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Direction of Arrival Estimation using a Root-MUSIC Algorithm

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***Abstract*—An array antenna system with innovative signal processing can enhance the resolution of a signal direction of arrival (DOA) estimation. Super resolution algorithms take advantage of array antenna structures to better process the incoming signals. They also have the ability to identify multiple targets. This paper explores the eigen-analysis category of super resolution algorithm. A class of Multiple Signal Classification (MUSIC) algorithms known as a root-MUSIC algorithm is presented in this paper.**

The root-MUSIC method is based on the eigenvectors of the sensor array correlation matrix. It obtains the signal estimation by examining the roots of the spectrum polynomial. The peaks in the spectrum space correspond to the roots of the polynomial lying close to the unit circle.

Statistical analysis of the performance of the processing algorithm and processing resource requirements are discussed in this paper. Extensive computer simulations are used to show the performance of the algorithms.

***Index Terms*— Array antenna, Direction of arrival estimation, Signal processing.**

I. Introduction

Accurate estimation of a signal direction of arrival (DOA) has received considerable attention in communication and radar systems of commercial and military applications. Radar, sonar, and mobile communication are a few examples of the many possible applications. For example, in defense application, it is important to identify the direction of a possible threat. One example of commercial application is to identify the direction of a emergency cell phone call such that the rescue team can be dispatched to the proper location.

DOA estimation using a fixed antenna has many limitations. Its resolution is limited by the antenna's mainlobe beamwidth. Antenna mainlobe beamwidth is inversely proportional to its physical size. Improving the accuracy of angle measurement by increasing the physical aperture of the receiving antenna is not always a practical option. Certain systems such as a missile seeker or aircraft antenna have physical size limitations; therefore they have relatively wide mainlobe beamwidth. Consequently, the resolution is quite poor. Also, if there are multiple signals falling in the antenna mainlobe, it is difficult to distinguish between them.

Instead of using a single antenna, an array antenna system with innovative signal processing can enhance the resolution of signal DOA. An array sensor system has multiple sensors distributed in space. This array configuration provides spatial samplings of the received waveform. A sensor array has better performance than the single sensor in signal reception and parameter estimation. Its superior spatial resolution provides a means to estimate the direction of arrival (DOA) of multiple signals. A sensor array also has applications in interference rejection [1], electronic steering [2], multi-beam forming [3], etc. This technology is now widely used in communications, radar, sonar, seismology, radio astronomy ,etc.

There are many different super resolution algorithms including spectral estimation, model based, and eigen-analysis to name a few [4,5,6]. In this paper, we concentrate the discussion on the application of estimating the DOA of multiple signals. The focuses are on a class of Multiple Signal Classification (MUSIC) algorithms known as root-MUSIC and an extension of root-MUSIC. We present detailed MATLAB simulation results for each algorithm.

II. Array Sensor Systems

We use an array antenna with a 16 element uniform linear array (ULA) in this paper. Fig. 1 shows the general configuration for a ULA antenna having M elements arranged along a straight line with the distance between sensor elements, be d = $\lambda/2$, where λ is the incoming signal wavelength. The angle of the incoming signal, θ, is measured relative to the antenna bore sight.

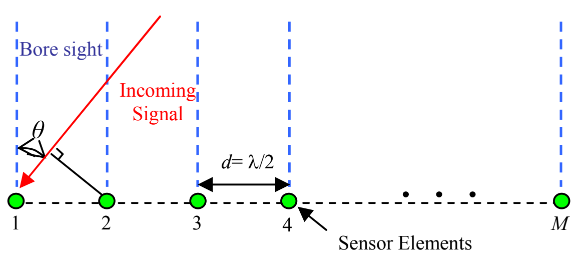

Figure 1. ULA Antenna configuration

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For the a conventional antenna, the main lobe beam width (MLBW) of an antenna manner is given by, in radians

$$\text{MLBW} = k\frac{\lambda}{D} \tag{1}$$

where D is the diameter of the antenna array and k is a proportionality constant, for most case $k \approx 1$ [6].

III. Root-MUSIC Algorithm

The root-MUSIC method relies on the following properties of the array correlation matrix: the space spanned by its eigenvectors may be partitioned into two orthogonal subspaces, namely the signal plus noise subspace and the noise only subspace; the steering vectors corresponding to the directional sources are orthogonal to the noise subspace [7]. The MxM correlation matrix that contains L number of incoming signals is formed by

$$\mathbf{R} = \mathbf{SDS^H} + \sigma^2\mathbf{I} \tag{2}$$

where σ^2 is the variance of the Gaussian white noise, **D** is the signal power matrix and **S** is the signal direction matrix

$$\mathbf{D} = diag[P_1, P_2, ...P_L] \tag{3}$$

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-j\beta(\theta_1)} & e^{-j\beta(\theta_2)} & \cdots & e^{-j\beta(\theta_L)} \\ \cdots & \cdots & \cdots & \cdots \\ e^{-j(M-1)\beta(\theta_1)} & e^{-j(M-1)\beta(\theta_2)} & \cdots & e^{-j(M-1)\beta(\theta_L)} \end{bmatrix} \tag{4}$$

and the phase delay between sensor elements is

(4)

$$\beta(\theta_i) = \frac{2\pi d}{\lambda}\sin(\theta_i) \tag{5}$$

Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_M$ be the eigenvalues of the correlation matrix **R**, and $\upsilon_1 \geq \upsilon_2 \geq ... \geq \upsilon_L$ be the eigenvalues for $\mathbf{SDS^H}$. Then from (2)

$$\lambda_i = \begin{cases} \upsilon_i + \sigma^2 & i = 1, 2, ...L \\ \sigma^2 & i = L+1, ...M \end{cases} \tag{6}$$

For high signal to noise ratios (SNR) $\upsilon_i \gg \sigma^2$. The eigenvalues can be used to determine the number of sources that are detected by counting the number of comparatively large eigenvalues. Alternatively, Ref. [7] suggests a more rigorous approach to determining the number of incoming sources that provides better detection performance when the incoming SNR is not as high. For the purposes of this, the incoming SNR is chosen to be sufficiently high as to not be in a situation where the source number detection is ambiguous.

Let $\mathbf{q_1}$, $\mathbf{q_2}$, … , $\mathbf{q_M}$ be the eigenvectors associated with the decreasing ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_M$. From (6) the first L eigenvectors will span the signal plus noise subspace and the remaining M-L eigenvectors will span the noise only subspace, $\mathbf{Q_N}$. By eigen-analysis we can represent the M-L smallest eigenvectors as

$$\mathbf{Rq_i} = \sigma^2\mathbf{q_i} \qquad i = L+1, ..., M \tag{7}$$

Using (7) in (2), can be rewritten as

$$\mathbf{SDS^H q_i} = 0 \qquad i = L+1, ..., M \tag{8}$$

Since **S** is a full column rank matrix and **D** is diagonal, (8) becomes

$$\mathbf{S^H q_i} = 0 \qquad i = L+1, ..., M \tag{9}$$

or more explicitly

$$\mathbf{s_k}^{\mathbf{H}}\mathbf{q_i} = 0 \qquad \begin{matrix} i = L+1, ..., M \\ k = 1, ..., L \end{matrix} \tag{10}$$

Equation (10) proves the orthogonality between the signal plus noise and the noise only subspaces. This is important because it shows that the angle of the incoming signals can be found by searching for signal direction vectors that, when projected onto the noise only subspace, give a zero result. Following this idea, if a polynomial, $J(z)$, is constructed such that

$$J(z) = \mathbf{v^H Q_N Q_N{}^H v} = 0 \tag{11}$$

where the steering vector **v** is

$$\mathbf{v} = \begin{bmatrix} 1 & z^{-1} & z^{-2} & \cdots & z^{-(M-1)} \end{bmatrix}^T \tag{12}$$

and

$$z = e^{j\frac{2\pi d}{\lambda}\sin(\theta)} \tag{13}$$

Then the roots of $J(z)$ contain the directional information of the incoming signals. Ideally, the roots of $J(z)$ would be on the unit circle at locations determined by the directions of the incoming signals; however, due to the presence of noise the roots may not necessarily be on the unit circle. In this case, the L closest roots to the unit circle are the roots that correspond to the L incoming signals [9]. These selected roots, by themselves, do not directly represent the incoming angle. For each root, the incoming angle is found by solving (13).

$$\theta_k = \arcsin\left[\frac{\lambda}{2\pi d}\arg(z_k)\right] \tag{14}$$

Obviously, when the root-MUSIC algorithm is implemented there is no prior knowledge of the incoming signal directions or signal powers needed to construct the correlation matrix using (2). Therefore the correlation matrix must be estimated using only the information available from the sensor array. There are several methods commonly used to perform this estimation such as temporal averaging, spatial smoothing or, a hybrid combination of both temporal averaging and spatial smoothing [8]. In this paper, we use only the temporal averaging method.

The estimated correlation matrix using the temporal averaging method with k snapshots is given as

$$\Phi = \mathrm{E}[\mathbf{A}^H \mathbf{A}] \tag{15}$$

where the incoming data matrix **A** is

$$\mathbf{A} = \begin{bmatrix} u_1(1) & u_1(2) & \dots & u_1(k) \\ u_2(1) & u_2(2) & \dots & u_2(k) \\ \dots & \dots & \dots & \dots \\ u_M(1) & u_M(2) & \dots & u_M(k) \end{bmatrix}. \tag{16}$$

with $u_i(k)$ being the i^{th} sensor output at time *k*.

The estimated correlation matrix, Φ, asymptotically approaches the correlation matrix, **R** as the number of snapshots increases. Therefore in order to have an accurate estimation of the correlation matrix the observation time must be sufficiently long. The long observation times are not ideal for radar signal processing applications; however there are many applications where this does not pose a problem. Correlation matrix estimation techniques like the spatial smoothing method are better suited for use in time sensitive systems.

A. Sensor Spacing and Phase Sensitivity

The root-MUSIC algorithm assumes that each sensor is perfectly spaced relative to the other sensors in the array. While this holds true for the theoretical case, perfect sensor spacing is difficult to achieve when the algorithm is actually implemented even with modern construction techniques. The method by which the algorithm is modified to model these variations is rather straight forward. The sensor spacing problem is characterized by Fig.3.

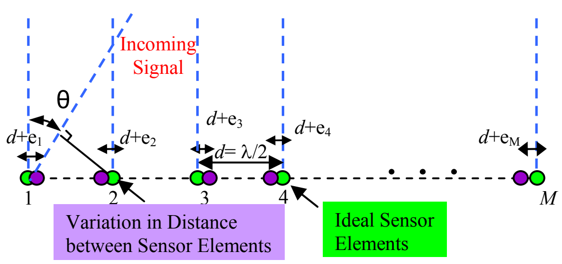


Figure 3. Non-ideal ULA sensor spacing

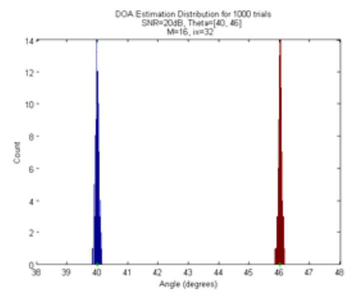
The spacing error of each sensor, e_i, is a Gaussian random variable added to the ideal spacing. Taking this error into account, (5) is used to create the phase shift between sensor elements for the incoming signals becomes

$$\beta(\theta_i) = \frac{2\pi(d+e_i)}{\lambda}\sin(\theta_i) \tag{17}$$

When these data are applied to the root-MUSIC algorithm, (14) no longer has the correct value for *d* and obviously will return a result that has some increased error.

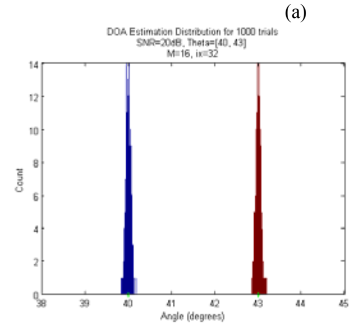
B. Root-Music Simulation Results

Statistical results, Fig.4 , show that two incoming signals are clearly identified even as the separation between the two signals is well below the conventional main lobe beam width. It can be seen that as the spacing between signals decreases the variance of the estimates increases. The average estimation error is in all cases nearly zero.



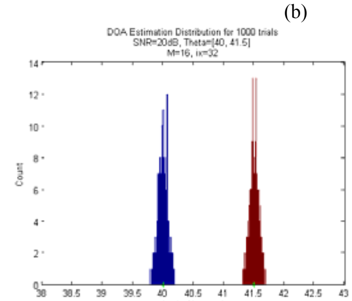
1 Beam Width Separation:
Average Estimated Angle = 39.9972 46.0041
Average Estimation Error= 0.0028 -0.0041
Mean Square Error = 0.0009 0.0012

(a)



1/2 Beam Width Separation:
Average Estimated Angle = 39.9988 42.9995
Average Estimation Error= 0.0012 0.0005
Mean Square Error = 0.0023 0.0025

(b)



1/4 Beam Width Separation:
Average Estimated Angle = 39.9999 41.4987
Average Estimation Error= 0.0001 0.0013
Mean Square Error = 0.0047 0.0042

(c)

Figure 4. Statistical comparison for 2 signals at various separation angles

Fig.5 shows the effect of increasing the number of snap shots used for the temporal averaging correlation matrix estimation. The estimation variance decreases with increased observation times. The average estimation error does not seem to be very sensitive to the observation time

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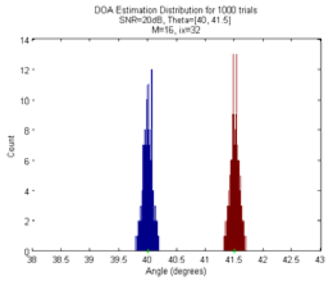
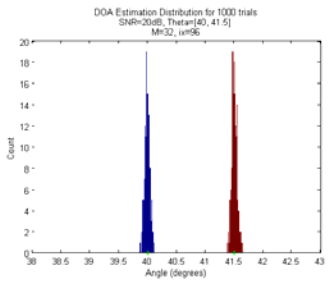


1/4 Beam Width Separation
32 Snap Shots:
Average Estimated Angle =
39.9999 41.4987
Average Estimation Error=
0.0001 0.0013
Mean Square Error =
0.0047 0.0042

(a)



1/4 Beam Width Separation
96 Snap Shots:
Average Estimated Angle =
39.9992 41.5008
Average Estimation Error=
0.0008 -0.0008
Mean Square Error =
0.0015 0.0016

(b)

Figure 5. Variation of snap shot comparison

In Fig.6, We compare the simulation results of ideally spaced sensor elements against the results of a simulation where the sensor elements have a 1% random variance in their spacing; in other words $\sigma = \lambda / 200$.

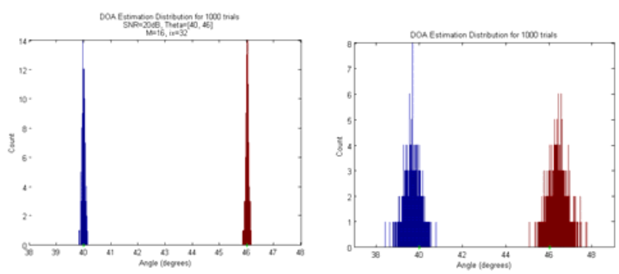


Figure 6. Comparison of Non-Ideal Sensor Spacing

Increasing the amount of spacing variance from 1% to 5% shows an increased error variance in Fig.7. It is worth noting that while the performance of the algorithm decreases with increased sensor spacing error the algorithm is still able to successfully distinguish both incoming signal directions.

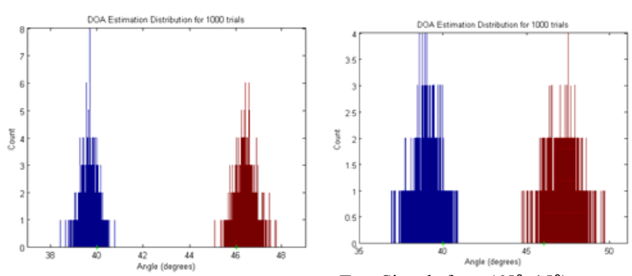


Figure 7. MUSIC Spectrum with Two Signals from (65°, 15°) and (65°, 25°)

The simulation results of the root-MUSIC algorithm clearly demonstrate the ability to resolve multiple targets with separation angles smaller then the main lobe beam width of the array thus proving its super-resolution capabilities.

The algorithm does exhibit a rather strong sensitivity to the positional accuracy of the sensor placement; however with proper array calibration these effects could be minimized.

IV. Conclusion

We have presented the root-MUSIC method based on the eigenvector of the sensor array correlation matrix to estimate angle of incoming signals. We give extensive computer simulation results to demonstrate the performance of the algorithms, which enhance the DOA estimation.

The simulation results of the root-MUSIC algorithm show the following results

1. The capability to resolve multiple targets with separation angles smaller the main lobe beam width of the array antenna.
2. The estimation variance can be reduced by increasing the number of snapshots in correlation matrix estimation
3. The estimation variance increases as the angle separation between signals becomes smaller
4. The estimation variance depends on the direction of the signal. A signal coming from the bore sight has minimum estimation variance.

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