,p_v}) - r(C_2)b(P_{p_h,p_v})$ and $q_2 = r(C_1)b(P_{p_{g+1},p_v}) - b(C_1)r(P_{p_{g+1},p_v})$.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_h,p_v} is obtained, namely the path $(1, d_4 + d(p_1, p_v))$. Using this path, we get $q_1 = 4n - 3 - 4(d_4 + d(p_1, p_v))$. The path P_{p_{g+1},p_v} is obtained, namely the path $(0, d_3 + d(p_1, p_v))$. Using this path, we get $q_2 = d_3 + d(p_1, p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 4n - 3 + 4(d_3 - d_4) \\ 4n^2 - 7n + 3 + 4n(d_3 - d_4) - 3d_3 + 4d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 4n^2 - 3n + 4n(d_3 - d_4) + d_3 + d(p_1, p_v) \quad (32)$$

for every point p_v on the path $p_1 \to p_e$.

Furthermore, the arcs in cycle C_2 are placed alternately. In a non-Hamilton $D^{(2)}$ two-coloured digraph with five red arcs, the first arc lies in cycle C_1, namely arc $p_h \to p_{h+1}$ with $1 \leq h \leq n-1$. The second arc, third arc, fourth arc, and fifth arc are located alternately in cycle C_2, namely $p_e \to p_{e+1}$, $p_f \to p_{f+1}$, $p_{f+2} \to p_{f+3}$, and $p_{f+4=g} \to p_{g+1}$ with $1 \leq e < e+1 < f < f+1 < f+2 < f+3 < g \leq 4n+1$. Let d_1 represent the distance from p_{e+1} to p_1, d_2 represent the distance from p_{f+1} to p_1, d_3 represent the distance from p_{g+1} to p_1, and d_4 represent the distance from p_{h+1} to p_1.

Conjecture III.2. *Given $\mathcal{D}^{(2)}$, a non-Hamiltonian two-cycle primitive two-coloured digraph with length n and $4n+1$. If $\mathcal{D}^{(2)}$ has four alternating red arcs at C_2, then for every $v = 1, 2, \ldots, 4n+1$ it follows*
$\text{expin}(p_v, D^{(2)}) =$

$$\begin{cases} 16n^2 + 4n(d_3 - d_1) + d_3 + d(p_1, p_v), \\ \quad \text{for } d_3 \geq d_4,\ d_3 - d_4 \leq n + 1,\ d_4 \leq n - 1 \\ 16n^2 + 4n(d_4 - d_1) + d_4 + d(p_1, p_v), \\ \quad \text{for } d_3 < d_4 \\ 4n^2 - 3n + 4n(d_3 - d_4) + d_3 + d(p_1, p_v), \\ \quad \text{for } d_3 > d_4,\ d_3 - d_4 \geq n + 1,\ d_4 \leq n - 2. \end{cases}$$

The following are the steps in making conjecture III.2. Assume that $\text{expin}(p_v, \mathcal{D}^{(2)})$ for every $v = 1, 2, \ldots, 4n+1$ is obtained using path (a_v, i_v). The step will be divided into three cases as follows.

Case 1.4 : $d_3 \geq d_4,\ d_3 - d_4 \leq n + 1,\ d_4 \leq n - 1$.
The first step is to show that $\text{expin}(p_v, D^{(2)}) \geq 16n^2 + 4n(d_3 - d_1) + d_3 + d(p_1, p_v)$. Look at the P_{p_e,p_v} and P_{p_{g+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_e,p_v}) - r(C_2)b(P_{p_e,p_v})$ and $q_2 = r(C_1)b(P_{p_{g+1},p_v}) - b(C_1)r(P_{p_{g+1},p_v})$.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_e,p_v} is obtained, namely the path $(4, d_1 - 3 + d(p_1, p_v))$. Using this path, we get $q_1 = 16n - 4(d_1 + d(p_1, p_v))$. The path P_{p_{g+1},p_v} is obtained, namely the path $(0, d_3 + d(p_1, p_v))$. Using this path, we get $q_2 = d_3 + d(p_1, p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 16n + 4(d_3 - d_1) \\ 16n^2 + 4n(d_3 - d_1) - 16n + 4d_1 - 3d_3 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 16n^2 + 4n(d_3 - d_1) + d_3 + d(p_1, p_v) \quad (33)$$

for every point p_v on the path $p_1 \to p_e$.

Case 2.4 : $d_3 < d_4$.
The first step is to show that $\text{expin}(p_v, D^{(2)}) \geq 16n^2 + 4n(d_4 - d_1) + d_4 + d(p_1, p_v)$. Look at the P_{p_e,p_v} and P_{p_{h+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_e,p_v}) - r(C_2)b(P_{p_e,p_v})$ and $q_2 = r(C_1)b(P_{p_{h+1},p_v}) - b(C_1)r(P_{p_{h+1},p_v})$.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_e,p_v} is obtained, namely the path $(4, d_1 - 3 + d(p_1, p_v))$. Using this path, we get $q_1 = 16n - 4(d_1 + d(p_1, p_v))$. The path P_{p_{h+1},p_v} is obtained, namely the path $(0, d_4 + d(p_1, p_v))$. Using this path, we get $q_2 = d_4 + d(p_1, p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 16n + 4(d_4 - d_1) \\ 16n^2 + 4n(d_4 - d_1) - 16n + 4d_1 - 3d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 16n^2 + 4n(d_4 - d_1) + d_4 + d(p_1, p_v) \quad (34)$$

for every point p_v on the path $p_1 \to p_e$.

Case 3.4 : $d_3 > d_4,\ d_3 - d_4 \geq n + 1,\ d_4 \leq n - 2$.
The first step is to show that $\text{expin}(p_v, D^{(2)}) \geq 4n^2 - 3n + 4n(d_3 - d_4) + d_3 + d(p_1, p_v)$. Look at the P_{p_h,p_v} and P_{p_{g+1},p_v} paths and define $q_1 = b(C_2)r(P_{p_h,p_v}) - r(C_2)b(P_{p_h,p_v})$ and $q_2 = r(C_1)b(P_{p_{g+1},p_v}) - b(C_1)r(P_{p_{g+1},p_v})$.

The point p_v is on the path $p_1 \to p_e$. The path P_{p_h,p_v} is obtained, namely the path $(1, d_4 + d(p_1, p_v))$. Using this path, we get $q_1 = 4n - 3 - 4(d_4 + d(p_1, p_v))$. The path P_{p_{g+1},p_v} is obtained, namely the path $(0, d_3 + d(p_1, p_v))$. Using this path, we get $q_2 = d_3 + d(p_1, p_v)$. Based on Lemma II.2, we get

$$\begin{bmatrix} a_v \\ i_v \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 4n - 3 + 4(d_3 - d_4) \\ 4n^2 - 7n + 3 + 4n(d_3 - d_4) - 3d_3 + 4d_4 + d(p_1, p_v) \end{bmatrix}.$$

Thus

$$\text{expin}(p_v, D^{(2)}) \geq 4n^2 - 3n + 4n(d_3 - d_4) + d_3 + d\,(p_1, p_v) \tag{35}$$

for every point p_v on the path $p_1 \to p_e$.

IV. Conclusion

The inner local exponent of two-coloured digraphs with cycles length n and $4n+1$ have been carried out. The inner local exponent is specialized in two-coloured digraphs with consecutive red arcs and alternating at C_2. The theorems and conjectures show three inner local exponent patterns for two-coloured digraphs with n and $4n+1$ cycle lengths. This research is significant to complete so that generalizations can be made for cases n and $kn+1$.

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