Smoothing Connected Bézier Curves and Surfaces through Optimal Adjustment of the Control Points

Juncheng Li and Chengzhi Liu

Abstract—**In this paper, we aim to improve the order of the continuity to smooth two connected Bézier curves and surfaces when they satisfy lower-order continuity. It is necessary to adjust their control points to smooth two connected Bézier curves and surfaces. Because each adjusted control point is always expected not to be far from the original position, we use the approximate distance functions to optimize the control points that need to be changed. Based on solving bi-objective minimizations, we give the methods for smoothing two connected Bézier curves from C^{r-1} to $C^r(r=1, 2)$. Then we extend the approaches to the cases of surfaces, which include smoothing two connected Bézier surfaces in the *u* direction, in the *u* and *v*, and in the *v* direction. Some numerical examples show that the proposed methods are effective and easy to implement, so they can be used to smooth complex curves and surfaces connected by Bézier curves and surfaces.**

Index Terms—**Bézier curve and surface, connection, control points, optimization, smoothness**

I. INTRODUCTION

The Bézier curve [1] has long been an essential module for geometric design and shape representation in most computer-aided design systems. Due to the Bézier curve being established by a linear combination of the Bernstein basis functions and control points, it has simple and intuitive expressions and many excellent properties. However, using a segment of the Bézier curve is often difficult to construct complex geometric objects. People always need to connect multiple segments of Bézier curves to generate complex curves in practical applications. The same is true for the tensor product Bézier surface.

The connected Bézier curves should satisfy certain continuity conditions to make the composite curves have overall smoothness. There are two metrics for the smooth continuity between connected parametric curves: parametric continuity (also called C^n continuity) and geometric continuity (also called G^n continuity) [2]. The smooth continuity between connected Bézier curves we are concerned with here is parametric continuity. Since the connection of multiple Bézier curves can be decomposed into step-by-step connections of two Bézier curves, we only need to consider the problem of smoothing two connected Bézier curves. Let us focus on the issues: if two connected Bézier curves satisfy lower-order continuity, how to improve the order of the continuity to smooth the connected Bézier curves? Theoretically, we can adjust the control points of the connected Bézier curves to let them reach higher-order continuity, but the adjustment of control points without specific objectives may not meet the needs of practical applications. Therefore, we are interested in how to smooth two connected Bézier curves by optimizing their control points according to the given objectives.

In recent years, some objective functions have been proposed for modifying the shape of curves in many kinds of literature. The internal energy of curves is a widely used objective function, which has three forms: the stretch energy, the strain energy, and the curvature variation energy [3, 4]. Without providing an exhaustive survey, we list some literature where internal energy has been successfully applied to smooth the shape of curves [3-9]. The external energy of curves is another common objective function for modifying the shape of curves [10-12]. As an external energy of curves, the curve attractor was applied to smooth two connected curves defined by control points in [11]. The curve attractor used in [11] can be regarded as the distance between adjusted control points and their original positions. It is noted that the methods in [11] took the total distance between all adjusted control points with their original positions as the minimization target. Different from [11], we deem that the distance between each adjusted control point with its original position should be minimized synchronously, instead of minimizing the total distance between all adjusted control points with their original positions. In this paper, we consider using multi-objective distance minimization to optimize the control points for smoothing two connected Bézier curves and surfaces in this paper.

The remainder of this paper is organized as follows. In Section II, we present the methods for smoothing two connected Bézier curves by optimizing their control points. In Section III, we extend the approaches to the tensor product Bézier surfaces. In Section IV, we show some numerical examples of the proposed methods. Finally, we give a conclusion in Section V.

Manuscript received March 26, 2023; revised July 17, 2023.

This work was supported by the Hunan Provincial Natural Science Foundation of China under Grant 2021JJ30373, and the National Natural Science Foundation of China under Grant 12101225.

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II. Smoothing connected Bézier curves

Given two Bézier curves

$$C_1(t)=\sum_{i=0}^{m} b_{i,m}(t)\boldsymbol{p}_i\ ,\ C_2(t)=\sum_{j=0}^{n} b_{j,n}(t)\boldsymbol{q}_j\ , \tag{1}$$

where $0\le t\le 1$, $b_{k,M}(t)=\binom{M}{k}(1-t)^{M-k}t^k$, $\boldsymbol{p}_i$ $(i=0,1,\cdots,m)$ and $\boldsymbol{q}_j$ $(j=0,1,\cdots,n)$ are respective control points.

The problem discussed here can be described as follows: Assume that $C_1(t)$ and $C_2(t)$ satisfy $C^{r\text{-}1}(r\ge 1)$ continuity, we aim to smooth the two connected curves from $C^{r\text{-}1}$ to C^r by optimizing their control points. Since C^1 or C^2 can meet the needs of most practical engineering problems, we only consider smoothing the two connected curves from $C^{r\text{-}1}$ to C^r (r=1, 2).

A. From C^0 to C^1

Since $C_1(t)$ and $C_2(t)$ satisfy C^0 continuity, it must have

$$C_1(1)=C_2(0)\ . \tag{2}$$

By direct calculation from (1) and (2), we have

$$\boldsymbol{p}_m=\boldsymbol{q}_0\triangleq\boldsymbol{d}\ . \tag{3}$$

To make the two curves satisfy C^1 continuity, it is necessary to have

$$C_1'(1)=C_2'(0)\ . \tag{4}$$

Then from (1) and (4), we obtain

$$m(\boldsymbol{d}-\boldsymbol{p}_{m-1})=n(\boldsymbol{q}_1-\boldsymbol{d})\ . \tag{5}$$

Hence, we only need to adjust the control points $\boldsymbol{p}_{m-1}$ and $\boldsymbol{q}_1$ to make (5) hold for smoothing the two connected Bézier curves from C^0 to C^1, see Fig. 1 for an illustration.

Let $\tilde{\boldsymbol{p}}_{m-1}$ and $\tilde{\boldsymbol{q}}_1$ be the adjusted position of the control point $\boldsymbol{p}_{m-1}$ and $\boldsymbol{q}_1$, respectively. Then, from (5) we have

$$\tilde{\boldsymbol{q}}_1=\frac{m+n}{n}\boldsymbol{d}-\frac{m}{n}\tilde{\boldsymbol{p}}_{m-1}\ . \tag{6}$$

Oftentimes, each adjusted control point is expected to be close from its original position. Then we can get a bi-objective minimization as follows,

$$\min\quad \left(D_1(\tilde{\boldsymbol{p}}_{m-1}),D_2(\tilde{\boldsymbol{q}}_1)\right)^{\mathrm{T}}\ , \tag{7}$$

where $D_1(\tilde{\boldsymbol{p}}_{m-1}):=\left\|\tilde{\boldsymbol{p}}_{m-1}-\boldsymbol{p}_{m-1}\right\|^2$ represent the approximate distance between $\tilde{\boldsymbol{p}}_{m-1}$ and $\boldsymbol{p}_{m-1}$, $D_2(\tilde{\boldsymbol{q}}_1):=\left\|\tilde{\boldsymbol{q}}_1-\boldsymbol{q}_1\right\|^2$ represent the approximate distance between $\tilde{\boldsymbol{q}}_1$ and $\boldsymbol{q}_1$.

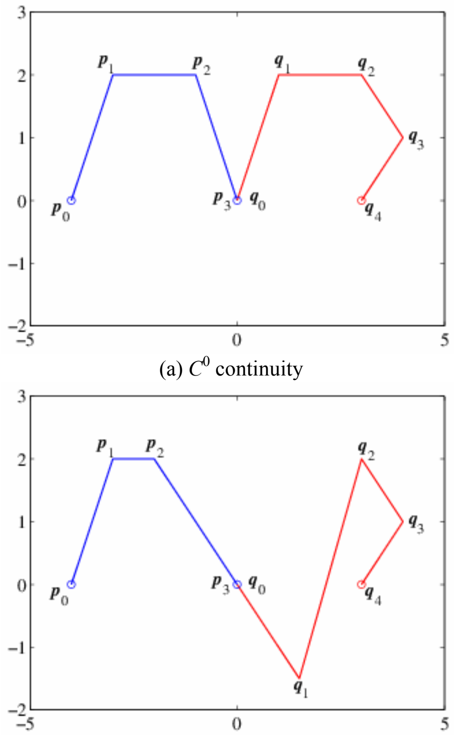
From (6), we can rewrite (7) as

$$\min\quad \left(D_1(\tilde{\boldsymbol{p}}_{m-1}),D_2(\tilde{\boldsymbol{p}}_{m-1})\right)^{\mathrm{T}}\ , \tag{8}$$

where

$$D_1(\tilde{\boldsymbol{p}}_{m-1})=\left\|\tilde{\boldsymbol{p}}_{m-1}-\boldsymbol{p}_{m-1}\right\|^2\ ,$$

$$D_2(\tilde{\boldsymbol{p}}_{m-1})=\left\|\frac{m+n}{n}\boldsymbol{d}-\frac{m}{n}\tilde{\boldsymbol{p}}_{m-1}-\boldsymbol{q}_1\right\|^2\ .$$



(a) C^0 continuity

(b) C^1 continuity

Fig. 1. The control points of two connected Bézier curves from C^0 to C^1

Generally, (8) can be transformed into a single objective minimization as follows,

$$\min\quad D(\tilde{\boldsymbol{p}}_{m-1})=\lambda D_1(\tilde{\boldsymbol{p}}_{m-1})+(1-\lambda)D_2(\tilde{\boldsymbol{p}}_{m-1})\ , \tag{9}$$

where λ $(0\le\lambda\le 1)$ is the weight. If the optimal solution of (9) is unique, then it must be the non-inferior solution of (8) [13].

Before solving (9), the value of λ needs to be determined. To this end, we consider the following single objective minimizations,

$$\min\quad D_1(\tilde{\boldsymbol{p}}_{m-1})\ , \tag{10}$$

$$\min\quad D_2(\tilde{\boldsymbol{p}}_{m-1})\ . \tag{11}$$

Clearly, the solution of (10), denoted by $\tilde{\boldsymbol{p}}_{m-1}^1$, is $\tilde{\boldsymbol{p}}_{m-1}^1=\boldsymbol{p}_{m-1}$. The solution of (11), denoted by $\tilde{\boldsymbol{p}}_{m-1}^2$, should satisfy $\frac{m+n}{n}\boldsymbol{d}-\frac{m}{n}\tilde{\boldsymbol{p}}_{m-1}^2=\boldsymbol{q}_1$, that is $\tilde{\boldsymbol{p}}_{m-1}^2=\frac{m+n}{m}\boldsymbol{d}-\frac{n}{m}\boldsymbol{q}_1$.

Then we determine the value of λ by using the sorting algorithm [13] overall described in Algorithm 1.

Algorithm 1. *Determining the value of the weight for smoothing connected Bézier curves.*

1. Compute $\delta_i^j := D_i(\tilde{\boldsymbol{p}}_{m-1}^j) - D_i(\tilde{\boldsymbol{p}}_{m-1}^i)$, $i, j = 1, 2$.

2. Let $g_i := \sum_{j=1}^{2} \delta_i^j$, $i = 1, 2$.

3. Compute $\lambda_i := g_i/(g_1 + g_2)$, $i = 1, 2$.

4. If $\lambda_1 \geq \lambda_2$*, the weight is taken as* $\lambda = \lambda_2$*, else the weight is taken as* $\lambda = \lambda_1$.

Because $D_1(\tilde{\boldsymbol{p}}_{m-1}^1) = 0$ and $D_2(\tilde{\boldsymbol{p}}_{m-1}^2) = 0$, in Step1 we have $\delta_1^1 = 0$, $\delta_1^2 = D_1(\tilde{\boldsymbol{p}}_{m-1}^2) \geq 0$, $\delta_2^1 = D_2(\tilde{\boldsymbol{p}}_{m-1}^1) \geq 0$, and $\delta_2^2 = 0$. Then in Step2, we have $g_1 = D_1(\tilde{\boldsymbol{p}}_{m-1}^2)$ and $g_2 = D_2(\tilde{\boldsymbol{p}}_{m-1}^1)$. In Step 3, we can easily get the initial weights of the two targets which satisfy that $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$. In Step 4, we choose the final weight of (9).

After determining the value of the weight, the solution of (9) can be given by the following theorem.

Theorem 1. *Given two Bézier curves* $\boldsymbol{C}_1(t)$ *and* $\boldsymbol{C}_2(t)$ *that satisfy* C^0 *continuity. For smoothing the two connected curves from* C^0 *to* C^1*, the adjusted control points* $\tilde{\boldsymbol{p}}_{m-1}$ *and* $\tilde{\boldsymbol{q}}_1$ *are given by*

$$\begin{cases} \tilde{\boldsymbol{p}}_{m-1} = \dfrac{\lambda n^2 \boldsymbol{p}_{m-1} + (1-\lambda)\left(m(m+n)\boldsymbol{d} - mn\boldsymbol{q}_1\right)}{(1-\lambda)m^2 + \lambda n^2}, \\ \tilde{\boldsymbol{q}}_1 = \dfrac{m+n}{n}\boldsymbol{d} - \dfrac{m}{n}\tilde{\boldsymbol{p}}_{m-1}, \end{cases} \tag{12}$$

where λ *is determined by Algorithm 1, and* $\boldsymbol{d} = \boldsymbol{p}_m$.

Proof. The gradients of $D(\tilde{\boldsymbol{p}}_{m-1})$ expressed in (9) can be calculated by

$$\begin{aligned} \frac{\partial D(\tilde{\boldsymbol{p}}_{m-1})}{\partial \tilde{\boldsymbol{p}}_{m-1}} &= \lambda \frac{\partial D_1(\tilde{\boldsymbol{p}}_{m-1})}{\partial \tilde{\boldsymbol{p}}_{m-1}} + (1-\lambda)\frac{\partial D_2(\tilde{\boldsymbol{p}}_{m-1})}{\partial \tilde{\boldsymbol{p}}_{m-1}} \\ &= 2\lambda(\tilde{\boldsymbol{p}}_{m-1} - \boldsymbol{p}_{m-1}) - \\ &\quad 2(1-\lambda)\frac{m}{n}\left(\frac{m+n}{n}\boldsymbol{d} - \frac{m}{n}\tilde{\boldsymbol{p}}_{m-1} - \boldsymbol{q}_1\right). \end{aligned} \tag{13}$$

Since (9) has a unique optimal solution solved by $\dfrac{\partial D(\tilde{\boldsymbol{p}}_{m-1})}{\partial \tilde{\boldsymbol{p}}_{m-1}} = \boldsymbol{0}$, then we can get (12) by computing from (13) and (6).

B. From C^1 *to* C^2

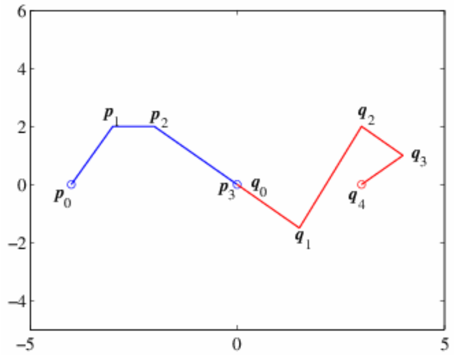
Since $\boldsymbol{C}_1(t)$ and $\boldsymbol{C}_2(t)$ satisfy C^1 continuity, (3) and (5) must hold. In order to make the two connected curves satisfy C^2, it is also necessary to have

$$\boldsymbol{C}_1''(1) = \boldsymbol{C}_2''(0). \tag{14}$$

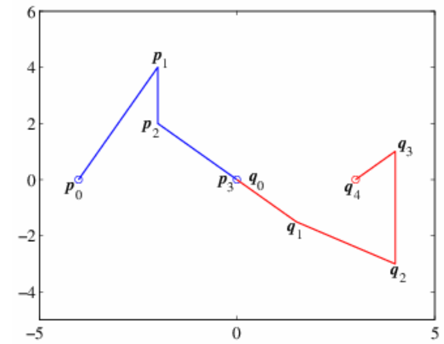
By computing from (1) and (14), we have

$$m(m-1)(\boldsymbol{p}_{m-2} - 2\boldsymbol{p}_{m-1} + \boldsymbol{p}_m) = n(n-1)(\boldsymbol{q}_0 - 2\boldsymbol{q}_1 + \boldsymbol{q}_2). \tag{15}$$

Because $\boldsymbol{p}_{m-1}$, $\boldsymbol{p}_m = \boldsymbol{q}_0$ and $\boldsymbol{q}_1$ should keep unchanged for ensuring the two curves satisfy C^1 continuity, we only need to adjust the control points $\boldsymbol{p}_{m-2}$ and $\boldsymbol{q}_2$ to make (15) hold for smoothing the two connected Bézier curves from C^1 to C^2, see Fig. 2 for an illustration.



(a) C^1 continuity



(b) C^2 continuity

Fig. 2. The control points of two connected Bézier curves from C^1 to C^2

According to the process of smoothing the two connected curves from C^0 to C^1, we give the following theorem without derivation.

Theorem 2. *Given two Bézier curves* $\boldsymbol{C}_1(t)$ *and* $\boldsymbol{C}_2(t)$ *that satisfy* C^1 *continuity. For smoothing the two connected curves from* C^1 *to* C^2*, the adjusted control points* $\tilde{\boldsymbol{p}}_{m-2}$ *and* $\tilde{\boldsymbol{q}}_2$ *are given by*

$$\begin{cases} \tilde{\boldsymbol{p}}_{m-2} = \dfrac{n(n-1)\left(\lambda n(n-1)\boldsymbol{p}_{m-2} - (1-\lambda)m(m-1)(\boldsymbol{e} - \boldsymbol{q}_2)\right)}{\lambda n^2(n-1)^2 + (1-\lambda)m^2(m-1)^2}, \\ \tilde{\boldsymbol{q}}_2 = \boldsymbol{e} + \dfrac{m(m-1)}{n(n-1)}\tilde{\boldsymbol{p}}_{m-2}, \end{cases} \tag{16}$$

where λ *is determined concerning Algorithm 1, and* $\boldsymbol{e} := \dfrac{m(m-1) + 2m(n-1) + n(n-1)}{n(n-1)}\boldsymbol{d} - \dfrac{2m(m-1) + 2m(n-1)}{n(n-1)}\boldsymbol{p}_{m-1}$.

III. Extension to the Surfaces

Given two tensor product Bézier surfaces

$$\begin{cases} \boldsymbol{S}_1(u,v) = \sum_{i=0}^{m_1}\sum_{j=0}^{n_1} b_{i,m_1}(u) b_{i,n_1}(v) \boldsymbol{p}_{i,j}, \\ \boldsymbol{S}_2(u,v) = \sum_{i=0}^{m_2}\sum_{j=0}^{n_2} b_{i,m_2}(u) b_{i,n_2}(v) \boldsymbol{q}_{i,j}, \end{cases} \tag{17}$$

where $0 \leq u, v \leq 1$, $\boldsymbol{p}_{i,j}$ and $\boldsymbol{q}_{i,j}$ are respective control points.

Since the surface has u and v directions, there are three ways to connect two Bézier surfaces: connected in the u direction, in the u and v directions, and in the v direction. In each direction, we consider smoothing the two connected surfaces from C^{r-1} to C^r (r=1, 2).

A. Smoothing connected Bézier surfaces in the u direction

Let us first consider smoothing two connected Bézier surfaces from C^0 to C^1 in the u direction. Since $\boldsymbol{S}_1(u,v)$ and $\boldsymbol{S}_2(u,v)$ satisfy C^0 continuity in the u direction, it must have

$$\boldsymbol{S}_1(u,1) = \boldsymbol{S}_2(u,0). \tag{18}$$

By computing from (17) and (18), we have

$$\begin{cases} m_1 = m_2, \\ \boldsymbol{p}_{i,n_1} = \boldsymbol{q}_{i,0} \triangleq \boldsymbol{u}_i, i = 0,1,\cdots,m_1. \end{cases} \tag{19}$$

In order to make the two surfaces satisfy C^1 continuity in the u direction, the following additional condition needs to be met by adopting the Faux method [14, 15],

$$\left.\frac{\partial \boldsymbol{S}_1(u,v)}{\partial v}\right|_{v=1} = \left.\frac{\partial \boldsymbol{S}_2(u,v)}{\partial v}\right|_{v=0}. \tag{20}$$

The physical meaning of (20) is that the cross-boundary tangencies of the two surfaces on their common boundary should be the same. Then from (17) and (20) we obtain

$$\frac{\boldsymbol{u}_i - \boldsymbol{p}_{i,n_1-1}}{n_2} = \frac{\boldsymbol{q}_{i,1} - \boldsymbol{u}_i}{n_1}, \ i = 0,1,\cdots,m_1. \tag{21}$$

That means we only need to adjust the control points $\boldsymbol{p}_{i,n_1-1}$, $\boldsymbol{q}_{i,1}$ $(i = 0,1,\cdots,m_1)$ to make (21) hold for smoothing the two connected surfaces from C^0 to C^1 in the u direction. Let $\tilde{\boldsymbol{p}}_{i,n_1-1}$ and $\tilde{\boldsymbol{q}}_{i,1}$ be the adjusted position of the control point $\boldsymbol{p}_{i,n_1-1}$ and $\boldsymbol{q}_{i,1}$, respectively. Then, from (21) we have

$$\tilde{\boldsymbol{q}}_{i,1} = \frac{n_1+n_2}{n_2}\boldsymbol{u}_i - \frac{n_1}{n_2}\tilde{\boldsymbol{p}}_{i,n_1-1}, \ i = 0,1,\cdots,m_1. \tag{22}$$

Similar to the curves, the bi-objective minimizations can be obtained as follows,

$$\min \ \left(E_{i,1}(\tilde{\boldsymbol{p}}_{i,n_1-1}), E_{i,2}(\tilde{\boldsymbol{p}}_{i,n_1-1})\right)^{\mathrm{T}}, \tag{23}$$

where

$$E_{i,1}(\tilde{\boldsymbol{p}}_{i,n_1-1}) = \left\|\tilde{\boldsymbol{p}}_{i,n_1-1} - \boldsymbol{p}_{i,n_1-1}\right\|^2,$$

$$E_{i,2}(\tilde{\boldsymbol{p}}_{i,n_1-1}) = \left\|\frac{n_1+n_2}{n_2}\boldsymbol{u}_i - \frac{n_1}{n_2}\tilde{\boldsymbol{p}}_{i,n_1-1} - \boldsymbol{q}_{i,1}\right\|^2, \ i = 0,1,\cdots,m_1.$$

We transform (23) into the following minimizations,

$$\min \ E_i(\tilde{\boldsymbol{p}}_{i,n_1-1}) := \omega_i E_{1,i}(\tilde{\boldsymbol{p}}_{i,n_1-1}) + (1-\omega_i)E_{2,i}(\tilde{\boldsymbol{p}}_{i,n_1-1}), \tag{24}$$

where $i = 0,1,\cdots,m_1$, ω_i $(0 \le \omega_i \le 1)$ are the weights.

Similarly, the values of ω_i $(i = 0,1,\cdots,m_1)$ need to be determined before solving (24). We consider the following single-objective minimizations to this end,

$$\min \ E_{1,i}(\tilde{\boldsymbol{p}}_{i,n_1-1}), \tag{25}$$

$$\min \ E_{2,i}(\tilde{\boldsymbol{p}}_{i,n_1-1}), \tag{26}$$

where $i = 0,1,\cdots,m_1$.

It is clear that the solution of (25), denoted by $\tilde{\boldsymbol{p}}^1_{i,n_1-1}$, is $\tilde{\boldsymbol{p}}^1_{i,n_1-1} = \boldsymbol{p}_{i,n_1-1}$, $i = 0,1,\cdots,m_1$. The solution of (26), denoted by $\tilde{\boldsymbol{p}}^2_{i,n_1-1}$, should satisfy $\frac{n_1+n_2}{n_2}\boldsymbol{u}_i - \frac{n_1}{n_2}\tilde{\boldsymbol{p}}^2_{i,n_1-1} = \boldsymbol{q}_{i,1}$, that is $\tilde{\boldsymbol{p}}^2_{i,n_1-1} = \frac{n_1+n_2}{n_1}\boldsymbol{u}_i - \frac{n_2}{n_1}\boldsymbol{q}_{i,1}$, $i = 0,1,\cdots,m_1$.

Then we use the sorting algorithm [13] to determine the values of ω_i $(i = 0,1,\cdots,m_1)$. The overall algorithm is described in Algorithm 2.

Algorithm 2. *Determining the values of the weights for smoothing connected Bézier surfaces in the u direction.*

1. Compute $\varepsilon^j_{k,i} := E_{k,i}(\tilde{\boldsymbol{p}}^j_{i,n_1-1}) - E_{k,i}(\tilde{\boldsymbol{p}}^k_{i,n_1-1})$, $k, j = 1,2$.

2. Let $h_{k,i} := \sum_{j=1}^{2} \varepsilon^j_{k,i}$, $k = 1,2$.

3. Compute $\xi_{k,i} := h_{k,i}/(h_{1,i} + h_{2,i})$, $k = 1,2$.

4. If $\xi_{1,i} \ge \xi_{2,i}$, *the weight is taken as* $\omega_i = \xi_{2,i}$, *else the weight is taken as* $\omega_i = \xi_{1,i}$.

Because $E_{1,i}(\tilde{\boldsymbol{p}}^1_{i,n_1-1}) = 0$ and $E_{2,i}(\tilde{\boldsymbol{p}}^2_{i,n_1-1}) = 0$, in Step1 we have $\varepsilon^1_{1,i} = 0$, $\varepsilon^2_{1,i} = E_{1,i}(\tilde{\boldsymbol{p}}^2_{i,n_1-1}) \ge 0$, $\varepsilon^1_{2,i} = E_{2,i}(\tilde{\boldsymbol{p}}^1_{i,n_1-1}) \ge 0$, and $\varepsilon^2_{2,i} = 0$. Then in Step2, we have $h_{1,i} = E_{1,i}(\tilde{\boldsymbol{p}}^2_{i,n_1-1})$ and $h_{2,i} = E_{2,i}(\tilde{\boldsymbol{p}}^1_{i,n_1-1})$. In Step 3, we can easily get the initial weights of the two targets which satisfy that $\xi_{1,i}, \xi_{2,i} \ge 0$ and $\xi_{1,i} + \xi_{2,i} = 1$. In Step 4, we choose the final weight of (24).

After determining the values of the weights, we can obtain the solution of (24) given by the following theorem.

Theorem 3. *Given two Bézier surfaces* $\boldsymbol{S}_1(u,v)$ *and* $\boldsymbol{S}_2(u,v)$ *that satisfy* C^0 *continuity in the u direction. For smoothing the two connected surfaces from* C^0 *to* C^1 *in the u direction, the adjusted control points* $\tilde{\boldsymbol{p}}_{i,n_1-1}$ *and* $\tilde{\boldsymbol{q}}_{i,1}$ *are given by*

$$\begin{cases} \tilde{\boldsymbol{p}}_{i,n_1-1} = \dfrac{\omega_i n_2^2 \boldsymbol{p}_{i,n_1-1} + (1-\omega_i)\left(n_1(n_1+n_2)\boldsymbol{u}_i - n_1 n_2 \boldsymbol{q}_{i,1}\right)}{(1-\omega_i)n_1^2 + \omega_i n_2^2}, \\ \tilde{\boldsymbol{q}}_{i,1} = \dfrac{n_1+n_2}{n_2}\boldsymbol{u}_i - \dfrac{n_1}{n_2}\tilde{\boldsymbol{p}}_{i,n_1-1}, \end{cases} \tag{27}$$

where ω_i *are determined by Algorithm 2, and* $\boldsymbol{u}_i = \boldsymbol{p}_{i,n_1}$, $i = 0,1,\cdots,m_1$.

Proof. The gradients of $E_i(\tilde{\boldsymbol{p}}_{i,n_1-1})$ expressed in (24) can be calculated by

$$\begin{aligned}\frac{\partial E_i(\tilde{\boldsymbol{p}}_{i,n_1-1})}{\partial \tilde{\boldsymbol{p}}_{i,n_1-1}} &= \omega_i \frac{\partial E_{1,i}(\tilde{\boldsymbol{p}}_{i,n_1-1})}{\partial \tilde{\boldsymbol{p}}_{i,n_1-1}} + (1-\omega_i)\frac{\partial E_{2,i}(\tilde{\boldsymbol{p}}_{i,n_1-1})}{\partial \tilde{\boldsymbol{p}}_{i,n_1-1}} \\ &= 2\omega_i(\tilde{\boldsymbol{p}}_{i,n_1-1} - \boldsymbol{p}_{i,n_1-1}) - \\ &\quad 2(1-\omega_i)\frac{n_1}{n_2}\left(\frac{n_1+n_2}{n_2}\boldsymbol{u}_i - \frac{n_1}{n_2}\tilde{\boldsymbol{p}}_{i,n_1-1} - \boldsymbol{q}_{i,1}\right),\end{aligned} \tag{28}$$

where $i = 0,1,\cdots,m_1$.

Since (24) has a unique optimal solution solved by $\frac{\partial E_i(\tilde{\boldsymbol{p}}_{i,n_1-1})}{\partial \tilde{\boldsymbol{p}}_{i,n_1-1}} = \boldsymbol{0}$, then we can get (27) by computing from (28) and (22).

Then let us consider smoothing connected Bézier surfaces from C^1 to C^2 in the u direction. Since $\boldsymbol{S}_1(u,v)$ and $\boldsymbol{S}_2(u,v)$ satisfy C^1 continuity in the u direction, (19) and (21) must hold. In order to make the two surfaces satisfy C^2 continuity in the u direction, the additional condition can be simplified to [14, 15]

$$\left.\frac{\partial^2 \boldsymbol{S}_1(u,v)}{\partial v^2}\right|_{v=1} = \left.\frac{\partial^2 \boldsymbol{S}_2(u,v)}{\partial v^2}\right|_{v=0}. \tag{29}$$

The physical meaning of (29) is that the second-order cross-boundary tangencies of the two surfaces on their common boundary should be the same. By computing from (17) and (29), we have

$$\begin{aligned}&n_1(n_1-1)(\boldsymbol{p}_{i,n_1-2} - 2\boldsymbol{p}_{i,n_1-1} + \boldsymbol{p}_{i,n_1}) \\ &= n_2(n_2-1)(\boldsymbol{q}_{i,0} - 2\boldsymbol{q}_{i,1} + \boldsymbol{q}_{i,2}),\end{aligned} \tag{30}$$

where $i = 0,1,\cdots,m_1$.

Because $\boldsymbol{p}_{i,n_1-1}$, $\boldsymbol{p}_{i,n_1} = \boldsymbol{q}_{i,0}$, $\boldsymbol{q}_{i,1}$ $(i = 0,1,\cdots,m_1)$ should keep unchanged for ensuring the two surfaces satisfy C^1 continuity, we only need to adjust the control points $\boldsymbol{p}_{i,n_1-2}$ and $\boldsymbol{q}_{i,2}$ $(i = 0,1,\cdots,m_1)$ to make (30) hold for smoothing the two connected surfaces from C^1 to C^2 in the u direction.

According to the process of smoothing two connected surfaces from C^0 to C^1 in the u direction, we give the following theorem without derivation.

Theorem 4. *Given two Bézier surfaces $\boldsymbol{S}_1(u,v)$ and $\boldsymbol{S}_2(u,v)$ that satisfy C^1 continuity in the u direction. For smoothing the two connected surfaces from C^1 to C^2 in the u direction, the adjusted control points $\tilde{\boldsymbol{p}}_{i,n_1-2}$ and $\tilde{\boldsymbol{q}}_{i,2}$ are given by*

$$\begin{cases}\tilde{\boldsymbol{p}}_{i,n_1-2} = \dfrac{n_2(n_2-1)\left(\omega_i n_2(n_2-1)\boldsymbol{p}_{i,n_1-2} - (1-\omega_i)n_1(n_1-1)(\boldsymbol{f}_i - \boldsymbol{q}_{i,2})\right)}{\omega_i n_2^2(n_2-1)^2 + (1-\omega_i)n_1^2(n_1-1)^2}, \\ \tilde{\boldsymbol{q}}_{i,2} = \boldsymbol{f}_i + \dfrac{n_1(n_1-1)}{n_2(n_2-1)}\tilde{\boldsymbol{p}}_{i,n_1-2},\end{cases} \tag{31}$$

where ω_i are determined referring to Algorithm 2, and

$$\boldsymbol{f}_i := \frac{n_1(n_1-1) + 2n_1(n_2-1) + n_2(n_2-1)}{n_2(n_2-1)}\boldsymbol{u}_i - \frac{2n_1(n_1-1) + 2n_1(n_2-1)}{n_2(n_2-1)}\boldsymbol{p}_{i,n_1-1},\ i = 0,1,\cdots,m_1.$$

B. Smoothing connected Bézier surfaces in the other two directions

Similar to the derivation process of smoothing two connected surfaces from C^{r-1} to C^r (r=1, 2) in the u direction, the following conclusions about smoothing two connected surfaces from C^{r-1} to C^r (r=1, 2) in the other two directions can be proved.

Theorem 5. *Given two Bézier surfaces $\boldsymbol{S}_1(u,v)$ and $\boldsymbol{S}_2(u,v)$ that satisfy C^0 continuity in the u and v directions. For smoothing the two connected surfaces from C^0 to C^1 in the u and v directions, the adjusted control points $\tilde{\boldsymbol{p}}_{i,n_1-1}$ and $\tilde{\boldsymbol{q}}_{1,j}$ are given by*

$$\begin{cases}\tilde{\boldsymbol{p}}_{i,n_1-1} = \dfrac{\omega_i m_2^2 \boldsymbol{p}_{i,n_1-1} + (1-\omega_i)\left(n_1(n_1+m_2)\boldsymbol{v}_i - n_1 m_2 \boldsymbol{q}_{1,j}\right)}{(1-\omega_i)n_1^2 + \omega_i m_2^2}, \\ \tilde{\boldsymbol{q}}_{1,j} = \dfrac{n_1+m_2}{m_2}\boldsymbol{v}_i - \dfrac{n_1}{m_2}\tilde{\boldsymbol{p}}_{i,n_1-1},\end{cases} \tag{32}$$

where ω_i are determined referring to Algorithm 2, and $\boldsymbol{v}_i = \boldsymbol{p}_{i,n_1}$, $i = j = 0,1,\cdots,m_1$.

Theorem 6. *Given two Bézier surfaces $\boldsymbol{S}_1(u,v)$ and $\boldsymbol{S}_2(u,v)$ that satisfy C^1 continuity in the u and v directions. For smoothing the two connected surfaces from C^1 to C^2 in the u and v directions, the adjusted control points $\tilde{\boldsymbol{p}}_{i,n_1-2}$ and $\tilde{\boldsymbol{q}}_{2,j}$ are given by*

$$\begin{cases}\tilde{\boldsymbol{p}}_{i,n_1-2} = \dfrac{m_2(m_2-1)\left(\omega_i m_2(m_2-1)\boldsymbol{p}_{i,n_1-2} - (1-\omega_i)n_1(n_1-1)(\boldsymbol{g}_i - \boldsymbol{q}_{2,j})\right)}{\omega_i m_2^2(m_2-1)^2 + (1-\omega_i)n_1^2(n_1-1)^2}, \\ \tilde{\boldsymbol{q}}_{2,j} = \boldsymbol{g}_i + \dfrac{n_1(n_1-1)}{m_2(m_2-1)}\tilde{\boldsymbol{p}}_{i,n_1-2},\end{cases} \tag{33}$$

where ω_i are determined referring to Algorithm 2, and

$$\boldsymbol{g}_i := \frac{n_1(n_1-1) + 2n_1(m_2-1) + m_2(m_2-1)}{m_2(m_2-1)}\boldsymbol{v}_i - \frac{2n_1(n_1-1) + 2n_1(m_2-1)}{m_2(m_2-1)}\boldsymbol{p}_{i,n_1-1},\ i = j = 0,1,\cdots,m_1.$$

Theorem 7. *Given two Bézier surfaces $\boldsymbol{S}_1(u,v)$ and $\boldsymbol{S}_2(u,v)$ that satisfy C^0 continuity in the v direction. For smoothing the two connected surfaces from C^0 to C^1 in the v direction, the adjusted control points $\tilde{\boldsymbol{p}}_{m_1-1,j}$ and $\tilde{\boldsymbol{q}}_{i,j}$ are given by*

$$\begin{cases} \tilde{\boldsymbol{p}}_{m_1-1,j} = \dfrac{\omega_j m_2^2 \boldsymbol{p}_{m_1-1,j} + (1-\omega_j)\left(m_1(m_1+m_2)\boldsymbol{r}_j - m_1 m_2 \boldsymbol{q}_{1,j}\right)}{(1-\omega_j)m_1^2 + \omega_j m_2^2}, \\ \tilde{\boldsymbol{q}}_{1,j} = \dfrac{m_1+m_2}{m_2}\boldsymbol{r}_j - \dfrac{m_1}{m_2}\tilde{\boldsymbol{p}}_{m_1-1,j}, \end{cases} \tag{34}$$

where ω_j *are determined referring to Algorithm 2, and* $\boldsymbol{r}_j = \boldsymbol{p}_{m_1,j}$, $j = 0,1,\cdots,n_1$.

Theorem 8. *Given two Bézier surfaces* $\boldsymbol{S}_1(u,v)$ *and* $\boldsymbol{S}_2(u,v)$ *that satisfy* C^1 *continuity in the v direction. For smoothing the two connected surfaces from* C^1 *to* C^2 *in the v direction, the adjusted control points* $\tilde{\boldsymbol{p}}_{m_1-2,j}$ *and* $\tilde{\boldsymbol{q}}_{2,j}$ *are given by*

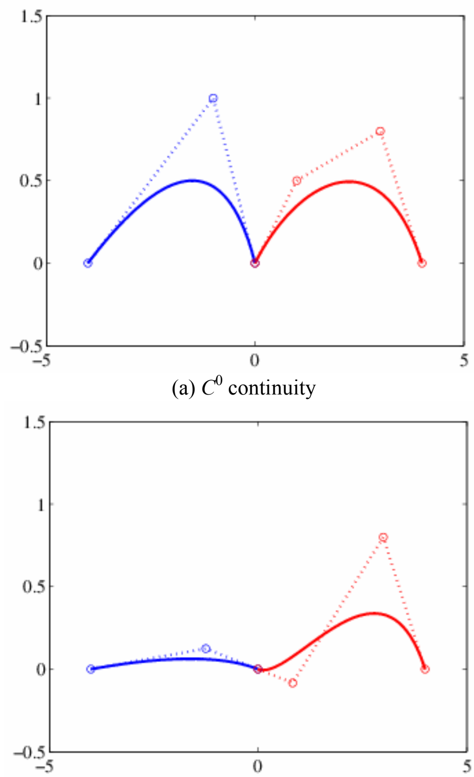
$$\begin{cases} \tilde{\boldsymbol{p}}_{m_1-2,j} = \dfrac{m_2(m_2-1)\left(\omega_j m_2(m_2-1)\boldsymbol{p}_{m_1-2,j} - (1-\omega_j)m_1(m_1-1)(\boldsymbol{h}_j - \boldsymbol{q}_{2,j})\right)}{\omega_j m_2^2(m_2-1)^2 + (1-\omega_j)m_1^2(m_1-1)^2}, \\ \tilde{\boldsymbol{q}}_{2,j} = \boldsymbol{h}_j + \dfrac{m_1(m_1-1)}{m_2(m_2-1)}\tilde{\boldsymbol{p}}_{m_1-2,j}, \end{cases} \tag{35}$$

where ω_j *are determined referring to Algorithm 2, and*

$$\boldsymbol{h}_j := \frac{m_1(m_1-1) + 2m_1(m_2-1) + m_2(m_2-1)}{m_2(m_2-1)}\boldsymbol{r}_j - \frac{2m_1(m_1-1) + 2m_1(m_2-1)}{m_2(m_2-1)}\boldsymbol{p}_{m_1-1,j}, \quad j = 0,1,\cdots,n_1.$$

IV. NUMERICAL EXAMPLES

In this section, we present several examples to illustrate the effectiveness of the proposed methods.



(a) C^0 continuity

(b) C^1 continuity

Fig. 3. Smoothing two connected Bézier curves from C^0 to C^1

Example 1. Given a quadratic Bézier curve $\boldsymbol{C}_1(t)$ with control points

$$\boldsymbol{p}_0 = (-4,0),\ \boldsymbol{p}_1 = (-1,1),\ \boldsymbol{p}_2 = (0,0),$$

and a cubic Bézier curve $\boldsymbol{C}_2(t)$ with control points

$$\boldsymbol{q}_0 = (0,0),\ \boldsymbol{q}_1 = (1,1/2),$$
$$\boldsymbol{q}_2 = (3,4/5),\ \boldsymbol{q}_3 = (4,0).$$

It implies that the two curves satisfy C^0 continuity. Let us smooth the two connected Bézier curves from C^0 to C^1. From (12), we can obtain the adjusted control points are

$$\tilde{\boldsymbol{p}}_1 = (-5/4,1/8),\ \tilde{\boldsymbol{q}}_1 = (5/6,-1/12).$$

The original and smoothed curves generated by the proposed method are shown in Fig. 3.

Example 2. Given a cubic Bézier curve $\boldsymbol{C}_1(t)$ with control points

$$\boldsymbol{p}_0 = (-4,0,0),\ \boldsymbol{p}_1 = (-3,1,1),$$
$$\boldsymbol{p}_2 = (-1,1,0),\ \boldsymbol{p}_3 = (0,0,-1),$$

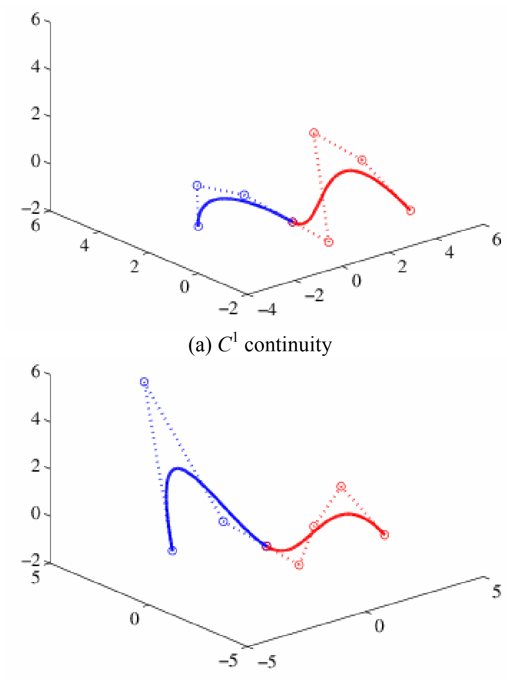
and a quartic Bézier curve $\boldsymbol{C}_2(t)$ with control points

$$\boldsymbol{q}_0 = (0,0,-1),\ \boldsymbol{q}_1 = (3/4,-3/4,-7/4),$$
$$\boldsymbol{q}_2 = (3,2,1),\ \boldsymbol{q}_3 = (4,1,0),\ \boldsymbol{q}_4 = (5,0,-2).$$

It implies that the two curves satisfy C^1 continuity. Let us smooth the two connected Bézier curves from C^1 to C^2. From (16), we can obtain the adjusted control points are

$$\tilde{\boldsymbol{p}}_1 = (-1,5,9/2),\ \tilde{\boldsymbol{q}}_2 = (2,0,-3/4).$$

The original and smoothed connection curves generated by the proposed method are shown in Fig. 4.



(a) C^1 continuity

(b) C^2 continuity

Fig. 4. Smoothing two connected Bézier curves from C^1 to C^2

Example 3. Given a 2×3 Bézier surface $\boldsymbol{S}_1(u,v)$ with control points

$$\boldsymbol{p}_{0,0} = (-5,-5,-50),\ \boldsymbol{p}_{0,1} = (-4,-5,-40),$$
$$\boldsymbol{p}_{0,2} = (-3,-5,-20),\ \boldsymbol{p}_{0,3} = (-2,-5,-14),$$
$$\boldsymbol{p}_{1,0} = (-5,-4,-40),\ \boldsymbol{p}_{1,1} = (-4,-4,-20),$$

$$p_{1,2}=(-3,-4,-12)\,,\ p_{1,3}=(-2,-4,-10)\,,$$
$$p_{2,0}=(-5,-3,-35)\,,\ p_{2,1}=(-4,-3,-18)\,,$$
$$p_{2,2}=(-3,-3,-10)\,,\ p_{2,3}=(-2,-3,-8)\,,$$

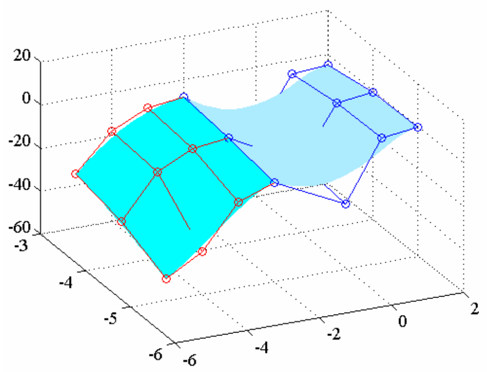
and a 2×3 Bézier surface $S_2(u,v)$ with control points

$$q_{0,0}=(-2,-5,-14)\,,\ q_{0,1}=(0,-5,-30)\,,$$
$$q_{0,2}=(1,-5,-2)\,,\ q_{0,3}=(2,-5,0)\,,$$
$$q_{1,0}=(-2,-4,-10)\,,\ q_{1,1}=(0,-4,-28)\,,$$
$$q_{1,2}=(1,-4,-3)\,,\ q_{1,3}=(2,-4,-1)\,,$$
$$q_{2,0}=(-2,-3,-8)\,,\ q_{2,1}=(0,-3,-32)\,,$$
$$q_{2,2}=(1,-3,-6)\,,\ q_{2,3}=(2,-3,-5)\,.$$

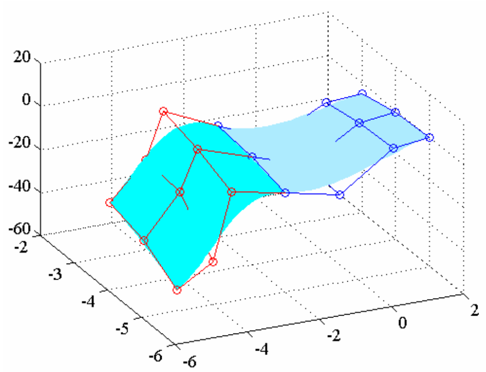
It implies that the two surfaces satisfy C^0 continuity in the u direction. Let us smooth the two connected Bézier surfaces from C^0 to C^1 in the u direction. The adjusted control points computed by the proposed method are

$$\tilde{p}_{0,2}=(-7/2,-5,-9)\,,\ \tilde{p}_{1,2}=(-7/2,-4,-2)\,,$$
$$\tilde{p}_{2,2}=(-7/2,-3,3)\,,\ \tilde{q}_{0,1}=(-1/2,-5,-19)\,,$$
$$\tilde{q}_{1,1}=(-1/2,-4,-18)\,,\ \tilde{q}_{2,1}=(-1/2,-3,-19)\,.$$

Fig. 5 shows the original and smoothed connection surfaces generated by the proposed method.



(a) C^0 continuity



(b) C^1 continuity

Fig. 5. Smoothing two connected Bézier surfaces from C^0 to C^1 in the u direction

Example 4. Given a 3×2 Bézier surface $S_1(u,v)$ with control points

$$p_{0,0}=(-5,-5,-50)\,,\ p_{0,1}=(-5,-4,-40)\,,$$
$$p_{0,2}=(-5,-3,-35)\,,\ p_{1,0}=(-4,-5,-40)\,,$$
$$p_{1,1}=(-4,-4,-20)\,,\ p_{1,2}=(-4,-3,-18)\,,$$
$$p_{2,0}=(-3,-5,-25)\,,\ p_{2,1}=(-3,-4,-12)\,,$$
$$p_{2,2}=(-3,-3,-10)\,,\ p_{3,0}=(-2,-5,-14)\,,$$
$$p_{3,1}=(-2,-4,-10)\,,\ p_{3,2}=(-2,-3,-8)\,,$$

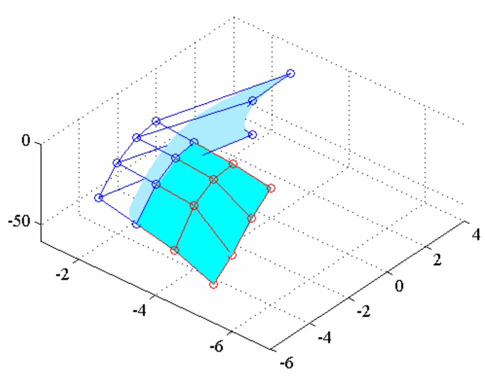
and a 3×2 Bézier surface $S_2(u,v)$ with control points

$$q_{0,0}=(-5,-3,-35)\,,\ q_{0,1}=(-5,-2,-30)\,,$$
$$q_{0,2}=(-1,-4,0)\,,\ q_{1,0}=(-4,-3,-18)\,,$$
$$q_{1,1}=(-4,-2,-16)\,,\ q_{1,2}=(0,-3,-12)\,,$$
$$q_{2,0}=(-3,-3,-10)\,,\ q_{2,1}=(-3,-2,-8)\,,$$
$$q_{2,2}=(1,-3,-6)\,,\ q_{3,0}=(-2,-3,-8)\,,$$
$$q_{3,1}=(-2,-2,-6)\,,\ q_{3,2}=(3,-3,-5)\,.$$

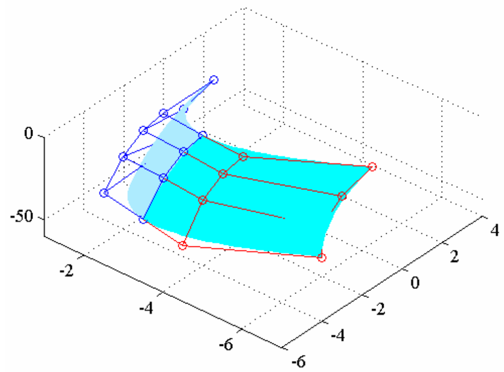
It implies that the two surfaces satisfy C^1 continuity in the u direction. Let us smooth the two connected Bézier surfaces from C^1 to C^2 in the u direction. The adjusted control points computed by the proposed method are

$$\tilde{p}_{0,0}=(-3,-13/2,-35)\,,\ \tilde{p}_{1,0}=(-2,-6,-30)\,,$$
$$\tilde{p}_{2,0}=(-1,-6,-39/2)\,,\ \tilde{p}_{3,0}=(1/2,-6,-27/2)\,,$$
$$\tilde{q}_{0,2}=(-3,-5/2,-15)\,,\ \tilde{q}_{1,2}=(-2,-2,-22)\,,$$
$$\tilde{q}_{2,2}=(-1,-2,-23/2)\,,\ \tilde{q}_{3,2}=(1/2,-2,-11/2)\,.$$

Fig. 6 shows the original and smoothed connection surfaces generated by the proposed method.



(a) C^1 continuity



(b) C^2 continuity

Fig. 6. Smoothing two connected Bézier surfaces from C^1 to C^2 in the u direction

Example 5. Given a 2×3 Bézier surface $S_1(u,v)$ with control points

$$p_{0,0}=(-5,-5,-50)\,,\ p_{0,1}=(-4,-5,-40)\,,$$
$$p_{0,2}=(-3,-5,-20)\,,\ p_{0,3}=(-2,-5,-14)\,,$$
$$p_{1,0}=(-5,-4,-40)\,,\ p_{1,1}=(-4,-4,-20)\,,$$
$$p_{1,2}=(-3,-4,-12)\,,\ p_{1,3}=(-2,-4,-10)\,,$$
$$p_{2,0}=(-5,-3,-35)\,,\ p_{2,1}=(-4,-3,-18)\,,$$
$$p_{2,2}=(-3,-3,-10)\,,\ p_{2,3}=(-2,-3,-8)\,,$$

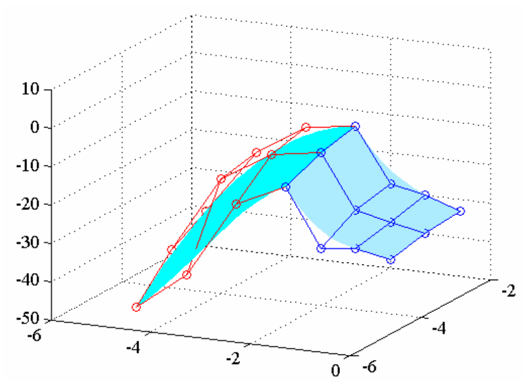
and 3×2 Bézier surface $S_2(u,v)$ with control points

$$q_{0,0}=(-2,-5,-14),\ q_{0,1}=(-2,-4,-10),$$
$$q_{0,2}=(-2,-3,-8),\ q_{1,0}=(-2,-4,-35),$$
$$q_{1,1}=(-2,-3,-30),\ q_{1,2}=(-2,-2,-28),$$
$$q_{2,0}=(-2,-3,-40),\ q_{2,1}=(-2,-2,-38),$$
$$q_{2,2}=(-2,-1,-36),\ q_{3,0}=(-2,-2,-48),$$
$$q_{3,1}=(-2,-1,-46),\ q_{3,2}=(-2,0,-45).$$

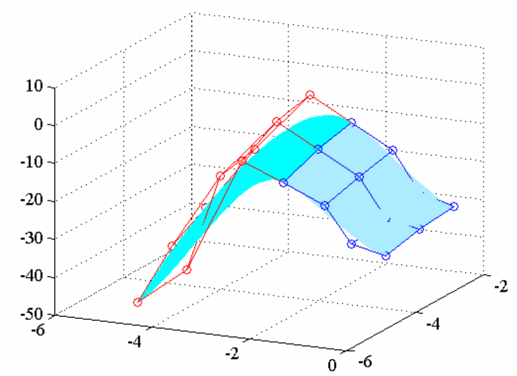
It implies that the two surfaces satisfy C^0 continuity in the u and v directions. Let us smooth the two connected Bézier surfaces from C^0 to C^1 in the u and v directions. The adjusted control points computed by the proposed method are

$$\tilde{p}_{0,2}=(-5/2,-11/2,-13/2),\ \tilde{p}_{1,2}=(-5/2,-9/2,-1),$$
$$\tilde{p}_{1,3}=(-5/2,-7/2,1),\ \tilde{q}_{1,0}=(-3/2,-9/2,-43/2),$$
$$\tilde{q}_{1,1}=(-3/2,-7/2,-19),\ \tilde{q}_{1,2}=(-3/2,-5/2,-17).$$

The original and smoothed connection surfaces generated by the proposed method are shown in Fig. 7.



(a) C^0 continuity



(b) C^1 continuity

Fig. 7. Smoothing two connected Bézier surfaces from C^0 to C^1 in the u and v directions

Example 6. Given a 3×2 Bézier surface $S_1(u,v)$ with control points

$$p_{0,0}=(-5,-5,-48),\ p_{0,1}=(-5,-4,-40),$$
$$p_{0,2}=(-5,-3,-35),\ p_{1,0}=(-4,-5,-40),$$
$$p_{1,1}=(-4,-4,-20),\ p_{1,2}=(-4,-3,-18),$$
$$p_{2,0}=(-3,-5,-25),\ p_{2,1}=(-3,-4,-12),$$
$$p_{2,2}=(-3,-3,-10),\ p_{3,0}=(-2,-5,-12),$$
$$p_{3,1}=(-2,-4,-10),\ p_{3,2}=(-2,-3,-8),$$

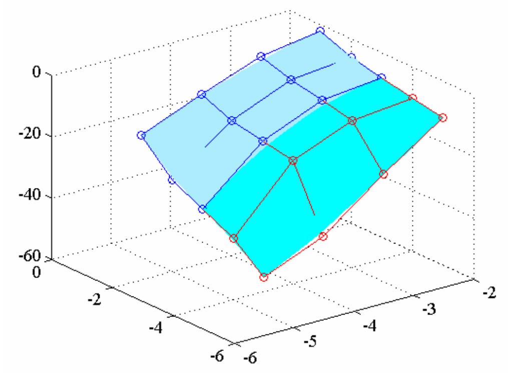
and a 2×3 Bézier surface $S_2(u,v)$ with control points

$$q_{0,0}=(-5,-3,-35),\ q_{0,1}=(-4,-3,-18),$$
$$q_{0,2}=(-3,-3,-10),\ q_{0,3}=(-2,-3,-8),$$
$$q_{1,0}=(-5,-2,-30),\ q_{1,1}=(-4,-2,-16),$$
$$q_{1,2}=(-3,-2,-8),\ q_{1,3}=(-2,-2,-6),$$
$$q_{2,0}=(-5,-1,-20),\ q_{2,1}=(-4,-1,-12),$$
$$q_{2,2}=(-3,-1,-5),\ q_{2,3}=(-2,-1,-2).$$

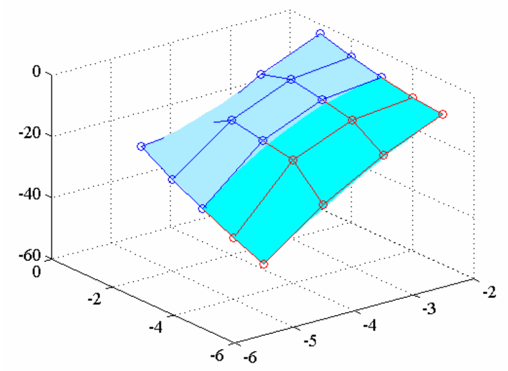
It implies that the two surfaces satisfy C^1 continuity in the u and v directions. Let us smooth the two connected Bézier surfaces from C^1 to C^2 in the u and v directions. The adjusted control points computed by the proposed method are

$$\tilde{p}_{0,0}=(-5,-5,-44),\ \tilde{p}_{1,0}=(-4,-5,-30),$$
$$\tilde{p}_{2,0}=(-3,-5,-19),\ \tilde{p}_{3,0}=(-2,-5,-11),$$
$$\tilde{q}_{2,0}=(-5,-1,-24),\ \tilde{q}_{2,1}=(-4,-1,-22),$$
$$\tilde{q}_{2,2}=(-3,-1,-11),\ \tilde{q}_{2,3}=(-2,-1,-3).$$

The original and smoothed connection surfaces generated by the proposed method are shown in Fig. 8.



(a) C^1 continuity



(b) C^2 continuity

Fig. 8. Smoothing two connected Bézier surfaces from C^1 to C^2 in the u and v directions

Example 7. Given a 3×2 Bézier surface $S_1(u,v)$ with control points

$$p_{0,0}=(-5,-5,-50),\ p_{0,1}=(-5,-4,-40),$$
$$p_{0,2}=(-5,-3,-35),\ p_{1,0}=(-4,-5,-40),$$
$$p_{1,1}=(-4,-4,-20),\ p_{1,2}=(-4,-3,-18),$$
$$p_{2,0}=(-3,-5,-25),\ p_{2,1}=(-3,-4,-12),$$
$$p_{2,2}=(-3,-3,-10),\ p_{3,0}=(-2,-5,-14),$$
$$p_{3,1}=(-2,-4,-10),\ p_{3,2}=(-2,-3,-8),$$

and a 3×2 Bézier surface $S_2(u,v)$ with control points

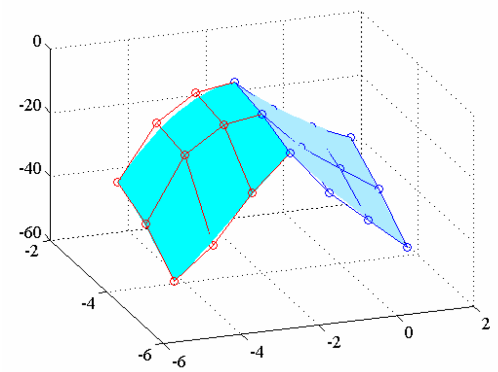
$$q_{0,0}=(-2,-5,-14),\ q_{0,1}=(-2,-4,-10),$$
$$q_{0,2}=(-2,-3,-8),\ q_{1,0}=(-1,-5,-28),$$

$$q_{1,1} = (-1,-4,-22),\ q_{1,2} = (-1,-3,-18),$$
$$q_{2,0} = (0,-5,-38),\ q_{2,1} = (0,-4,-30),$$
$$q_{2,2} = (0,-3,-25),\ q_{3,0} = (1,-5,-48),$$
$$q_{3,1} = (1,-4,-38),\ q_{3,2} = (1,-3,-30).$$

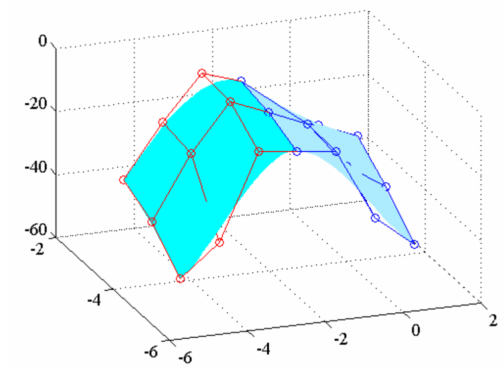
It implies that the two surfaces satisfy C^0 continuity in the v direction. Let us smooth the two connected Bézier surfaces from C^0 to C^1 in the v direction. The adjusted control points computed by the proposed method are

$$\tilde{p}_{2,0} = (-3,-5,-25/2),\ \tilde{p}_{2,1} = (-3,-4,-5),$$
$$\tilde{p}_{2,2} = (-3,-3,-4),\ \tilde{q}_{1,0} = (-1,-5,-31/2),$$
$$\tilde{q}_{1,1} = (-1,-4,-15),\ \tilde{q}_{1,2} = (-1,-3,-12).$$

Fig. 9 shows the original and smoothed connection surfaces generated by the proposed method.



(a) C^0 continuity



(b) C^1 continuity

Fig. 9. Smoothing two connected Bézier surfaces from C^0 to C^1 in the v direction

Example 8. Given a 2×3 Bézier surface $S_1(u,v)$ with control points

$$p_{0,0} = (-5,-5,-48),\ p_{0,1} = (-4,-5,-40),$$
$$p_{0,2} = (-3,-5,-20),\ p_{0,3} = (-2,-5,-16),$$
$$p_{1,0} = (-5,-4,-40),\ p_{1,1} = (-4,-4,-20),$$
$$p_{1,2} = (-3,-4,-12),\ p_{1,3} = (-2,-4,-10),$$
$$p_{2,0} = (-5,-3,-35),\ p_{2,1} = (-4,-3,-18),$$
$$p_{2,2} = (-3,-3,-10),\ p_{2,3} = (-2,-3,-8),$$

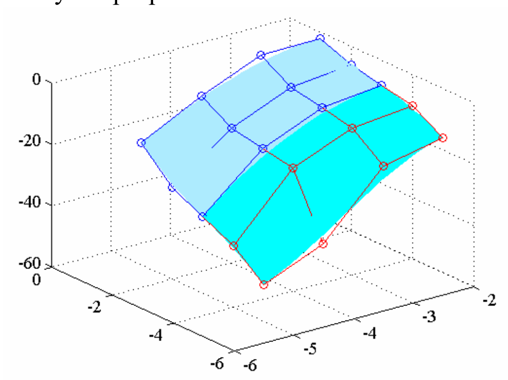
and a 2×3 Bézier surface $S_2(u,v)$ with control points

$$q_{0,0} = (-5,-3,-35),\ q_{0,1} = (-4,-3,-18),$$
$$q_{0,2} = (-3,-3,-10),\ q_{0,3} = (-2,-3,-8),$$
$$q_{1,0} = (-5,-2,-30),\ q_{1,1} = (-4,-2,-16),$$
$$q_{1,2} = (-3,-2,-8),\ q_{1,3} = (-2,-2,-6),$$
$$q_{2,0} = (-5,-1,-20),\ q_{2,1} = (-4,-1,-10),$$
$$q_{2,2} = (-3,-1,-2),\ q_{2,3} = (-2,-1,-2).$$

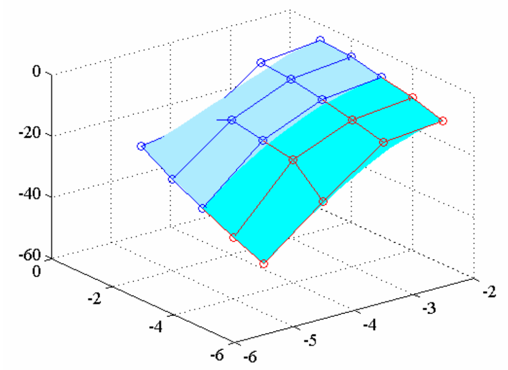
It implies that the two surfaces satisfy C^1 continuity in the v direction. Let us smooth the two connected Bézier surfaces from C^1 to C^2 in the v direction. The adjusted control points computed by the proposed method are

$$\tilde{p}_{0,0} = (-5,-5,-44),\ \tilde{p}_{0,1} = (-4,-5,-29),$$
$$\tilde{p}_{0,2} = (-3,-5,-15),\ \tilde{p}_{0,3} = (-2,-5,-13),$$
$$\tilde{q}_{2,0} = (-5,-1,-24),\ \tilde{q}_{2,1} = (-4,-1,-21),$$
$$\tilde{q}_{2,2} = (-3,-1,-7),\ \tilde{q}_{2,3} = (-2,-1,-5).$$

Fig. 10 shows the original and smoothed connection surfaces generated by the proposed method.



(a) C^1 continuity



(b) C^2 continuity

Fig. 10. Smoothing two connected Bézier surfaces from C^1 to C^2 in the v direction

V. Conclusion

In this paper, we focus on smoothing connected Bézier curves and surfaces from lower-order continuity to higher continuity by optimizing their control points. Specifically, we present the methods for smoothing two connected Bézier curves from C^{r-1} to C^r (r=1, 2). Then we extend the methods to two connected Bézier surfaces. The cases of surfaces include smoothing two connected Bézier surfaces in the u direction, smoothing two connected Bézier surfaces in the u and v directions, and smoothing two connected Bézier surfaces in the v direction. All the methods are proposed on the base of bi-objective minimizations. These methods can make each adjusted control point not far from its original position as much as possible. Numerical examples show the effectiveness of the proposed methods. Since people always need to connect Bézier curves and surfaces to construct complex curves and surfaces, the proposed methods would help improve the smooth continuity of connected Bézier curves and surfaces, which is often encountered in computer aided design. However, we do not present the methods of directly smoothing two connected Bézier curves and surfaces from C^0 to C^2. We have tried to use the proposed ideology to inquire into this problem, but the effect is unsatisfactory. This will be our next study issue.

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