Controllability Approach to H_∞ Control Problem of Linear Time-Varying Switched Systems

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Abstract—**This paper considers a H_∞ control problem of a class of linear uncertain time-varying switched system via controllability approach. We show that the solution of this problem can be verified by the global null-controllability of linear control systems. The feedback stabilizing controllers for the problem are constructed via the solutions of certain Riccati differential equations.**

Keywords: Time-varying switched system, H_∞ control, stabilization, controllability, Riccati differential equation

1 Introduction

In the last decade, the H_∞ control problem for linear uncertain systems has attracted increasing attention, and still many questions remain unsolved. The standard H_∞ control problem is to find conditions that guarantee the existence of a feedback controller stabilizing given system and satisfies a prescribed $\gamma-$suboptimal level on peturbations/uncertainties. In the H_∞ control problem for linear autonomous systems, the appropriate methods make use of Lyapunov-Krasovskii function approach and the sufficient conditions are obtained via solving either linear matrix inequalities (LMIs), or algebraic Riccati-type equations. However, this approach may not be readily applied to time-varying systems due to the difficulties of solving time-dependent LMIs. For linear time-varying systems, the investigation of the stability and control problem becomes more complicated. For instance, in contrast to linear autonomous systems, the stability (and instability) of linear time-varying systems may not be determined from the spectral property of their system matrix $A(t)$. It was shown that the real parts of eigenvalues of system matrix $A(t)$ for every t are negative does not imply the asymptotic stability, and there is a linear time-varying system stable with positive eigenvalues. In spite of this, for linear time-varying systems, in the literature to date, there are several papers dealing with stability and control problem. Stabilization problem of linear time-varying systems is also of great interest recently. In order to find H_∞ controller for linear time-varying systems, the state-space approach might be used as well as via Riccati differential equations. However, the problem of existence of solutions of Riccati differential equations is still under active investigations. One of the efficient approaches to this problem is the controllability approach introduced by Kalman [4]. Some sufficient conditions for global stabilization of linear time-varying systems using controllability assumption are given in [3, 8]. The controllability approach was also used in [2, 6] as a systematic method for solving the solution of partial differential equations. To the best of our knowledge, surprisingly few conditions have so far been established for the H_∞ control of uncertain linear time-varying systems. Therefore, finding new conditions for the problem is of interest.

Recently, the study of stability and control of switched systems has become a very popular topics. The main reason is there are many physical models which are governed by more than one dynamical systems and these systems are changed depending on time and the state of the system. In other words, a switched system consists of a family of differential equations (or difference equations) and a switching rule which will determine which system is to be switched on certain time intervals. There are many approaches to study analysis of switched systems. An effective approach is the use of multiple Lyapunov functions.

Based on the above stated reasons, the study of H_∞ control problem for switched systems has attracted many researchers recently.

In this paper, we propose a controllability approach for studying the H_∞ control problem of linear uncertain time-varying switched systems. The feature of our paper is twofold. Firstly, we show that the solution to this problem can be verified by the global null-controllability of linear control systems. Then, we construct feedback stabilizing controllers for the problem via the solutions of Riccati differential equations (RDE).

The paper is organized as follows. Section 2 introduces the main notations, definitions and some auxiliary propositions needed for the proofs. The main result and an illustrated example of the result are given in Section 3. The paper ends with conclusions and cited references.

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2 Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all non-negative real numbers; N denotes the set of all positive integers; for each $k \in N$, I_k denotes the set $\{1, 2, \dots, k\}$; R^n denotes a $n-$dimensional Euclidean space with the norm $\|.\|$ and the inner product $\langle ., . \rangle$; $L_2([t, +\infty), R^n)$ denotes the set of all strongly measurable L_2-integrable R^n-valued functions on $[t, +\infty)$; I denotes the identity matrix. A matrix $Q \in M^{n\times n}$ is called positivetive semi-definite ($Q \geq 0$) if $\langle Qx, x\rangle \geq 0$, for all $x \in R^n$. If for some $c > 0, \langle Qx, x\rangle \geq c\|x\|^2$ for all $x \in R^n$, then Q is called positive definite ($Q > 0$). $A \geq B$ means $A - B \geq 0$; A matrix function $Q(t)$ is uniformly positive definite ($Q(t) \gg 0$) if

$$\exists c > 0 : \quad \langle Q(t)x, x\rangle \geq c\|x\|^2, \quad \forall (t, x) \in R^+ \times R^n.$$

Matrix A is called symmetric if $A = A^T$. It is well known that, if the matrix A is symmetric positive definite, then there is a matrix B such that $A = B^2$ and the matrix B is usually defined by $B = A^{\frac{1}{2}}$. $BM^+(0, +\infty)$ denotes the set of all symmetric non-negative definite matrix functions, which are continuous and bounded on R^+; $BMU^+(0, +\infty)$ denotes the set of all symmetric uniformly positive definite matrix functions, which are continuous and bounded in $t \in R^+$. Denote a partition of R^+ by $\tau = \{0 = t_0 < t_1 < \dots, \ \lim_{i\to+\infty} t_i = +\infty\}$.

Consider the following uncertain linear time-varying switched system with respect to a partition τ of R^+

$$\begin{aligned} \dot{x}(t) &= A_{\alpha_i}(t)x(t) + B_{\alpha_i}(t)u_{\alpha_i}(t) + B_{1\alpha_i}(t)w_{\alpha_i}(t), \quad (1)\\ z(t) &= C_{\alpha_i}(t)x(t) + D_{\alpha_i}(t)u_{\alpha_i}(t), \quad x(0) = x_0, \end{aligned}$$

where $t \in [t_{i-1}, t_i)$, $i \in N$, $x \in R^n$ is the state; $u_{\alpha_i}(t) \in R^m$ is the control; $w_{\alpha_i}(t) \in R^p$ is the uncertain input, $z(t) \in R^q$ is the observation output; $\alpha_i(t) : [t_{i-1}, t_i) \to I_N = \{1, 2, \dots, N\}$ is constant switching signal for each $i \in N$. For each $\alpha_i \in I_N$, $A_{\alpha_i}(t)$, $B_{\alpha_i}(t)$, $B_{1\alpha_i}(t)$, $C_{\alpha_i}(t)$, $D_{\alpha_i}(t) \in R^{n\times n}$ are given state matrix functions continuous and bounded on R^+. For a preassigned partition τ of R^+, the state matrices are activated at each t_i in accordance with a preassigned switching sequence $q = \{(s_0, t_0), (s_1, t_1), \dots, (s_i, t_i), \dots : s_i \in I_N$ and $\alpha_i(t) = s_i, \ t \in [t_{i-1}, t_i), \ i \in N\}$. For each $j \in I_N$, let $N_j(t)$ denote the number of times the subsystem j is activated on $[0, t)$. We say that the controls $u_j(t)$, $j \in I_N$, are admissible if $u_j(t) \in L_2([0, +\infty), R^m)$, and the uncertainty $w_j(t)$ is admissible if $w_j(t) \in L_2([0, +\infty), R^p)$ and $\sum_{i=1}^{+\infty} \int_{t_{i-1}}^{t_i} \|w_{\alpha_i}(t)\|^2 dt < +\infty$. For every initial state $x_0 \in R^n$, for every admissible controls $u_j(t)$, and admissible uncertainties $w_j(t)$, $j \in I_N$, linear control system (1) has a solution given by

$$\begin{aligned} x(t) &= U_{\alpha_i}(t, t_{i-1})x(t_{i-1}) \\ &+ \int_{t_{i-1}}^{t} U_{\alpha_i}(t, s)[B_{\alpha_i}(s)u_{\alpha_i}(s) + B_{1\alpha(i)}(s)w_{\alpha_i}(s)]ds, \end{aligned}$$

where $t \in [t_{i-1}, t_i)$, $U_{\alpha_i}(t, s)$ is the fundamental matrix solution of the linear time-varying system

$$\dot{x}(t) = A_{\alpha_i}(t)x(t), \quad t \in [t_{i-1}, t_i).$$

Definition 2.1. The switched system (1) is asymptotically stable if there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\lim_{t\to+\infty} \|x(t)\| = 0$.

Definition 2.2. Linear control switched system (1), where $w_j(t) = 0$, $j \in I_N$, is stabilizable if there exist a partition $\tau = \{0 = t_0 < t_1 < \dots, \ \lim_{n\to+\infty} t_n = +\infty\}$ of R^+, a switching sequence $q = \{(s_0, t_0), (s_1, t_1), \dots, (s_i, t_i), \dots : s_i \in I_N$ and $\alpha_i(t) = s_i, \ t \in [t_{i-1}, t_i), \ i \in N\}$, admissible feedback controls $u_j(t) = h_j(x(t))$, $j \in I_N$, where $h_j(.) : R^n \to R^m$ are feedback control functions, such that the zero solution of the closed-loop system

$$\dot{x}(t) = A_{\alpha_i}(t)x(t) + B_{\alpha_i}(t)h_{\alpha_i}(x(t)), \tag{2}$$

is asymptotically stable.

The standard H_∞ control problem for the switched system (1) is concerned with a stabilization problem and the existence of a $\gamma-$suboptimal level problem. In this paper, we consider the following H_∞ control problem, which guarantees the existence of a $\gamma-$suboptimal controller under the nonzero initial condition.

Definition 2.3. Given $\gamma > 0$. The H_∞ control problem for the switched system (1) has a solution if there are feedback controls $u_j(t) = h_j(x(t))$, $j \in I_N$, such that

(i) The closed-loop system (2) is asymptotically stable, namely, the control switched system (1), where $w_j(.) = 0$, $j \in I_N$, is stablizable.

(ii) There is a number $c_0 > 0$ such that

$$\sup \frac{\int_0^{+\infty} \|z(t)\|^2 dt}{c_0\|x_0\|^2 + \int_0^{+\infty} \|w(t)\|^2 dt} \leq \gamma, \tag{3}$$

where $w(t) = w_{\alpha_i}(t)$, $t \in [t_{i-1}, t_i)$ and the supremum is taken over all initial states x_0 and non-zero admissible uncertainties $w_j(t)$, $j \in I_N$. In this case we say that the feedback controls $u_j(t) = h_j(x(t)$, $j \in I_N$ stabilize the system (1).

In the sequel, we recall the concept of global controllability, which is concerned with the possibility of steering any state to an another state of the system in finite time. Consider the following linear time-varying control system, briefly denoted by $[A(t), B(t)]$,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in R^+.$$

Definition 2.4. System $[A(t), B(t)]$ is globally null-controllable (GNC) in a finite time $t > +\infty$ if for every initial state x_0, there is an admissible control $u(t)$ such that

$$U(T,0)x_0 + \int_0^T U(T,\tau)B(\tau)u(\tau)d\tau = 0.$$

The following controllability criterion given in [5] will be used later.

Proposition 2.2. *Assume that the matrix functions $A(t), B(t)$ are analytic on R^+. The system $[A(t), B(t)]$ is GNC in some finite time if*

$$\exists t_0 > 0 : \quad rank[M_1(t_0), M_2(t_0), ..., M_n(t_0)] = n, \quad (4)$$

where

$$M_1(t) = B(t), \quad M_k(t) = -A(t) + \frac{d}{dt}M_{k-1}(t),$$

$k = 2, ..., n-1$. Associated with the control system (2), we consider the following RDE

$$\dot{P}(t)+A^T(t)P(t)+P(t)A(t)-P(t)B(t)B^T(t)P(t)+Q(t) = 0. \quad (5)$$

Proposition 2.3. [7] *If system $[A(t), B(t)]$ is globally null-controllable in some finite time, then for any matrix $Q \in BM^+(0,+\infty)$, the RDE (5) has a solution $P \in BM^+(0,+\infty)$.*

Proposition 2.4 *For any matrix function $A(t)$ bounded on R^+, there exists $Q \in BM^+(0,+\infty)$ such that $Q(t) - A(t) \geq 0$.*

Proof. By the same arguments used in the proof of Proposition 2.4 in [8], the matrix $Q(t)$ is chosen as

$$Q(t) = \text{diag}\{q_1(t), q_2(t), ..., q_n(t)\},$$

where $q_i(t) \geq \max\{|q_i^0(t)|, 0\}$ and

$$q_i^0(t) = a_{ii}(t) + \frac{1}{4}\sum_{j\neq i}^n a_{ij}^2(t) + n - 1, i = 1, 2, ..., n.$$

We conclude this section with the following well-known technical results for later use.

Proposition 2.5. *Let Q, S are symmetric matrices of appropriate dimensions and $S > 0$. Then*

$$2\langle Qy, x\rangle - \langle Sw, w\rangle \leq \langle QS^{-1}Q^Tx, x\rangle, \quad \forall(x, y, w).$$

Proposition 2.6. *The matrix* $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, *where $a > 0$ or $c > 0$, is positive definite if $b^2 < ac$.*

3 Main result

In this section, sometimes for the sake of brevity, we will omit the variable t of matrix functions, if it does not cause any confusion.

Consider linear control system (1). For the sake of technical simplification, without loss of generality, as in [1, 9] we assume that

$$D_j^T(t)[C_j(t) \quad D_j(t)] = [I \quad 0], \quad j \in I_N, \ t \in R^+.$$

Given $\gamma > 0$. Let us set

$$A_{\gamma j}(t) = A_j(t) + \frac{1}{\gamma}B_{1j}(t)B_1^T(t) - B_j(t)B_j^T(t),$$

$$B_{\gamma_j}(t) = \left(B_j(t)B_j^T(t) - \frac{1}{\gamma}B_{1j}(t)B_{1j}^T(t)\right)^{\frac{1}{2}}, \ j \in I_N.$$

The following assumption will be used in the proof of the main theorem.

A. $B_j(t)B_j^T(t) - \frac{1}{\gamma}B_{1j}(t)B_{1j}^T(t) > 0, \ t \in R^+, \ j \in I_N.$

We first prove the following technical lemma.

Lemma 3.1. *Let*

$$\tau = \left\{\begin{matrix} 0 = t_0 < t_1 < \ldots, \ \lim_{n\to+\infty} t_n = +\infty, \\ \tau_1 := \sup_{i\in N}\{t_i - t_{i-1}\} < +\infty \end{matrix}\right\}$$

be a partition of R^+ and let

$$q = \left\{\begin{matrix} (s_0, t_0), (s_1, t_1), \ldots, (s_i, t_i), \ldots : s_i \in I_N \\ \text{and } \alpha_i(t) = s_i, \ t \in [t_{i-1}, t_i), \ i \in N \end{matrix}\right\}$$

be a switching sequence satisfying the following two conditions:

(i) For each $j \in I_N$, there exist matrix functions $X_j(t) \in BMU^+(0,+\infty)$, $R_j(t) \in BMU^+(0,+\infty)$, such that

$$\dot{X}_{\alpha_i} + A_{\alpha_i}^T X_{\alpha_i} + X_{\alpha_i}A_{\alpha_i} + C_{\alpha_i}^T C_{\alpha_i} + R_{\alpha_i}$$
$$-X_{\alpha_i}[B_{\alpha_i}B_{\alpha_i}^T - \frac{1}{\gamma}B_{1\alpha_i}B_{1\alpha_i}^T]X_{\alpha_i} \leq 0, \quad (6)$$

$t \in [t_{i-1}, t_i), \ i \in N$. *(ii) There exist $\alpha \in (0,1)$ and $s \in I_N$ such that the dwell-time $\tau_0 := \inf_{i\in I_N}\{t_i - t_{i-1}\} \geq \frac{\lambda_1}{\varepsilon}\ln\frac{\lambda_1}{\alpha\lambda_2}$ and $\beta := \sum_{i=1}^{+\infty}\alpha^{N_s(t_{i-1})} < +\infty$, where $0 < \varepsilon < \min_{j\in I_N}\{\inf_{t\in R^+}\lambda_{\min}(R_j(t))\}$,* $\lambda_1 := \max_{j\in I_N}\{\sup_{t\in R^+}\lambda_{\max}(X_j(t))\}$, and $\lambda_2 := \min_{j\in I_N}\{\inf_{t\in R^+}\lambda_{\min}(X_j(t))\}$.

Then, under partition τ and switching sequence q, the H_∞ control problem for the switched system (1) has a solution, where the admissible feedback controls are chosen as

$$u_j(t) = -B_j^T(t)X_j(t)x(t), \ j \in I_N. \quad (7)$$

Proof. Using the feedback control (7), for each $j \in I_N$, we consider the following Lyapunov functions for the closed-loop system (2) when $w_j(t) = 0$ and $h_j(x(t)) = -B_j^T(t)X_j(t)x(t)$:

$$V_j(t,x) = \langle X_j(t)x, x\rangle.$$

Since $X_j(t) \in BMU^+(0,+\infty)$, the condition (i) of Proposition 2.1 holds. To verify the condition (ii), taking the derivative of $V_j(t,.)$ along the solution $x(t)$ of the closed-loop system, we easily get

$$\begin{aligned}\dot{V}_j(t,x(t)) &= \langle(\dot{X}_j + A_j^T X_j + X_j A_j)x(t), x(t)\rangle \\ &\quad -2\langle X_j B_j B_j^T X_j x(t), x(t)\rangle.\end{aligned}$$

Using RDI (6), we have

$$\begin{aligned}\dot{V}_j(t,x(t)) &\le -\frac{1}{\gamma}\langle X_j B_{1j} B_{1j}^T X_j x(t), x(t)\rangle \\ &\quad -\langle X_j B_j B_j^T X_j x(t), x(t)\rangle \\ &\quad -\langle C_j^T C_j x(t), x(t)\rangle - \langle R_j x(t), x(t)\rangle. \quad (8)\end{aligned}$$

Since

$$\langle C_j^T C_j x(t), x(t)\rangle \ge 0,\ \langle X_j B_j B_j^T X_j x(t), x(t)\rangle \ge 0,$$
$$\langle X_j B_{1j} B_{1j}^T X_j x(t), x(t)\rangle \ge 0,\ t \in R^+,\ j \in I_N$$

and by assumption, $\langle R_j x(t), x(t)\rangle \ge \varepsilon\|x(t)\|^2$ which , by (8), gives

$$\dot{V}_j(t,x(t)) \le -\langle R_j(t)x(t), x(t)\rangle \le -\varepsilon\|x(t)\|^2,\ t \in R^+. \quad (9)$$

Note that (9) gives that, for each $j \in I_N$, $\dot{V}_j(t,x(t)) \le -\varepsilon\|x(t)\|^2$. Next, suupose that the system switches from state j to state i at the time $\tau \in \{t_i\}_{i\in N}$, namely, $\alpha(\tau^-) = j$ and $\alpha(\tau^+) = i$. Then, we have

$$\begin{aligned}V_i\left(\tau^+, x\left(\tau^+\right)\right) &= \langle X_i(\tau)x(\tau), x(\tau)\rangle \\ &\le \lambda_1 \|x(\tau)\|^2 \\ &= \frac{\lambda_1}{\lambda_2}\lambda_2\|x(\tau)\|^2 \\ &\le \frac{\lambda_1}{\lambda_2}\langle X_j(\tau)x(\tau), x(\tau)\rangle \\ &= \frac{\lambda_1}{\lambda_2}V_j\left(\tau^-, x\left(\tau^-\right)\right).\end{aligned}$$

Similarly, one may show that

$$\frac{\lambda_2}{\lambda_1}V_j\left(\tau^-, x\left(\tau^-\right)\right) \le V_i\left(\tau^+, x\left(\tau^+\right)\right) \quad (10)$$

Now, if $\nu \in \{t_i\}_{i\in N}$, is the time when the system switches from some state k to state j, then from (9), we obtain

$$\begin{aligned}\dot{V}_j(t,x(t)) &\le -\varepsilon\|x(t)\|^2 \\ &\le -\varepsilon\frac{1}{\lambda_1}V_j(t,x(t)).\end{aligned}$$

Thus,

$$\frac{1}{V_j(t,x(t))}dV_j(t,x(t)) \le -\frac{\varepsilon}{\lambda_1}dt. \quad (11)$$

By integrating (11) from ν to τ, we obtain

$$V_j(\tau^-, x(\tau^-)) \le V_j(\nu^+, x(\nu^+))e^{-\frac{\varepsilon}{\lambda_1}(\tau-\nu)}. \quad (12)$$

By using (12) and the estimation (10) as $V_i\left(\tau^+, x\left(\tau^+\right)\right) \le \frac{\lambda_1}{\lambda_2}V_j\left(\tau^-, x\left(\tau^-\right)\right)$ obtained above, we have

$$\begin{aligned}V_i\left(\tau^+, x\left(\tau^+\right)\right) &\le \frac{\lambda_1}{\lambda_2}V_j(\nu^+, x(\nu^+))e^{-\frac{\varepsilon}{\lambda_1}(\tau-\nu)} \\ &\le \frac{\lambda_1}{\lambda_2}V_j(\nu^+, x(\nu^+))e^{-\frac{\varepsilon}{\lambda_1}\tau_0} \\ &\le \frac{\lambda_1}{\lambda_2}V_j(\nu^+, x(\nu^+))\frac{\alpha\lambda_2}{\lambda_1}\end{aligned}$$

where we use assumption (ii) in the last inequality. Therefore,

$$V_i\left(\tau^+, x\left(\tau^+\right)\right) \le \alpha V_j(\nu^+, x(\nu^+)). \quad (13)$$

Let $N(t)$ denote the number of times the system is activated on $[0,t)$. (Without loss of generality, we may assume that $\lim_{t\to+\infty} N_j(t) = +\infty$ for each $j \in I_N$). Then, $\lim_{t\to+\infty} N(t) = +\infty$. Suppose that $\alpha(0) = i_0$ and $\alpha(t) = i$, then by taking (13) into account, we get

$$\lambda_2\|x(t)\|^2 \le V_i(t, x(t)) \le \alpha^{N(t)}V_{i_0}(0, x(0)). \quad (14)$$

This implies that, for any $x(0) \in R^n$,

$$\lim_{t\to+\infty}\|x(t)\| \to 0.$$

Therefore, the switched system (1) is asymptotically stable. Note that from (14), and condition (ii) of the theorem, we obtain

$$\begin{aligned}\lambda_2\int_0^{+\infty}\|x(t)\|^2 &\le \int_0^{+\infty}\alpha^{N_s(t)}V_{i_0}(0, x(0))\,dt \\ &\le V_{i_0}(0, x(0))\sum_{i=1}^{+\infty}\alpha^{N_s(t_{i-1})}(t_i - t_{i-1}) \\ &< \tau_1\beta V_{i_0}(0, x(0)) < +\infty.\end{aligned}$$

Hence, $x(t) \in L_2([0,+\infty), R^n)$ and so does $z(t)$. To complete the proof of the theorem, it remains to show the $\gamma-$suboptimal condition (3). To this end, we consider the relation

$$\begin{aligned}
&\textstyle\int_0^{t_k}[\|z(s)\|^2-\gamma\|w(s)\|^2]ds \\
&= \int_0^{t_k}\left[\|z(s)\|^2-\gamma\|w(s)\|^2+\dot V(s,x(s))\right]ds \\
&\quad -\int_0^{t_k}\dot V(s,x(s))ds, \\
&= \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\|z(s)\|^2-\gamma\|w(s)\|^2+\dot V_{\alpha(i)}(s,x(s))\right]ds \\
&\quad -\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\dot V_{\alpha_i}(s,x(s))ds, \\
&= \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\|z(s)\|^2-\gamma\|w(s)\|^2+\dot V_{\alpha_i}(s,x(s))\right]ds \\
&\quad -\sum_{i=1}^{k}[V_{\alpha_i}\left(t_i,x\left(t_i\right)\right)-V_{\alpha_i}\left(t_{i-1},x\left(t_{i-1}\right)\right)] \\
&\le \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\|z(s)\|^2-\gamma\|w(s)\|^2+\dot V_{\alpha_i}(s,x(s))\right]ds \\
&\quad +V_{\alpha_1}\left(t_0,x\left(t_0\right)\right)-V_{\alpha_k}\left(t_{k-1},x\left(t_{k-1}\right)\right) \\
&\quad +\sum_{i=1}^{k-1}\left[V_{\alpha_{i+1}}\left(t_i,x\left(t_i\right)\right)-V_{\alpha_i}\left(t_i,x\left(t_i\right)\right)\right] \\
&\le \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\|z(s)\|^2-\gamma\|w(s)\|^2+\dot V_{\alpha_i}(s,x(s))\right]ds \\
&\quad +V_{\alpha_1}\left(t_0,x\left(t_0\right)\right)+\sum_{i=1}^{k-1}\left(1-\frac{\lambda_2}{\lambda_1}\right)V_{\alpha_i}\left(t_i,x\left(t_i\right)\right) \\
&\quad -V_{\alpha_k}\left(t_{k-1},x\left(t_{k-1}\right)\right) \text{ by (3.5)} \\
&\le \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\|z(s)\|^2-\gamma\|w(s)\|^2+\dot V_{\alpha_i}(s,x(s))\right]ds \\
&\quad +V_{\alpha_1}\left(t_0,x\left(t_0\right)\right)+\sum_{i=1}^{k-1}\left(1-\frac{\lambda_2}{\lambda_1}\right)\lambda_1\|x(t_i)\|^2 \\
&\quad -V_{\alpha_k}\left(t_{k-1},x\left(t_{k-1}\right)\right) \\
&\le \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\|z(s)\|^2-\gamma\|w(s)\|^2+\dot V_{\alpha_i}(s,x(s))\right]ds \\
&\quad +V_{\alpha_1}\left(t_0,x\left(t_0\right)\right)-V_{\alpha_k}\left(t_{k-1},x\left(t_{k-1}\right)\right) \\
&\quad +\sum_{i=1}^{k-1}\left(1-\frac{\lambda_2}{\lambda_1}\right)\frac{\lambda_1}{\lambda_2}\alpha^{N_s(t_i)}V_{i_0}\left(0,x\left(0\right)\right),
\end{aligned} \tag{15}$$

where we use (14) in the last inequality. Now, by taking the estimation of $\dot V_j(t,x(t))$ as

$$\begin{aligned}
\dot V_j(t,x(t)) &\le -\varepsilon\|x(t)\|^2-\langle C_j^TC_jx(t),x(t)\rangle \\
&- \langle X_jB_jB_j^TX_jx(t),x(t)\rangle \\
&- \frac{1}{\gamma}\langle X_jB_{1j}B_{1j}^TX_jx(t),x(t)\rangle \\
&+ 2\langle X_jB_{1j}w(t),x(t)\rangle
\end{aligned} \tag{16}$$

and by substituting

$$\begin{aligned}
&\|z\left(t\right)\|^2 \\
&= \langle[C_{\alpha_i}^T(t)C_{\alpha_i}(t)+X_{\alpha_i}(t)B_{\alpha_i}(t)B_{\alpha_i}^T(t)X_{\alpha_i}(t)]x(t),x(t)\rangle,
\end{aligned}$$

$t\in[t_{i-1},t_i)$ into inequality (16), we obtain

$$\begin{aligned}
&\textstyle\int_0^{t_k}[\|z(s)\|^2-\gamma\|w(s)\|^2]ds \\
&\le \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\Big[[-\varepsilon\|x(s)\|^2-\frac{1}{\gamma}\langle X_{\alpha_i}B_{1\alpha_i}B_{1\alpha_i}^TX_{\alpha_i}x(s),x(s)\rangle \\
&\quad +2\langle X_{\alpha_i}B_{1\alpha_i}w(s),x(s)\rangle-\gamma\langle w(s),w(s)\rangle\Big]ds \\
&\quad +V_{i_0}\left(t_0,x\left(t_0\right)\right)+\sum_{i=1}^{k-1}\left(1-\frac{\lambda_2}{\lambda_1}\right)\frac{\lambda_1}{\lambda_2}\alpha^{N_s(t_i)}V_{i_0}\left(0,x\left(0\right)\right) \\
&\quad -V_{\alpha_k}\left(t_{k-1},x\left(t_{k-1}\right)\right).
\end{aligned} \tag{17}$$

By Proposition 2.5, we have

$$2\langle X_{\alpha_i}B_{1\alpha_i}w(s),x(s)\rangle-\gamma\langle w,w\rangle\le\frac{1}{\gamma}\langle X_{\alpha_i}B_{1\alpha_i}B_{1\alpha_i}^TX_{\alpha_i}x,x\rangle.$$

Then, from (15) and (17), we obtain

$$\begin{aligned}
&\textstyle\int_0^{t_k}[\|z(s)\|^2-\gamma\|w(s)\|^2]ds \\
&\le -\varepsilon\int_0^{t_k}\|x(s)\|^2ds+V_{i_0}\left(t_0,x\left(t_0\right)\right)+ \\
&\quad \sum_{i=1}^{k-1}\left(1-\frac{\lambda_2}{\lambda_1}\right)\frac{\lambda_1}{\lambda_2}\alpha^{N_s(t_i)}V_{i_0}\left(0,x\left(0\right)\right) \\
&\quad -V_{\alpha_k}\left(t_{k-1},x\left(t_{k-1}\right)\right).
\end{aligned}$$

By letting $k\to+\infty$, we finally obtain

$$\begin{aligned}
&\textstyle\int_0^{+\infty}[\|z(t)\|^2-\gamma\|w(t)\|^2]dt \\
&\le V_{i_0}\left(0,x\left(0\right)\right)\left(1+\tau_1\beta\left(\frac{\lambda_1}{\lambda_2}-1\right)\right)
\end{aligned}$$

and since $V_{i_0}\left(0,x\left(0\right)\right)\le\|X_{i_0}\left(0\right)\|\,\|x\left(0\right)\|^2$, we get

$$\begin{aligned}
&\textstyle\int_0^{+\infty}[\|z(t)\|^2dt \\
&\le \gamma\left\{\begin{array}{c}\int_0^{+\infty}\|w(t)\|^2dt \\ +\frac{\|X_{i_0}(0)\|\|x(0)\|^2\left(1+\tau_1\beta\left(\frac{\lambda_1}{\lambda_2}-1\right)\right)}{\gamma}\end{array}\right\}.
\end{aligned}$$

Setting $c_0=\frac{\|X_{i_0}(0)\|\left(1+\tau_1\beta\left(\frac{\lambda_1}{\lambda_2}-1\right)\right)}{\gamma}$ in the last inequality we obtain

$$\frac{\int_0^{+\infty}\|z(t)\|^2dt}{c_0\|x\left(0\right)\|^2+\int_0^{+\infty}\|w(t)\|^2dt}\le\gamma,$$

where $\int_0^{+\infty}\|w(t)\|^2dt = \sum_{i=1}^{+\infty}\int_{t_{i-1}}^{t_i}\|w_{\alpha_i}(t)\|^2dt$, for all $x\left(0\right)$ and non-zero admissible $w_j(t)\in$

$L_2([0,+\infty), R^p), j \in I_N$. This completes the proof of the lemma.

Based on Lemma 3.1, we obtain the following main result.

Theorem 3.1. *Assume that the condition A holds and linear control systems* $[A_{\gamma j}(t), B_{\gamma j}(t)]$, $j \in I_N$, *are GNC in some finite time. Let* τ *be a partition of* R^+ *and* q *a switching sequence such that the dwell-time* $\tau_0 := \inf_{i\in N}\{t_i - t_{i-1}\} \geq \frac{\lambda_1}{\varepsilon}\ln\frac{\lambda_1}{\alpha\lambda_2}$ *where* $\varepsilon > 0$, $\alpha \in (0,1)$ *satisfying* $\beta := \sum_{i=1}^{+\infty}\alpha^{N_s(t_{i-1})} < +\infty$ *for some* $s \in I_N$, *and* $\tau_1 := \sup_{i\in N}\{t_i - t_{i-1}\} < +\infty$ *where* $\lambda_1 := \max_{j\in I_N}\{\sup_{t\in R^+}\lambda_{\max}(P_j(t))\}$, $\lambda_2 := \min_{j\in I_N}\{\inf_{t\in R^+}\lambda_{\min}(P_j(t))\}$. *Then, under partition* τ *and switching sequence* q, *the* H_∞ *control problem for the switched system (2.1) has a solution . Moreover, the feedback stabilizing controls are*

$$u_j(t) = -B_j^T(t)[P_j(t) + I]x(t),\ t \in R^+,\ j \in I_N$$

where $P_j(t) \in BM^+(0,+\infty)$ *is a solution of RDE*
$\dot{P}_j(t) + A_{\gamma j}^T(t)P_j(t) + P_j(t)A_{\gamma j}(t) - P_j(t)B_{\gamma j}(t)B_{\gamma j}^T(t)P_j(t)$
$+Q_j(t) = 0$,
$t \in R^+,\ j \in I_N$ *and* $Q_j(t) \geq 0$ *is a matrix function satisfying*

$$Q_j(t) \geq A_j(t) + A_j^T(t) + C_j^T(t)C_j(t) + \varepsilon I,\ t \in R^+,\ j \in I_N.$$

Remark 3.1. Note that the problem of solving Riccati differential equations is in general still complicated, however some various efficient approaches to solving this problem can be found.

The following simple procedure can be applied to solve the H_∞ control problem for the switched system (1).

Step 1. Given $\gamma > 0$, find the matrices $A_{\gamma j}(t), B_{\gamma j}(t)$, $j \in I_N$.

Step 2. Check the assumptions A, conditions (i) and (ii) of Lemma 3.1.

Step 3. Check the global null-controllability of linear systems $[A_{\gamma j}(t), B_{\gamma j}(t)]$, $j \in I_N$, by Proposition 2.2.

Step 4. Find solutions $P_j(t),\ j \in I_N$ of RDE given in Theorem 3.1 and the feedback stabilizing controls are given by

$$u_j(t) = -B_j^T(t)[P_j(t) + I]x(t),\ t \in R^+,\ j \in I_N.$$

4 Conclusions

In this paper, we have shown that the H_∞ control conditions for linear time-varying switched systems has a solution if some appropriate linear control systems are globally null-controllable. The feedback stabilizing controllers are designed via the solutions of a Riccati differential equations. Numerical example is given to illustrate the effectiveness and validity of our main results.

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