A Variable Structure Model Reference Adaptive Control For MIMO Systems

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Abstract— **A Variable Structure Model Reference Adaptive Controller using state variables is proposed for multi input–multi output systems. Adaptation law is of variable structure type and switching functions are designed based on stability requirements. Global exponential stability is proved based on Lyapunov criterion. Transient behavior is analyzed using sliding mode control and shows perfect model following at a finite time.**

Keywords— Adaptive control, MIMO system, Model reference, Variable structure.

I. INTRODUCTION

Model Reference Adaptive Controller with conventional continuous adaptation laws has been investigated extensively in the literatures in two main branches: one assuming full state accessibility [7], and the other assuming accessibility of input and output [7]. Continuous adaptation laws are in the form of pure integral actions and have some problems such as:
Transient behavior is difficult to analysis.
Only global (but not asymptotic) stability has been guaranteed.
Undesirable transient responses and tracking performance.
Lack of robustness.
The variable structure systems (VSS) have been studied in great details in the literatures [4, 6, 8]. The basic concept of the variable structure control is that of sliding mode control. Switching control functions are generally designed to generate sliding surfaces, or sliding modes, in the state space [8]. When this is attained, the switching functions keep the trajectory on the sliding surfaces and the closed loop system becomes insensitive, to a certain extent, to parameter variations and disturbances.
The variable structure model reference adaptive controller has been investigated in the literatures in two main branches: one assuming full state accessibility [5], and the other assuming accessibility of input and output [2,3,9,10]. variable structure systems (VSS) have been studied in great details in the literatures.

II. PROBLEM DEFINITION

Consider a linear time variant plant with unknown parameters, which their bounds are known. Let the plant be of n-th order with accessible states and described by the differential equation

$$\dot{X} = AX + BU \tag{1}$$

where $n \times n$ matrix A is unknown and of full rank, $n \times m$ matrix B is unknown and partitioned as
$B = \begin{bmatrix} b_1 & b_2 & ... & b_m \end{bmatrix}$. U is a m-dimensional control vector, and (A,B) is controllable.
The reference model is characterized by the linear time invariant differential equation

$$\dot{X}_M = A_M X_M + B_M V \tag{2}$$

where A_M is a $n \times n$ asymptotically stable matrix, B_M is a $n \times m$ known matrix and partitioned as
$B_M = \begin{bmatrix} b_{M1} & b_{M2} & ... & b_{Mm} \end{bmatrix}$, and V is a m-dimensional input vector with bounded elements. The purpose is to find control U such that the state error

$$e = X - X_M \tag{3}$$

exponentially tends to zero in a finite time.

III. VARIABLE STRUCTURE ADAPTIVE CONTROLLER

The matrix B_m of the reference model can be chosen as

$$B_M = B Q^* \tag{4}$$

where Q^* is a known $m \times m$ diagonal matrix. It is further assumed that an unknown $m \times n$ matrix θ^* exists such that

$$A + B \Theta^* = A_M \tag{5}$$

A. Stability, And Switching Function Design

The control U to the plant, is generated introducing control law

$$U = \Phi \, (\Psi \, X + V) \tag{6}$$

where m×n feedback matrix Ψ, with the elements ψ_{ij}, and m×m matrix Φ, with the elements ϕ_k are adjusted using VS approach by designing switching functions ψ_{ij} and ϕ_k as described in the followings.

Subtracting equation (2) from (1), and using equations (4), (5) and (6), the error equation is obtained as

$$\dot{e} = A_M e + B_M \, (\Psi - \Theta^*) X + B_M \, (Q^{*-1} - \Phi^{-1}) \, U \tag{7}$$

Consider a Lyapunov function of the form

$$\Lambda = e^T P e \tag{8}$$

where P is a positive definite symmetric matrix which satisfies the Lyaponov equation

$$A_M^T P + P A_M = -Q_0 \tag{9}$$

where Q_0 is a positive definite symmetric matrix.

Differentiating (8) with respect to time along the trajectory (7) yields

$$\dot{\Lambda} = -e^T Q_0 e + 2 e^T P B_M \, (\Psi - \Theta^*) \, X$$
$$+ 2 e^T \, P B_M \, (Q^{*-1} - \Phi^{-1}) \, U$$

Assuming

$$\hat{U} = (\Psi \, X + V) \; , \quad U = \Phi \, \hat{U}$$

We have

$$\dot{\Lambda} = -e^T Q_0 e$$
$$+ 2 \sum_{i=1}^{m} \left[\, e^T P b_{M\,i} \, \sum_{j=1}^{n} (\psi_{ij} - \theta_{ij}^*) \, x_j \, \right]$$
$$+ 2 \sum_{k=1}^{m} \left[\, e^T P b_{M\,k} \, (q_k^{*-1} \, \phi_k - 1) \, \hat{u}_k \, \right] \tag{10}$$

Now, introducing the switching functions ψ_{ij} as

$$\psi_{ij} = -\overline{\theta_{ij}} \, \text{sgn} \, (e^T P b_{M\,i} \, x_j), \qquad \overline{\theta_{ij}} > \left| \theta_{ij}^* \right| \tag{11}$$

And the switching functions ϕ_k as

$$\phi_k = -\overline{q_k} \, \text{sgn} \, (e^T P b_{M\,k} \, \hat{u}_k) \, \text{sgn}(q^*{}_k), \quad \overline{q_k} > \left| q_k^* \right| \tag{12}$$

and substitute into (10), yields

$$\dot{\Lambda} = -e^T Q_0 e$$
$$- 2 \sum_{i=1}^{m} \left\{ \, \sum_{j=1}^{n} \left[\, \overline{\theta_{ij}} \left| e^T P b_{M\,i} \, x_j \right| + \theta_{ij}^* \, (e^T P b_{M\,i} \, x_j) \, \right] \right\}$$
$$- 2 \sum_{k=1}^{m} \left[\, \overline{q_k} \left| q_k^* \right|^{-1} \left| e^T P b_{M\,k} \, \hat{u}_k \right| \right.$$
$$\left. + \left(e^T P b_{M\,k} \, \hat{u}_k \right) \text{sgn}(q^*{}_k) \, \right] \tag{13}$$

The terms in the summations are always positive, therefore $\dot{\Lambda} < 0$ and regarding (8) it can be concluded that $\| e \|$ decreases at least exponentially.

B. Existence Of Sliding Mode

Here it is shown that the hyper surfaces

$$S_k = e^T P \, b_{M\,k} = 0 \tag{14}$$

are always sliding hyper surfaces for the system. Let examine the following reaching conditions [15, 9].

$$S_k \dot{S}_k < 0, \qquad \left| \dot{S}_k \right| > 0 \tag{15}$$

Regarding equations (7) and (11), we have

$$S_k \dot{S}_k = S_k \, b_{M\,k}{}^T P A_M e + S_k \, b_{M\,k}{}^T P B_M \, (\Psi - \Theta^*) \, X$$
$$+ S_k \, b_{M\,k}{}^T P B_M \, (Q^{*-1} - \Phi^{-1}) \, U$$
$$= S_k \, b_{M\,k}{}^T P A_M e$$
$$+ S_k \sum_{i=1}^{m} \left[\, b_{M\,k}{}^T P \, b_{M\,i} \, \sum_{j=1}^{n} (\psi_{ij} - \theta_{ij}^*) \, x_j \, \right]$$
$$+ S_k \sum_{i=1}^{m} \left[\, b_{M\,k}{}^T P \, b_{M\,i} \, (q_i^{*-1} \, \phi_i - 1) \, \hat{u}_i \, \right]$$

Substituting from (11) and (12)

$$S_k \dot{S}_k = S_k \, b_{M\,k}{}^T P A_M e$$
$$+ S_k \sum_{i=1}^{m} \left[\, b_{M\,k}{}^T P \, b_{M\,i} \, \sum_{j=1}^{n} \left[-\overline{\theta_{ij}} \, \text{sgn} \, (S_i \, x_j) - \theta_{ij}^* \right] x_j \, \right]$$
$$+ S_k \sum_{i=1}^{m} \left\{ b_{M\,k}{}^T P b_{M\,i} \left[q_i^{*-1} (-\overline{q_i} \, \text{sgn}(S_i \, \hat{u}_i) \, \text{sgn}(q_i^*) - 1 \right] \hat{u}_i \right\}$$

$$S_k \dot{S}_k = S_k \, b_{Mk}{}^T PA_M e$$
$$- |S_k| \sum_{i=1}^{m} \left[\, b_{Mk}{}^T \, P \, b_{Mi} \, \text{sgn} \, (S_i S_k) \sum_{j=1}^{n} \right.$$
$$\left[\overline{\theta_{ij}} + \theta_{ij}^* \, \text{sig}(e^T P \, b_{Mj}) \right] |x_j| \, \Big]$$
$$- |S_k| \sum_{i=1}^{m} \left[\, b_{Mk}{}^T \, P \, b_{Mi} \, \text{sgn} \, (S_i S_k) \right.$$
$$\left[\left| q_i^* \right|^{-1} \overline{q_i} - 1 \right] |\hat{u}_i| \, \Big]$$
$$\left\langle \, |S_k| \left\langle \, \varepsilon_1 \|e\| - \sum_{i=1}^{m} \left[\, b_{Mk}{}^T \, P \, b_{Mi} \, \text{sgn} \, (S_i S_k) \sum_{j=1}^{n} \right. \right. \right.$$
$$\left[\overline{\theta_{ij}} + \theta_{ij}^* \, \text{sig}(e^T P \, b_{Mj}) \right] |x_j| \, \Big]$$
$$- \sum_{i=1}^{m} \left[\, b_{Mk}{}^T \, P \, b_{Mi} \, \text{sgn} \, (S_i S_k) \right.$$
$$\left[\left| q_i^* \right|^{-1} \overline{q_i} - 1 \right] |\hat{u}_i| \, \Big] \, \Big\rangle$$

$$(16)$$

where ε_1 is a positive constant. Now we note that

$$S_k \dot{S}_i = b_{Mk}{}^T Pe \, b_{Mi}{}^T Pe = b_{Mk}{}^T Pe \, e^T Pb_{Mi}$$

The matrix ee^T is a positive semi-definite matrix, hence $Pee^T P$ is also positive semi-definite, and therefore $S_k \dot{S}_i = b_{Mk}{}^T Pe \, e^T Pb_{Mi}$ should has the same sign as $b_{Mk}{}^T P \, b_{Mi}$ or should be zero. Hence the terms in the summation in (16) are always positive or partly zero, since $\|e\|$ exponentially tends to zero, and if we define a new vector

$$Z^T = \left[X^T \; \hat{U}^T \right]$$

it can be concluded that if we have

$$\|Z(t)\| > \lambda > 0, \quad \forall t > t_0 \tag{17}$$

then there exists $T > t_0$ such that

$$S_k \dot{S}_k < 0, \quad \forall t \geq T \tag{18}$$

Condition $\left| \dot{S}_k \right| > 0$ in the relation (15) is considered here. One can writes

$$\dot{S}_k = b_{Mk}{}^T P\dot{e}$$
$$= b_{Mk}{}^T PA_M e + b_{Mk}{}^T PB_M (\Psi - \Theta^*) X$$
$$+ b_{Mk}{}^T PB_M (Q^{*-1} - \Phi^{-1}) U$$

$$= S_k \, b_{Mk}{}^T PA_M e$$
$$- \sum_{i=1}^{m} \left[\, b_{Mk}{}^T \, P \, b_{Mi} \sum_{j=1}^{n} (\psi_{ij} - \theta_{ij}^*) \, x_j \, \right]$$
$$- \sum_{i=1}^{m} \left[\, b_{Mk}{}^T \, P \, b_{Mi} \, (q_i^{*-1} \phi_i - 1) \, \hat{u}_i \, \right]$$

Substituting from (11) and (12)

$$\dot{S}_k = b_{Mk}{}^T PA_M e$$
$$- \sum_{i=1}^{m} \left[\, b_{Mk}{}^T \, P \, b_{Mi} \sum_{j=1}^{n} \left[\overline{\theta_{ij}} \, \text{sgn} \, (S_i \, x_j) + \theta_{ij}^* \right] x_j \, \right]$$
$$- \sum_{i=1}^{m} \left\{ b_{Mk}{}^T P b_{Mi} \left[q_i^{*-1} (\overline{q_i} \, \text{sgn}(S_i \, \hat{u}_i) \, \text{sgn}(q_i^*) + 1 \right] \hat{u}_i \right\}$$

$$(19)$$

since $\|e\|$ exponentially tends to zero, and consider to equation (16), it can be seen at least one of the terms in the summation is nonzero and we have

$$\left| \dot{S} \right| > 0, \quad \text{for } t > T \tag{20}$$

Relations (18) and (20) mean that for all t >T, the surface

$$S_k = e^T P \, b_{Mk} = 0 \quad \text{is a sliding surface.}$$

C. Average Control

When sliding mode is occurred, actually the average control $\overline{U}$ can be used [8], which elements are the outputs of m first order filters

$$\tau_i \, \overline{u}_i + \overline{u}_i = u_i \quad, \; i = 1, 2, \dots, m \tag{21}$$

where time constants τ_i are sufficiently small.
The block diagram is presented in figure 1.

IV. SIMULATION RESULTS

In this section, simulation results are presented to show the performance of the proposed schemes and comparing these with the conventional schemes. The example consists of a time variant system with parameter variation.

Example

Consider a time-variant system described by

$$\dot{X} = AX + BU \tag{22}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -9.81(1+.1\sin t) \\ .1\sin t & .0000157(1+.1\sin t) & .1\sin t \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ -\alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad 0 \langle \alpha \langle 2 \tag{23}$$

Choose a reference model described by equation

$$\dot{X}_M = A_M X_M + B_M V \tag{24}$$

where

$$A_M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -9.8 \\ .85 & 1 & -5 \end{bmatrix}, \quad B_M = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{25}$$

hence, from (4) and (5) it can be concluded that

$$Q^* = \begin{bmatrix} 1/\alpha & 0 \\ 0 & 1/\alpha \end{bmatrix}, \quad \psi_{11}^* = \psi_{12}^* = 0,$$

$$\psi_{13}^* = -0.981\sin(t), \quad \psi_{21}^* = 0.85 - 0.1\sin(t),$$

$$\psi_{22}^* = 1 - 0.0000157(1 - 0.1\sin(t)),$$

$$\psi_{23}^* = -5 - 0.1\sin(t) \tag{26}$$

therefore, with regard to equation (11) and (12) we can choose

$$\overline{\Theta} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1.1 & 5.2 \end{bmatrix}, \quad \overline{q_1} = \overline{q_2} = 1.1 \tag{27}$$

Thus the VS-MRA Controller can be designed using equations (11) and (6) for unfiltered, or equations (11) and (30-31) for filtered control. System was simulated, responses was compared with the responses of system with conventional adaptation law as $\dot{\theta} = -B^T p e X^T$, and control law

$U = \theta X + Q^* V$. The input signal was

.

$$V = \begin{bmatrix} 1 + \sin(2\pi t) \\ 1 + \sin(2\pi t) \end{bmatrix},$$

and initial conditions was $X(0) = [0 \; 0 \; 0]^T$, and

$X_M(0) = [0 \; 0 \; 0]^T$. Responses are presented in figure 2, for filtered control.

V. CONCLUSIONS

A variable structure model reference adaptive controller has been proposed for MIMO systems. Structure of the switching functions was designed based on exponential stability requirements. Magnitude of the switching functions then can be determined using reaching conditions of sliding mode.

Relation (5) is used to determine the bounds on elements θ_{ij}^*, and these elements must be limited.

This controller has some significant advantages compared to the conventional model reference adaptive controller. Global exponential stability is proved without requirements on persistence of excitation. Then it was shown that, it is always possible to introduce sliding mode into the system. Transient behavior was analyzed and showed perfect model following at a finite time. Insensitivity with respect to input disturbances was investigated and showed preference to the conventional schemes. Simulation was presented to clear the theoretical results.

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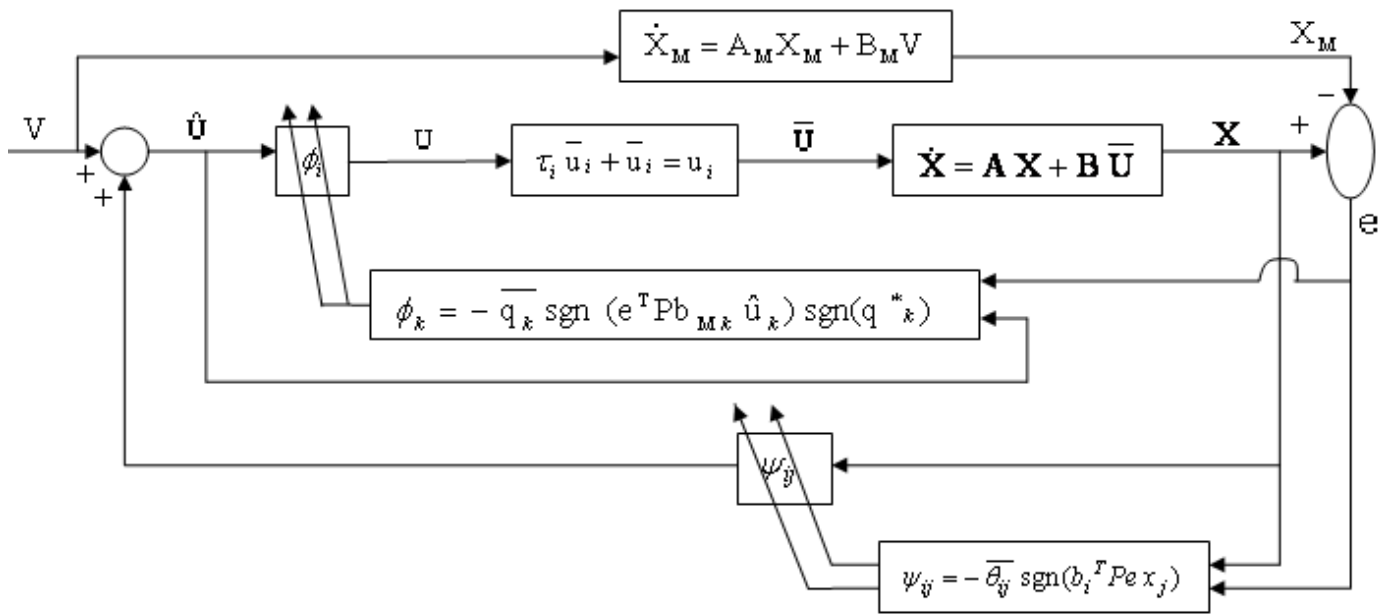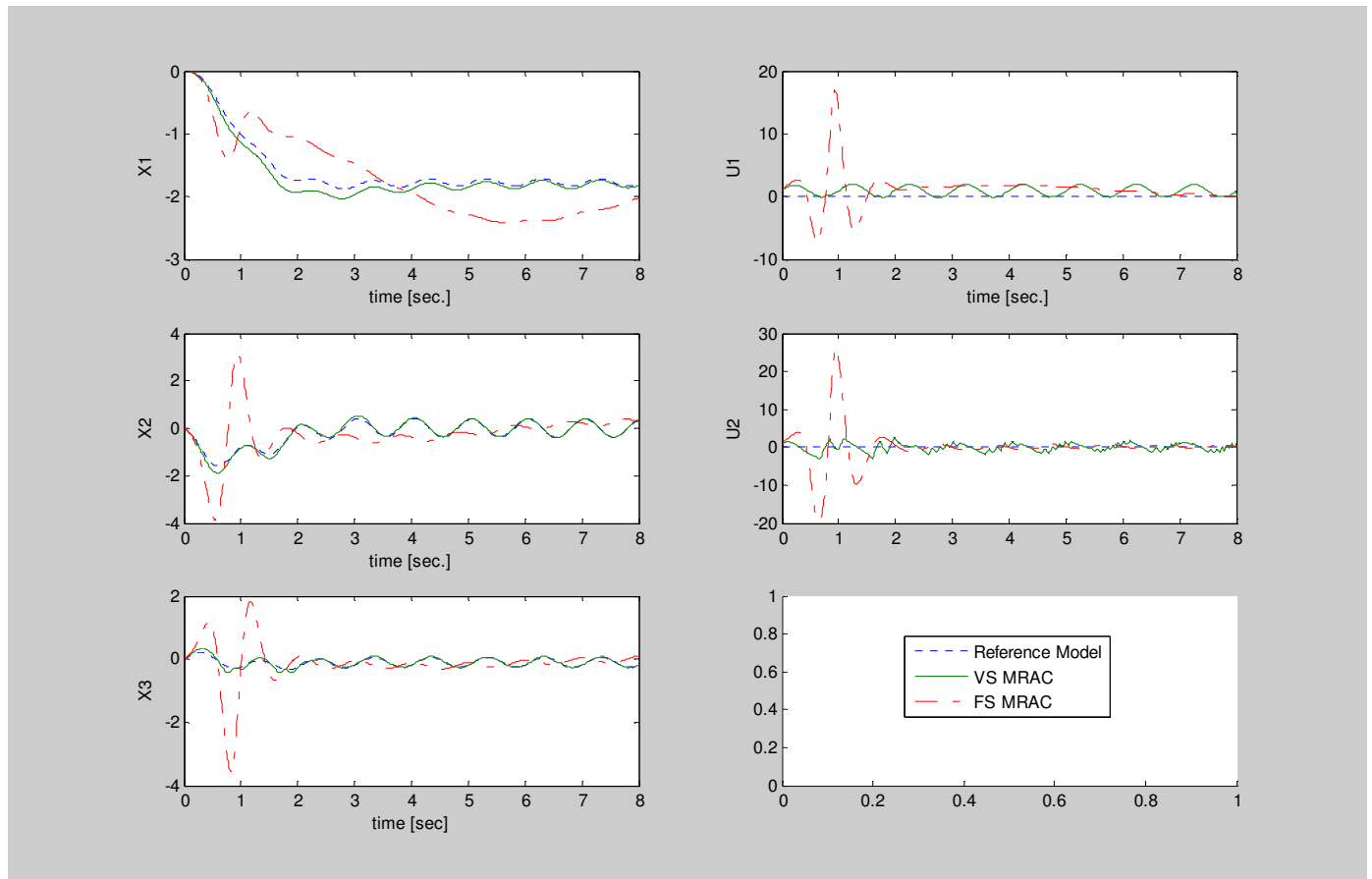
Fig. 1 The block diagram.

Fig. 2 Responses and controls.