Fixed-Time Trajectory Tracking Control of Nonholonomic WMRs Subject to Spatial Constraint

Tianjiao Chen, Zixing Zhao, Qu Zhao, Yi Yao, Runzhi Zhou and Yanling Shang

***Abstract*—The design of fixed-time trajectory tracking controller for nonholonomic wheeled mobile robots (WMRs) subject to spatial constraint is addressed in this paper. Firstly, a tan-type Barrier Lyapunov Function (BLF) that equates to the classical one for unconstrained systems is exploited. Then, by employing the adding a power integrator technique and switching control strategy, a state feedback controller is successfully constructed to forces the error dynamic states of the closed-loop system (CLS) to zero in a given fixed time without violation of the constraint. Finally, simulation results are given to confirm the efficacy of the presented control scheme.**

***Index Terms*—wheeled mobile robots, spatial constraint, fixed-time, trajectory tracking.**

I. Introduction

WHEELED mobile robots (WMRs) have been playing a crucial role in various applications such as entertainment, security, rescue missions, spacial missions and assistant health-care because of their simplicity, efficiency and flexibility [1-3]. An important feature of WMRs is that the number of control inputs is less than the number of degree of freedom, which leads to the control of WMRs challenging. As pointed out by Brockett in [4], there is not any smooth (or even continuous) time-invariant state feedback to stabilize such category of nonlinear systems. To give this difficulty a solution, a number of control approaches have been proposed, which mainly are time-varying feedback [5-7] and discontinuous time-invariant feedback [8,9] With these valid approaches, lots of important results on asymptotical control have been established over the last years, see, e.g., [10-16] and the references therein.

However, in many practical applications it be very desirable that system trajectories converge to the equilibrium in finite time because finite-time stable system retains not only faster convergence, but also better robustness and disturbance rejection properties [17-20]. Motivated by this, the finite-time control of nonholonomic systems has attained significant amount of interests and efforts over the last years [21-27]. But the existing finite-time control results suffer from two shortcomings: one is that convergence rate is relatively slow when the system states are far away from the equilibrium points, and the other is that the settling time heavily relies on initial system conditions. To address these two shortcomings, the idea of fixed-time stability that the involved settling time function is irrespective of initial system conditions was put forward in [28]. Soon afterwards, the research on fixed-time control has become a popular topic [29, 30]. As for fixed-time control of nonholonomic systems, some interesting results have recently been also reported [31]. Notwithstanding, the effect of state/output constraints is omitted in the above-mentioned works.

Note any constraint breach could result in system failure or performance deterioration, system failure or industrial accidents during operation[32–35]. Therefore, how to overcome the constraint is meaningful and crucial in practice. Motivated by the above observations, this paper focuses on solving the fixed-time trajectory tracking control of nonholonomic WMRs subject to spatial constraint which equates to the state/output constraints nonholonomic systems. The contributions are highlighted as follows. (i) Different from the existing asymptotic stabilization in [34] or finite-time stabilization in [36], fully taking into consideration of practical system requirements, both spatial constraint and fixed-time tracking are included to study the trajectory tracking problem of nonholonomic WMRs. (ii)To handle the obstacle caused by the spatial constraint, a new tan-type Barrier Lyapunov Function (BLF) that equates to the classical one for unconstrained systems is exploited. (iii) Based on the adding a power integrator technique and switching control strategy, a systematic design method is proposed to ensure the achievement of the performance requirements.

Notations. The notations adopted in this paper are fairly standard. Specifically, for a vector $z = (z_1, \ldots, z_n)^T \in \mathbb{R}^n$, define $\lceil z \rfloor^\delta$ as $\lceil z \rfloor^\delta = \mathrm{sign}(z)|z|^\delta$.

II. Problem Statement and Preliminaries

Consider the tracking control of nonholonomic WMRs in a restricted area shown in Fig.1. The following kinematic states are assumed as: (x, y) denotes the position of the center of mass of the robot, θ is the heading angle of the robot, v is the forward velocity while ω is the angular velocity of the robot,

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Tianjiao Chen is an undergraduate in the School of Automation, Nanjing Institute of Technology, Nanjing 211167, P. R. China (e-mail: 2697659505@qq.com).

Zixing Zhao is an undergraduate in the School of Automation, Nanjing Institute of Technology, Nanjing 211167, P. R. China (e-mail: zhaozixing@126.com).

Qu Zhao is an undergraduate in the School of Automation, Nanjing Institute of Technology, Nanjing 211167, P. R. China (e-mail: zhaoqu@126.com).

Yi Yao is an undergraduate in the School of Automation, Nanjing Institute of Technology, Nanjing 211167, P. R. China (e-mail: 2466451059@qq.com).

Runzhi Zhou is an undergraduate in the School of Computer engineering, Nanjing Institute of Technology, Nanjing 211167, P. R. China (e-mail: 1879177301@qq.com).

Yanling Shang is an associate professor in the School of Automation, Nanjing Institute of Technology, Nanjing 211167, P. R. China (Corresponding author, e-mail: hnnhsyl@126.com).

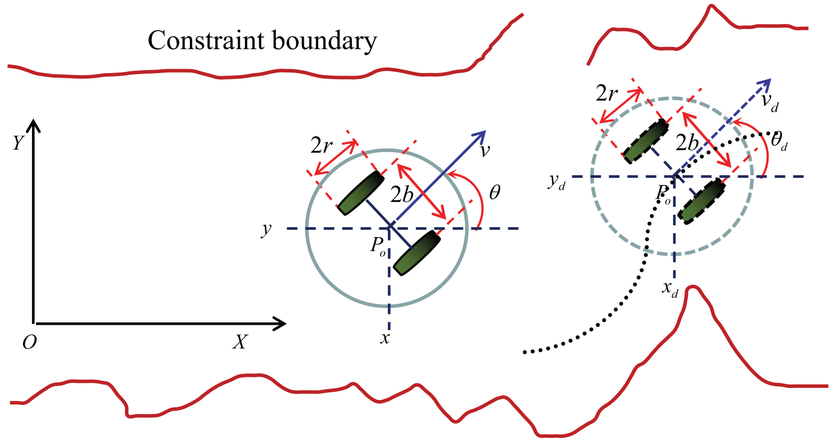


Fig. 1. Tracking control of nonholonomic WMRs.

r is the radius of the rear wheels and b is the distance between the rear wheels. Under the condition of pure rolling and non-slippage, the WMRs satisfy the following nonholonomic constraint

$$\dot{y}\cos\theta - \dot{x}\sin\theta = 0, \tag{1}$$

and it is easy to obtain that the following relationships describing the robots' motion

$$\begin{aligned} v &= \frac{v_R + v_L}{2}, \\ w &= \frac{v_R - v_L}{2b}, \end{aligned} \tag{2}$$

where system inputs v_L, v_R are the linear speeds of the left and right wheels, respectively. Further, the kinematic model of the robots is reformulated as

$$\begin{aligned} \dot{x} &= v\cos\theta, \\ \dot{y} &= v\sin\theta, \\ \dot{\theta} &= \omega, \end{aligned} \tag{3}$$

Introducing the following change of coordinates

$$\begin{aligned} &x_0 = x, \quad x_1 = y, \quad x_2 = \tan\theta, \\ &u_0 = v\cos\theta, \quad u_1 = w\sec^2\theta, \end{aligned} \tag{4}$$

system (3) is transformed into

$$\begin{aligned} \dot{x}_0 &= u_0, \\ \dot{x}_1 &= u_0 x_2, \\ \dot{x}_2 &= u_1. \end{aligned} \tag{5}$$

Similarly, the kinematic equations of target robot are represented by

$$\begin{aligned} \dot{x}_{0d} &= u_{0d}, \\ \dot{x}_{1d} &= u_{0d} x_{2d}, \\ \dot{x}_{2d} &= u_{1d}. \end{aligned} \tag{6}$$

To tackle the problem of trajectory tracking, define

$$x_{0e} = x_0 - x_{0d}, \quad x_{1e} = x_1 - x_{1d}, \quad x_{2e} = x_2 - x_{2d}, \tag{7}$$

Then, the tracking error dynamics are obtained as

$$\begin{aligned} \dot{x}_{0e} &= u_0 - u_{0d}, \\ \dot{x}_{1e} &= u_{0d} x_{2e} + (u_0 - u_{0d}) x_2, \\ \dot{x}_{2e} &= u_1 - u_{1d}. \end{aligned} \tag{8}$$

Clearly, the tracking of the spatial constrained WMRs is a state-constrained control problem. In this paper, we want to design control signals u_0, u_1 to achieve that x_{0e} x_{1e} and x_{2e} converge zero in a fixed time, and meanwhile all system tracking errors are kept in the predefined constrained regions

$$\Omega_{x_{ie}} = \{-k_i < x_{ie} < k_i\}, \quad i = 0, 1, \tag{9}$$

where k_i and k_i are positive funcitons.

The following assumptions, definitions and lemmas will serve as the basis of the coming control design and performance analysis.

Assumption 1. The reference control input $u_{0d}(t)$ satisfies that $u_{0d}^2(t) + \dot{u}_{0d}^2(t) \le m$ where constant $m > 0$, and it is also known, positive, and bounded, that is, there exist positive constants $\underline{u_{0d}}$, $\overline{u_{0d}}$ such that $0 < \underline{u_{0d}} < u_{0d}(t) < \overline{u_{0d}}$.

Assumption 2. The time-varying functions $k_i(t)$ $(i = 0, 1)$ are continuous differentiable and there are positive constants $\underline{k}_i$, $\overline{k}_i$ and $\overline{k}_{i3}$ such that $\underline{k}_i \le k_i(t) \le \overline{k}_i$, $|\dot{k}_i(t)| \le \overline{k}_{i3}$.

Definition 1[17]**.** Consider the nonlinear system

$$\dot{x} = f(t, x) \text{ with } f(t, 0) = 0, \quad x \in R^n, \tag{10}$$

where $f : R^+ \times U_0 \to R^n$ is continuous with respect to x on an open neighborhood U_0 of the origin $x = 0$. The equilibrium $x = 0$ of the system is (locally) uniformly finite-time stable if it is uniformly Lyapunov stable and finite-time convergent in a neighborhood $U \subseteq U_0$ of the origin. By "finite-time convergence," we mean: If, for any initial condition $x(t_0) \in U$ at any given initial time $t_0 \ge 0$, there is a settling time $T > 0$, such that every $x(t, t_0, x(t_0))$ of system (10) is defined with $x(t, t_0, x(t_0)) \in U/\{0\}$ for $t \in [t_0, T)$ and satisfies $\lim_{t\to T} x(t, t_0, x(t_0)) = 0$ and $x(t, t_0, x(t_0)) = 0$ for any $t \ge T$. If $U = U_0 = R^n$, the origin is a globally uniformly finite-time stable equilibrium.

Lemma 1[17]**.** Consider the nonlinear system described in (10). Suppose there is a C^1 function $V(t, x)$ defined on $\hat{U} \subseteq U_0 \times R$, where $\hat{U}$ is a neighborhood of the origin, class K functions π_1 and π_2, real numbers $c > 0$ and $0 < \alpha < 1$, for $t \in [t_0, T)$ and $x \in \hat{U}$ such that

$$\pi_1(|x|) \le V(t, x) \le \pi_2(|x|), \forall t \ge t_0, \forall x \in \hat{U},$$

and

$$\dot{V}(t,x)+cV^{\alpha}(t,x)\le 0, \forall t\ge t_0 \ , \forall x\in\hat{U}.$$

Then, the origin of (10) is finite-time stable with $T \le \frac{V^{1-\alpha}(t_0,x(t_0))}{c(1-\alpha)}$ for initial condition $x(t_0)$ in some open neighborhood $\hat{U}$ of the origin at initial time t_0. If $\hat{U}=U_0=R^n$ and π_1 and π_2 are class K_∞ functions, the origin of system (10) is globally finite-time stable.

Definition 2[28]**.** The origin of system (10) is said to be globally fixed-time stable if it is globally finite-time stable and the settling time function $T(x_0)$ is bounded, that is, there exists a positive constant T_{max} such that $T(x_0)\le T_{max}, \forall x_0\in R^n$.

Lemma 2[28]**.** Consider the nonlinear system (10). Suppose there exist a C^1, positive definite and radially unbounded function $V(x): R^n\to R$ and real numbers $c>0$, $d>0$, $0<\alpha<1$, $\gamma>1$, such that

$$\dot{V}(x)\le -cV^{\alpha}(x)-dV^{\gamma}(x), \ \ \forall x\in R^n.$$

Then, the origin of system (10) is globally fixed-time stable and the settling time $T(x_0)$ satisfies

$$T(x_0)\le T_{max}:=\frac{1}{c(1-\alpha)}+\frac{1}{d(\gamma-1)}, \ \ \forall x_0\in R^n.$$

Lemma 3[36]**.** For $x\in R$, $y\in R$, $p\ge 1$ and $c>0$ are constants, the following inequalities hold: (i) $|x+y|^p\le 2^{p-1}|x^p+y^p|$, (ii) $(|x|+|y|)^{1/p}\le |x|^{1/p}+|y|^{1/p}\le 2^{(p-1)/p}(|x|+|y|)^{1/p}$, (iii) $||x|-|y||^p\le ||x|^p-|y|^p|$, (iv) $|x|^p+|y|^p\le(|x|+|y|)^p$, (v) $|\lceil x\rceil^{1/p}-\lceil y\rceil^{1/p}|\le 2^{1-1/p}|x-y|^{1/p}$, (vi) $|[\lceil x\rceil^p-\lceil y\rceil^p|\le c|x-y||(|x-y|^{p-1}+|y|^{p-1}|$.

Lemma 4[36]**.** For any positive real numbers c,d and any real-valued function $\pi(x,y)>0$, one has

$$|x|^c|y|^d\le\frac{c}{c+d}\pi(x,y)|x|^{c+d}+\frac{d}{c+d}\pi^{-c/d}(x,y)|y|^{c+d}.$$

III. Fixed-time tracking control design

In this section, we give a constructive procedure for the design of fixed-time tracking controller for system (8). The overall controller design consists of two steps. First of all, a fixed-time tacking control law u_0 is put forward to force x_{0e} and $u_{0d}-u_{0d}$ tend to zero within a fixed time T_1. Then, under $x_{0e}=0$ and $u_{0d}-u_{0d}=0$ for $t\ge T_1$, control input u_1 is designed to force x_{1e} and x_{2e} convergent to zero a fixed time and keep in the predefined state-constrained region. In the second control stage, on time interval $[0,T_1]$, control law u_1 is reconstructed to keep that the solution of the whole closed-loop system will not escape.

A. Fixed-time tracking control of the x_{0e}-subsystem

To avoid the state x_{0e} violating the constraints, let us consider a candidate of asymmetric BLF function $V_0:\Omega_0\to\mathbb{R}$ is given as follows:

$$V_0=\frac{k_0^2}{\pi}\tan\left(\frac{\pi x_{0e}^2}{2k_0^2}\right). \tag{11}$$

Differentiating the function V_0 obtains that

$$\begin{aligned}\frac{\partial V_0}{\partial x_{0e}}&=\Lambda_0(x_{0e})x_{0e},\\ \frac{\partial V_0}{\partial k_0}&=\frac{2k_0}{\pi}\tan\left(\frac{\pi x_{0e}^2}{2k_0^2}\right)-\frac{1}{k_0}\Lambda_0(x_{0e})x_{0e}^2,\end{aligned} \tag{12}$$

with $\Lambda_0(x_{0e})$ defined as

$$\Lambda_{b0}(x_{0e})= \ \sec^2\left(\frac{\pi x_{0e}^2}{2k_0^2}\right), \tag{13}$$

Based on (12), the derivative of V_0 arrives

$$\begin{aligned}\dot{V}_0&=\frac{\partial V_0}{\partial x_{0e}}\dot{x}_{0e}+\frac{\partial V_0}{\partial k_0}\dot{k}_0\\ &=\Lambda_0(x_{0e})x_{0e}(u_0-u_{0d})+\frac{2k_0}{\pi}\tan\left(\frac{\pi x_{0e}^2}{2k_0^2}\right)\dot{k}_0\\ &\quad-\frac{1}{k_0}\Lambda_0(x_{0e})x_{0e}^2\dot{k}_0\\ &\le\Lambda_0(x_{0e})x_{0e}(u_0-u_{0d})+\frac{2}{k_0}\Lambda_0(x_{0e})x_{0e}^2|\dot{k}_0|\\ &\le\Lambda_0(x_{0e})\left(x_{0e}(u_0-u_{0d})+x_{0e}^2\varphi_0\right),\end{aligned} \tag{14}$$

where $\varphi_0\ge 2\overline{k}_{03}/\underline{k}_0$ is a positive constant.

Take

$$u_0=u_{0d}-l\lceil x_{0e}\rceil^{1-\tau}-l\lceil x_{0e}\rceil^{1+p-\tau}-x_{0e}\varphi_0, \tag{15}$$

where $l>0$, $0<\tau<1$ and $p>\tau$ are positive constants. Then, by substituting (15) into (14), one has

$$\dot{V}_0\le-\Lambda_0\left(l|x_{0e}|^{2-\tau}+l|x_{0e}|^{2+p-\tau}\right). \tag{16}$$

As a result, the following result can be established.

Theorem 1. For the x_{0e}-subsystem, suppose that $-k_0<x_{0e}(0)<k_0$, the fixed-time tracking control strategy (15) can guarantee that x_{0e} and $u_{0d}-u_{0d}$ tend to and keep zero within a fixed time T_1. Meantime, the desired state constraint $-k_0<x_{0e}(t)<k_0$ holds.

Proof. The proof is divided into two parts.

Part I: Verification of the constraint $-k_0<x_{0e}(t)<k_0$

From the definitions of V_0, we can easily verify that it is positive definite on $\Omega_{x_{0e}}$. This together with (16) renders that the origin of CLS is asymptotically stable. Therefore, for all $t\ge 0$, one has

$$V_0(t)\le V(0), \tag{17}$$

that is

$$\frac{\pi|x_{0e}|^{2-\tau}}{2k_0^{2-\tau}}\le\tan^{-1}\left(\frac{\pi(2-\tau)}{2k_0^{2-\tau}}V_0(0)\right)<\frac{\pi}{2}, \tag{18}$$

for all $t\ge 0$. As a result, the state x_{0e} will stay in the set $\Omega_{x_{0e}}$ and not violate the constraint.

Part II: Fixed-time stable analysis

Since the CLS is asymptotically stable at the origin is proved in Part I. From Definitions 1 and 2, to accomplish the fixed-time stability of the CLS, we take only to show that its bounded settling-time function exists here. Above all, by Lemma 3, it easily checks that

$$V_0\le\frac{2k_0^{2-\tau}}{\pi(2-\tau)}\tan\left(\frac{\pi|x_{0e}|^{2-\tau}}{2k_0^{2-\tau}}\right). \tag{19}$$

What is more, for all $x_{0e}\in\Omega_{x_{0e}}$, $0\le\frac{\pi|x_{0e}|^{2-\tau}}{2k_0^{2-\tau}}\le\frac{\pi}{2}$ since $2-\tau>1$. Then, according to the characteristics of tangent function, it is obtained that

$$\begin{aligned}\tan\left(\frac{\pi|x_{0e}|^{2-\tau}}{2k_0^{2-\tau}}\right)&\le\frac{\pi}{2k_0^{2-\tau}}\Lambda^{\frac{1}{2}}(x_{0e})|x_{0e}|^{2-\tau}\\ &\le\frac{\pi(2-\tau)}{2k_0^{2-\tau}}\Lambda^{\frac{1}{2}}(x_{0e})|x_{0e}|^{2-\tau},\end{aligned} \tag{20}$$

$$\begin{aligned}\tan\left(\frac{\pi|x_{0e}|^{2-\tau}}{2k_0^{2-\tau}}\right) &\le \frac{\pi}{2k_0^{2-\tau}}\Lambda(x_{0e})|x_{0e}|^{2-\tau} \\ &\le \frac{\pi(2-\tau)}{2k_0^{2-\tau}}\Lambda(x_{0e})|x_{0e}|^{2-\tau}.\end{aligned} \tag{21}$$

Noting the fact that $\Lambda(x_{0e}) \ge 1$ for all $x_{0e} \in \Omega_{x_{0e}}$ and $0 < 2/(2-\tau) < 1$, by (19), (21) and Lemma 3, it is deduced that

$$\begin{aligned}V_0^{\frac{2}{2-\tau}} &\le \frac{2k_0^{2-\tau}}{\pi(2-\tau)}\tan\left(\frac{\pi|x_{0e}|^{2-\tau}}{2k_0^{2-\tau}}\right) \\ &\le 2\Lambda(x_{0e})|x_{0e}|^2.\end{aligned} \tag{22}$$

On the other hand, observing $1 < (2+p)/(2-\tau) < 2$, by Lemma 3 and taking (19) and (20) into account, one can get

$$\begin{aligned}V_0^{\frac{2+p}{2-\tau}} &\le \frac{2k_0^{2-\tau}}{\pi(2-\tau)}\tan\left(\frac{\pi|x_{0e}|^{2-\tau}}{2k_0^{2-\tau}}\right) \\ &\le \left(\Lambda^{\frac{1}{2}}(x_{0e})|x_{0e}|^{2-\tau}\right)^{\frac{2+p}{2-\tau}} \\ &\le \Lambda^{\frac{2+p}{4-2\tau}}(x_{0e})|x_{0e}|^{2+p}\end{aligned} \tag{23}$$

Therefore, by considering (16), (22) and (23), it follows that

$$\dot{V}_0 \le -l2^{-1}V_0^{\alpha} - l2^{-\gamma}V_0^{\gamma}, \tag{24}$$

where $\alpha = 2/(2-\tau)$ and $\gamma = (2+p)/(2-\tau)$.

Thus, according to Lemma 2, we conclude that the equilibrium $x_{0e} = 0$ of the CLS is fixed-time stable and the settling time function T_1 satisfies

$$\begin{aligned}T_1 &\le \frac{2}{l(1-\alpha)} + \frac{2^{\gamma}}{l(\gamma-1)} \\ &= \frac{2(\tau-2)}{l\tau} + \frac{(2-\tau)2^{\frac{2+p}{2-\tau}}}{l(p+\tau)}\end{aligned} \tag{25}$$

Thus, the proof is completed.

B. Fixed-time tracking control of the (x_{1e}, x_{2e})-subsystem for $t \ge T_1$

In light of $u_0 - u_{0d} \equiv 0$ when $t \ge T_1$, it is easily known that the dynamic of (x_{1e}, x_{2e})-subsystem can be written as

$$\begin{aligned}\dot{x}_{1e} &= u_{0d}x_{2e}, \\ \dot{x}_{2e} &= u_1 - u_{1d}.\end{aligned} \tag{26}$$

for which, a fixed-time controller will be designed for u_1 by employing recursive technique.

Step 1. choose

$$V_1 = \frac{k_1^2}{\pi}\tan\left(\frac{\pi x_{1e}^2}{2k_1^2}\right). \tag{27}$$

Then, we have

$$\begin{aligned}\dot{V}_1 &= \frac{\partial V_1}{\partial x_{1e}}\dot{x}_{1e} + \frac{\partial V_1}{\partial k_1}\dot{k}_1 \\ &\le \Lambda(x_{1e})u_{0d}x_{1e}x_{2e} + \frac{2}{k_1}\Lambda(x_{1e})x_{1e}^2|\dot{k}_1| \\ &\le \Lambda(x_{1e})u_{0d}x_{1e}(x_{2e} - x_{2e}^*) \\ &\quad + \Lambda(x_{1e})u_{0d}x_{1e}(x_{2e} - x_{2e}^*) + \Phi(x_1)|x_{1e}|^2\varphi_1,\end{aligned} \tag{28}$$

where $\varphi_1 \ge 2k_{13}/k_{11}$ is a positive constant.

Select the virtual controller x_{2e}^* as

$$\begin{aligned}x_{2e}^* &= -\frac{1}{u_{0d}}\left(1 + \lambda + \lambda|\xi_1|^p + \varphi_1(1 + x_{1e}^2)^{1/2}\right)\lceil x_{1e}\rceil^{1-\nu} \\ &:= -\alpha_1(x_{1e})\lceil x_{1e}\rceil^{1-\nu},\end{aligned} \tag{29}$$

with design parameters$0 < \nu < 1/2$, $\lambda > 0$ and $q > \nu$ to be determined later.

Substituting (29) into (28), it can be obtained that

$$\begin{aligned}\dot{V}_1 &\le -\lambda\Lambda(x_{1e})(|x_{1e}|^{2-\nu} + |x_{1e}|^{2+p-\nu}) - |x_{1e}|^{2-\nu} \\ &\quad + \Lambda(x_{1e})x_{1e}u_{0d}(x_{2e} - x_{2e}^*).\end{aligned} \tag{30}$$

Step 2. Based on the virtual controller x_{2e}^*, we define $\xi_2 = \lceil x_{2e}\rceil^{\frac{1}{1-\nu}} - \lceil x_{2e}^*\rceil^{\frac{1}{1-\nu}}$ and choose the Lyapunov function $V_2 = V_1 + W_2$ with

$$W_2 = \int_{x_{2e}^*}^{x_{2e}} \left\lceil \lceil s\rceil^{\frac{1}{1-\nu}} - \lceil x_{2e}^*\rceil^{\frac{1}{1-\nu}} \right\rceil^{1+\nu} \mathrm{d}s. \tag{31}$$

Since

$$\begin{cases}\dfrac{\partial W_2}{\partial x_{2e}} = \lceil \xi_2\rceil^{1+\nu}, \\ \dfrac{\partial W_2}{\partial x_{1e}} = -(1+\nu)\dfrac{\partial\left(\lceil x_{2e}^*\rceil^{\frac{1}{1-\nu}}\right)}{\partial x_{1e}} \\ \qquad\qquad \times\displaystyle\int_{x_{2e}^*}^{x_{2e}} \left\lceil \lceil s\rceil^{\frac{1}{1-2\nu}} - \lceil x_{2e}^*\rceil^{\frac{1}{1-2\nu}}\right\rceil^{\nu} \mathrm{d}s\end{cases} \tag{32}$$

a direct calculation gives

$$\begin{aligned}\dot{V}_2 &\le -\lambda\Lambda(x_{1e})(|x_{1e}|^{2-\nu} + |x_{1e}|^{2+p-\nu}) - |x_{1e}|^{2-\nu} \\ &\quad + \Lambda(x_{1e})x_{1e}u_{0d}(x_{2e} - x_{2e}^*) \\ &\quad + \frac{\partial W_2}{\partial x_{1e}}\dot{x}_{1e} + \lceil \xi_{2e}\rceil^{1-2\nu}(u_1 - u_{1d}),\end{aligned} \tag{33}$$

We give the upper bound estimates for some terms in the right hand of (33).

First, by the definition of ξ_2, we have

$$\begin{aligned}|\lceil x_{2e}\rceil - \lceil x_{2e}^*\rceil| &= \left|\left(\lceil x_{2e}\rceil^{\frac{1}{1-\nu}}\right)^{r_2} - \left(\lceil x_{2e}^*\rceil^{\frac{1}{1-\nu}}\right)^{r_2}\right| \\ &\le 2^{\nu}\left|\lceil x_{2e}\rceil^{\frac{1}{1-\nu}} - \lceil x_{2e}^*\rceil^{\frac{1}{1-\nu}}\right|^{1-\nu} \\ &= 2^{\nu}|\xi_2|^{1-\nu}.\end{aligned} \tag{34}$$

Thus, from (34) and Lemma 4, we can obtain that

$$\begin{aligned}\Lambda(x_{1e})x_{1e}u_{0d}(x_{2e} - x_{2e}^*) &\le 2^{\nu}\Lambda(x_{1e})|u_{0d}x_{1e}|\xi_2|^{1-\nu} \\ &\le \frac{1}{3}|x_{1e}|^{2-\nu} + |\xi_2|^{2-\nu}h_{21},\end{aligned} \tag{35}$$

where $h_{21} \ge 0$ is a smooth function.

Secondly, note that

$$\begin{aligned}&-(1+\nu)\int_{x_{2e}^*}^{x_{2e}} \left\lceil \lceil s\rceil^{\frac{1}{1-\nu}} - \lceil x_{2e}^*\rceil^{\frac{1}{1-\nu}}\right\rceil^{\nu} \mathrm{d}s \\ &\le (1+\nu)|\xi_2|^{\nu}|x_{2e} - x_{2e}^*| \\ &\le (1+\nu)2^{\nu}|\xi_2|,\end{aligned} \tag{36}$$

and

$$\left|\frac{\partial\left(\lceil x_{2e}^*\rceil^{\frac{1}{1-\nu}}\right)}{\partial x_{1e}}\right| \le \left|\frac{\partial\left(\alpha_1^{\frac{1}{1-\nu}}\right)}{\partial x_{1e}}\right||x_{1e}| + \alpha_1 \le \gamma_{21}, \tag{37}$$

where $\gamma_{21} \geq 0$ is a smooth function.

Therefore, according to (9), (36), (37) and Lemma 4, we have

$$\frac{\partial W_2}{\partial x_{1e}} \dot{x}_{1e} \leq \frac{1}{3}|x_{1e}|^{2-\nu} + |\xi_2|^{2-\nu} h_{22}, \tag{38}$$

where $h_{22} \geq 0$ is a smooth function.

Substituting (35) and (38) into (34) yields

$$\begin{aligned} \dot{V}_2 \leq & -\lambda\Lambda(x_{1e})(|x_{1e}|^{2-\nu} + |x_{1e}|^{2+p-\nu}) \\ & + \lceil \xi_{2e} \rceil^{1+\nu}(u_1 - u_{1d}) + (h_{21} + h_{22})|\xi_2|^{2-\nu}. \end{aligned} \tag{39}$$

Designing the controller

$$u_1 = u_{1d} - (\lambda + \lambda|\xi_2|^p + h_{21} + h_{22}) \lceil \xi_2 \rceil^{1-2\nu}, \tag{40}$$

then the time derivative of V_2 becomes

$$\dot{V}_2 \leq -\lambda\Lambda(x_{1e})(|x_{1e}|^2 + |x_{1e}|^{2+p}) - \lambda(|\xi_2|^2 + |\xi_2|^{2+p}). \tag{41}$$

Consequently, the following result is obtained.

Theorem 2. If the controller u_1 of system (26) is specified by (40) then the closed-loop system is globally fixed-time stable without violating the constraint.

Proof. This proof follows the same line of that of Theorem 1.

C. *Tracking control of the* (x_{1e}, x_{2e})*-subsystem for* $[0, T_1)$

Next, it will be shown that system states of (8) will not escape to infinity in the time interval $[0, T_1)$. To proceed the coming control design, Assumption 1 is further limited as the following assumption:

Assumption 3. For u_{0d}, it further assumes that

$$u_{0d} > l\overline{k}_0^{-1-\tau} + l\overline{k}_0^{-1+p-\tau}\overline{k}_0\varphi_0. \tag{42}$$

It is easy to verify that (42) leads to $u_0(t) > 0$ for any $t \geq 0$. In addition, since u_{0d} is specified by (15), u_{0d}, u_{1d}, and x_{id} are bounded, therefore, there are positive constants d_i such that $|f_1| = |x_{2d}(u_0 - u_{0d})| < d_1$ and $|f_2| = |-u_{1d}| < d_2$. To facilitate the presentation, we rewritten system (26) as

$$\begin{aligned} \dot{x}_{1e} &= u_0 x_{2e} + f_1, \\ \dot{x}_{2e} &= u_1 + f_2, \end{aligned} \tag{43}$$

which is very similar to system (26) except for the presence of the terms f_i. But it does not cause much difficulty to use the above control design for generating a controller

$$u_1 = u_{1new}. \tag{44}$$

such that the closed-loop system states are bounded on the time interval $[0, T_1]$ and the desired state constraints are not violated.

By summarizing the above results, we can get the main results of this paper as follows:

Theorem 3. Consider the resulting closed-loop error system (8) under Assumptions 1–3, if the tracking control inputs u_0 and u_1 are actualized in the following way:

$$u_0 = u_0|_{(15)}, \tag{45}$$

$$u_1 = \begin{cases} u_1|_{(44)}, & t < T_1, \\ u_1|_{(40)}, & t \geq T_1, \end{cases} \tag{46}$$

then the states of the closed-loop error system are regulated to zero within a fixed-time while, at the same time constraint (9) is met.

IV. Simulation results

In this section, we illustrate the effectiveness of the proposed approach with the boundedness of $k_0(t) = k_1(t) = 1 + 0.1\sin 2t$, which satisfies the assumption made in this paper with $\underline{k}_0 = \underline{k}_1 = 0.9$, $\overline{k}_0 = \overline{k}_1 = 1.1$ and $\overline{k}_{03} = \overline{k}_{13} = 0.1$. To carry out the tracking simulation, the fixed time is picked as $T_1 = 2$, the control design parameters are configured as $u_{0d} = 2$, $u_{1d} = 1$, $l = 2$, $\lambda = 1$, $p = 2$ and $\tau = \nu = 1/3$. For different initial conditions: (i) $(x_{0e}(0), x_{1e}(0), x_{2e}(0))= (-0.2, 0.4, 1)$, (ii)$(x_{0e}(0), x_{1e}(0), x_{2e}(0))= (-0.4, 0.6, 5)$ and (iii) $(x_{0e}(0), x_{1e}(0), x_{2e}(0))= (-0.6, 0.9, 50)$, the responses of tracking errors are exhibited in Fig. 2. We can clearly observe that, when the initial value increases, the convergence time of the fixed-time control algorithm increase slowly and has upper constant-bounds, which demonstrates the effectiveness of the control method proposed in this paper.

V. Conclusion

This paper has studied the problem of fixed-time tracking control for nonholonomic WMRs subject to spatial constraint. Based on tan-type Barrier Lyapunov Function (BLF) and by skillfully using recursive technique, a systematic design procedure is given to render the error dynamics states to zero for any given fixed time while the constraint is not violated.

References

[1] R. M. Mahamood, "Improving the performance of adaptive PDPID control of two-link flexible robotic manipulator with ILC," *Engineering Letters*, vol.20, no.3, pp.259–270, 2012.

[2] A. Gorbenko and V. Popov, "Visual landmark selection for mobile robot navigation," *IAENG International Journal of Computer Science*, vol.40, no.3, pp.134–142, 2013.

[3] Y. L. Shang, D. H. Hou, and F. Z. Gao, "Global output feedback stabilization of nonholonomic chained form systems with communication delay," *IAENG International Journal of Applied Mathematics*, vol.46, no.3, pp. 367–371, 2016.

[4] R. W. Brockett, "Asymptotic stability and feedback stabilization," in: R.W. Brockett, R.S. Millman, and H.J. Sussmann (Eds.), *Differential Geometry Control Theory*, pp. 2961–2963, 1983.

[5] C. Samson, "Velocity and torque feedback control of a nonholonomic cart," *Advanced robot control*, Springer Berlin Heidelberg, pp. 125-151, 1991.

[6] C. Samson, "Control of chained systems application to path following and time-varying point-stabilization of mobile robots," *IEEE Transactions on Automatic Control*, vol.40, no.1, pp. 64-77, 1995.

[7] R. T. M'Closkey, R. M. Murray, "Exponential stabilization of driftless nonlinear control systems using homogeneous feedback," *IEEE Transactions on Automatic Control*, vol.42, no.2, pp. 614-628, 1997.

[8] A. Astolfi, "Discontinuous control of nonholonomic systems,"*Systems and Control Letters*, vol.27, no.1, pp. 37–45, 1996.

[9] Z. P. Jiang, "Robust exponential regulation of nonholonomic systems with uncertainties," *Automatica*, vol.36, no.2, pp.189–209, 2000.

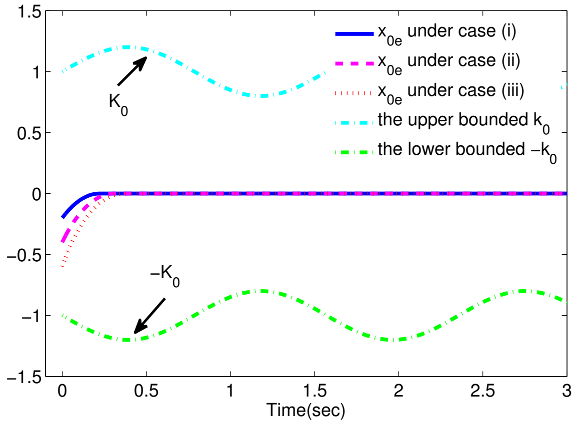
[10] S. S. Ge, Z. P. Wang, and T. H. Lee, "Adaptive stabilization of uncertain nonholonomic systems by state and output feedback," *Automatica*, vol.39, no.8, pp.1451–1460, 2003.

[11] Y. Q. Wu, G. L. Ju, and X. Y. Zheng, "Adaptive output feedback control for nonholonomic systems with uncertain chained form," *International Journal of Systems Science*, vol.41, no.12, pp. 1537–1547, 2010.

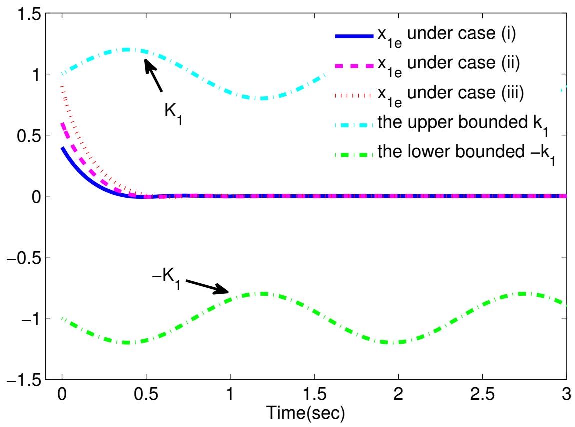
[12] F. Z. Gao, F. S. Yuan, and H. J. Yao, "Robust adaptive control for nonholonomic systems with nonlinear parameterization," *Nonlinear Analysis: Real World Applications*, vol.11, no.4, pp. 3242–3250, 2010.

[13] F. Z. Gao, F. S. Yuan, and Y. Q. Wu, "State-feedback stabilisation for stochastic non-holonomic systems with time-varying delays," *IET Control Theory and Applications*, vol.6, no.17, pp. 2593–2600, 2012.

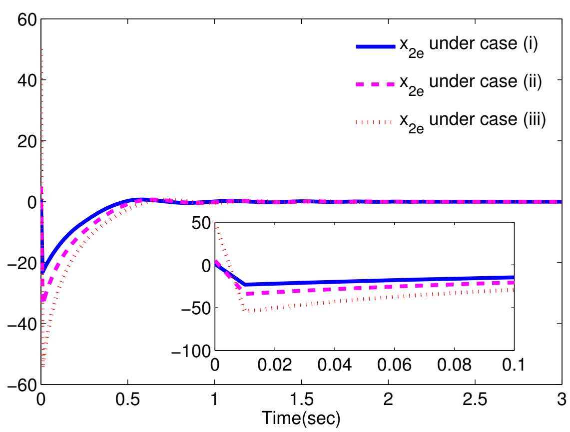
[14] F. Z. Gao, F. S. Yuan, H. J. Yao, and X. W. Mu, "Adaptive stabilization of high order nonholonomic systems with strong nonlinear drifts," *Applied Mathematical Modelling*, vol.35, no.9, pp.4222–4233, 2011.



(a) $x_{0e} = x_0 - x_{0d}$



(b) $x_{1e} = x_1 - x_{1d}$



(c) $x_{2e} = x_2 - x_{2d}$

Fig. 2. The responses of tracking errors.

[15] Y.Q. Wu, Y. Zhao, and J. B. Yu, "Global asymptotic stability controller of uncertain nonholonomic systems," *Journal of the Franklin Institute*, vol.350, no.5, pp. 1248–1263, 2013.
[16] J. Zhang and Y. Liu, "Adaptive stabilization of a class of high-order uncertain nonholonomic systems with unknown control coefficients," *International Journal of Adaptive Control and Signal Processing*, vol.27, no.5, pp.368–385, 2013.
[17] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol.38, no.3, pp. 751–766, 2000.
[18] F. Z. Gao, Y. Q. Wu, and Y. H. Liu, "Finite-time stabilization for a class of switched stochastic nonlinear systems with dead-zone input nonlinearities," *International Journal of Robust and Nonlinear Control*, vol.28, no.9, pp. 3239–3257, 2018.
[19] F. Z. Gao, Y.Q. Wu, Z. C. Zhang, and Y. H. Liu, "Finite-time stabilisation of a class of stochastic nonlinear systems with dead-zone input," *International Journal of Control*, vol.92, no.7, pp. 1541–1550, 2019.
[20] F. Z. Gao, X.C. Zhu, Y.Q. Wu, J. C. Huang, and H. S. Li, "Reduced-order observer-based saturated finite-time stabilization of high-order feedforward nonlinear systems by output feedback," *ISA Transactions*, vol.93, pp. 170–79, 2019.
[21] F. Z. Gao, Y. L. Shang, and F.S. Yuan, "Robust adaptive finite-time stabilization of nonlinearly parameterized nonholonomic systems," *Acta applicandae mathematicae*, vol.123, no.1, pp.157–173, 2013.
[22] Y. Q. Wu, F. Z. Gao, and Z. G. Liu, "Finite-time state feedback stabilization of nonholonomic systems with low-order nonlinearities," *IET Control Theory and Applications*, vol.9, no.10, pp.1553-1560, 2015.
[23] Y. L. Shang, D.H. Hou, and F. Z. Gao, "Finite-time output feedback stabilization for a class of uncertain high order nonholonomic systems," *Engineering Letters*, vol.25, no.1, pp. 39–45, 2017.
[24] F. Z. Gao, Y. Q. Wu, and Z. C. Zhang, "Finite-time stabilization of uncertain nonholonomic systems in feedforward-like form by output feedback," *ISA Transactions*, vol.59, pp.125–132, 2015.
[25] F. Z. Gao, X. C. Zhu, J. C. Huang, and X. L. Wen, "Finite-time state feedback stabilization for a class of uncertain high-order nonholonomic feedforward systems," *Engineering Letters*, vol.27, no.1, pp. 108–113, 2019.
[26] Y. L. Shang, J. C. Huang, X. X. Shi, and X. C. Zhu, "Global finite-time Stabilization for a class of upper-triangular systems subject to input saturation," *Engineering Letters*, vol. 28, no. 1, pp. 16–21, 2020.
[27] W. H. Zhang, Y. L. Shang, Q. Sun, and F. Z. Gao, "Finite-time stabilization of general stochastic nonlinear systems with application to a liquid-level system," *IAENG International Journal of Applied Mathematics*, vol. 51, no. 2, pp. 295-299, 2021.
[28] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Transactions on Automatic Control*, vol.57, no.8, pp. 2106–2110, 2012.
[29] F. Z. Gao, Y.Q. Wu, Z. C. Zhang, and Y. H. Liu, "Global fixed-time stabilization for a class of switched nonlinear systems with general powers and its application," *Nonlinear Analysis: Hybrid Systems*, vol.31, pp. 56-68, 2019.
[30] F. Z. Gao, Y.Q. Wu, and Z. C. Zhang, "Global fixed-time stabilization of switched nonlinear systems: a time-varying scaling transformation approach," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol.66, no.11, pp.1890-1894, 2019.
[31] Z. C. Yang, Y. L. Zhao and F. Z. Gao, "Fixed-time stabilization for a wheeled mobile robot with actuator dead-zones," *IAENG International Journal of Computer Science*, vol. 48, no.3, pp.514-518, 2021.
[32] K. D. Do, "Control of nonlinear systems with output tracking error constraints and its application to magnetic bearings," *International Journal of Control*, vol.83, no.6, pp. 1199–1216, 2010.
[33] Y. L. Shang, J. C. Huang, H. S. Li, and X. W. Wen, "Finite-time control design for nonholonomic mobile robots subject to spatial constraint," *IAENG International Journal of Applied Mathematics*, vol. 48, no. 4, pp. 449–454, 2018.
[34] Y. L. Shang, H. S. Li, and X. W. Wen, "Asymptotic stabilization of nonholonomic mobile robots with spatial constraint," *Engineering Letters*, vol. 26, no. 3, pp. 308–312, 2018.
[35] S. M. Zhang, Y. Gao, Z. C. Zhang, and D. G. Cao, "SDF-based tracking control for state-constrained nonholonomic systems with disturbances via relay switching control: theory and experiment," *International Journal of Adaptive Control and Signal Processing*, vol. 36, no.4, pp. 852–869, 2022.
[36] Y. L. Shang, and J. C. Huang, "Fixed-time stabilization of spatial constrained wheeled mobile robot via nonlinear mapping," *IAENG International Journal of Applied Mathematics*, vol. 50, no.4, pp.791–796, 2020.