Stability of a Class of Neural networks

Jiemin Zhao

Abstract—We give a concise result of global uniform asymptotic stability for the Hopfield model

\[
\begin{align*}
\frac{du_i(t)}{dt} &= -u_i(t) + a \int_0^{\tau_i(t)} e^{-\lambda s} ds,
\frac{du_j(t)}{dt} &= a \int_0^{\tau_i(t)} e^{-\lambda s} ds - u_j(t)
\end{align*}
\]

by means of the method of Liapunov function.

Index Terms—dynamics, neural network, Liapunov function, stability.

I. INTRODUCTION

The Hopfield model is well known in neural network. It may be described by the following system of ordinary differential equations

\[
\begin{align*}
C_i \frac{du_i(t)}{dt} &= \sum_{j=1}^{n} T_{ij} V_j - \frac{u_i(t)}{R_i} + I_i \\
 u_i &= g_i(V_i) \quad (i = 1, 2, \cdots, n),
\end{align*}
\]

where \( T_{ij} \) represents the electrical current input to cell \( i \) due to the present potential of cell \( j \), and \( T_{ij} \) is thus the synapse efficacy. Linear summing of inputs is assumed. \( I_i \) is any other (fixed) input current to neuron \( i \) [1].

Hopfield [1] investigate a two-neuron system. The system parameters are

\[ T_{12} - T_{21} = -1, \quad \lambda = 1.4 \quad \text{and} \quad g(u) = \frac{2}{\pi} \tan^{-1} \frac{\pi u}{2} \]

Energy contours are

\[ 0.449, 0.156, 0.017, \quad -0.003, \quad -0.023, \quad \text{and} \quad -0.041 \]

For a number of neuron models and their stability analysis,

we do not enumerate them. This paper presents a concise result of global uniform asymptotic stability for the Hopfield model

\[
\begin{align*}
\frac{du_i(t)}{dt} &= -u_i(t) + a \int_0^{\tau_i(t)} e^{-\lambda s} ds,
\frac{du_j(t)}{dt} &= a \int_0^{\tau_i(t)} e^{-\lambda s} ds - u_j(t)
\end{align*}
\]

where \( a = \text{const.} \in (0, 1) \). Such a concise result is useful to Hopfield model based neural network design in practical applications [1–13].

II. ANALYSIS AND COMPUTING

Let

\[
f(u_i) = a \int_0^{\tau_i} e^{-\lambda s} ds.
\]

Therefore,

\[
f(u_i) = -f(-u_i),
\]

\[
\frac{df(u_i)}{du_i} = ae^{-\lambda u_i} > 0 \quad \text{for} \; u_i \geq 0,
\]

\[
\frac{d^2f(u_i)}{du_i^2} = -2au_i e^{-\lambda u_i} < 0 \quad \text{for} \; u_i > 0,
\]

\[
\lim_{u_i \to \infty} f(u_i) = \frac{\sqrt{\pi}}{2}.
\]

The function \( u_2 = f(u_1) \) is illustrated in Fig.1.

![Fig.1](image-url)
From analysis and computing above, we construct Liapunov function

\[
V(u_1, u_2) = \frac{1}{2}(u_1 + u_2)^2 + \int_0^t [u - a \int_0^s e^{-s} ds] du + \int_0^t [u - a \int_0^s e^{-s} ds] du.
\]

Thus, if

\[
(u_1(t), u_2(t)) = (u_1(t_0, u_1_0), u_2(t_0, u_2_0))
\]
is a solution of system (1), then the derivative \( \frac{dV}{dt} \) of \( V \) along \((u_1(t), u_2(t))\) satisfies

\[
\frac{dV}{dt} = (u_1(t) + u_2(t)) \left( \frac{du_1(t)}{dt} + \frac{du_2(t)}{dt} \right) +
\]

\[
[u_1(t) - a \int_0^t e^{-s} ds] \frac{du_1(t)}{dt} +
\]

\[
[u_2(t) - a \int_0^t e^{-s} ds] \frac{du_2(t)}{dt}.
\]

Using (1), we obtain

\[
\frac{dV}{dt} = (u_1(t) + u_2(t)) \left[ -u_1(t) + a \int_0^t e^{-s} ds - u_2(t) \right] +
\]

\[
[u_1(t) - a \int_0^t e^{-s} ds] \frac{du_1(t)}{dt} +
\]

\[
[u_2(t) - a \int_0^t e^{-s} ds] \frac{du_2(t)}{dt}.
\]

Therefore,

\[
F(v_1) = v_1 - a \int_0^t e^{-s} ds.
\]

The function \( v_2 = F(v_1) \) is illustrated in Fig.2.
\[
[u'_i(t) - a \int_{0}^{\tau_i} e^{-r_i s} [u'_i(s) - a \int_{0}^{\tau_i} e^{-r_i s}] ds \leq 0.
\]

Furthermore, the system (1) is globally uniformly asymptotically stable [2–13].

III. MAIN RESULT

From analysis and computing above, we have result as follow:

The system (1) is globally uniformly asymptotically stable.

REFERENCES