Robust Mean-Variance Portfolio Selection Problem Including Fuzzy Factors

Takashi Hasuike and Hiroaki Ishii

Abstract—**This paper considers robust mean-variance portfolio selection problems including uncertainty sets and fuzzy factors. Since these problems are not well-defined problems due to fuzzy factors, it is hard to solve them directly. Therefore, introducing chance constraints, fuzzy goals and possibility measures, the proposed models are transformed into the deterministic equivalent problems. Furthermore, since it is difficult to solve them analytically and efficiently due to nonlinear programming problems, the solution method is constructed introducing a parameter and doing the equivalent transformations.**

Index Terms—**Portfolio selection problem, Robust optimization, Fuzzy optimization, Nonlinear programming**

I. Introduction

In recent investment markets, not only big companies and institutional investors but also individual investors called Day-Traders invest in stock, currency, property, etc.. Therefore, the role of investment theory becomes more and more important. Of course, it is easy to decide the most suitable financial assets allocation if decision makers can receive reliable information with respect to future returns a priori. However, there exist many cases that uncertainty from social conditions has a great influence on the future returns. In the real market, there are random factors derived from statistical analysis of historical data and ambiguous factors such as the psychological aspect of investors and lack of received efficient information. Under such uncertainty situations, they need to consider how to reduce a risk, and it becomes important whether they receive the greatest future profit.

Such a finance assets selection problem is generally called a portfolio selection problem, and various studies have been done till now. As for the research history on mathematical approach, Markowitz [24] has proposed mean-variance model and it has been central to research activity in the real financial field and numerous researchers have contributed to the development of modern portfolio theory (for instance, Luenberger [23], Steinbach [28]). On the other hand, many researchers have proposed models of portfolio selection problems which extended Markowitz model; Capital Asset Pricing Model (CAPM) (Sharpe [27], Lintner [21], Mossin [25]), mean-absolute-deviation model (Konno [19], Konno, et al. [20]), semi-variance model (Bawa [1]), safety-first model (Elton [6]), Value at Risk and conditional Value at Risk model (Rockfellar [26]), etc..

In such previous researches, expected future return and variance of each asset are assumed to be known, and in this case, the mean-variance model is equivalent to a quadratic convex programming problem. Therefore, its optimal portfolio is analytically obtained. However, decision makers may receive a lot of information and data in current market. However, it is almost impossible to estimate strict market parameters such as expected future return and variance, and to determine their random distribution. These distributions may be statistically determined as a confidence interval involving some error. Therefore, using these statistical distributions, it is more important to considering that decision makers optimize the problem in the worst case; i.e. robust optimization problem.

Recently, the robust optimization problem becomes a more active area of research, and there exists various studies (For example, [2, 3, 7, 10, 13]). Particularly, with respect to portfolio selection problems, there are some studies of robust portfolio selection problems determining optimal investment strategy using the robust approach (For example, [8, 22]). The expected return and variance of each asset are mainly estimated from historical data and occur according to random distributions derived from the statistical analysis. However, considering efficient or inefficient received information, the institution of expert decision maker and the existence of other random distribution, we need to consider that statistical distribution considering these conditions includes some ambiguity and is involved some flexibility. In this paper, we propose extensional models of robust portfolio selection problems including fuzzy factors.

Until now, there are some basic researches under various uncertainty conditions with respect to portfolio selection problems (Bilbao-Terol [4], Carlsson [5], Guo [9], Huang [11, 12], Inuiguchi [14, 15], Katagiri [17, 18], Tanaka [29, 30], Watada [31]). However, there are few models considering both uncertainty sets and ambiguity, simultaneously. Furthermore, there are no researches which are analytically extended and solved these types of portfolio selection problems. Since our proposal models are not well-defined problems, in this paper, we transform main problems into the deterministic equivalent problems and construct the analytical solution method of fuzzy robust portfolio selection problem as well as propose formulation of this model.

This paper is organized as follows. In Section 2, we introduce basic mean-variance portfolio selection problems minimizing the total variance and the total future return, respectively, and we formulate their robust models

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introducing the uncertainty sets. In Section 3, introducing fuzzy numbers to uncertainty sets of expected return and variance, we propose fuzzy extension models of robust mean-variance portfolio selection problems and construct the analytical solution method. Finally, in Section 4, we conclude this paper and discuss future research problems.

II. Formulation of Robust mean variance optimization problems

In this section, we consider basic portfolio selection problems and their robust models. First of all, we set the parameters in portfolio selection problems. We set the expected return of total future profit $E(\boldsymbol{r})$ and the total variance $\mathbf{Var}(\boldsymbol{r})$ as follows:

$$E(\boldsymbol{r})=\bar{\boldsymbol{r}}^t\phi,\quad \mathbf{Var}(\boldsymbol{r})=\phi^t\mathbf{V}\phi \tag{1}$$

where each notation means as follows:

$\boldsymbol{r}$: Future return vector assumed to be a random variable
$\bar{\boldsymbol{r}}$: Mean value vector of random variable $\boldsymbol{r}$
$\mathbf{V}$: Variance-covariance matrix of random variable $\boldsymbol{r}$
ϕ : Portfolio with respect to each asset $j,\ (j=1,2,...,n)$

From these notations, a mean-variance model Markowitz has proposed is formulated as the following problem:

$$\begin{array}{ll}\text{Minimize} & \phi^t\mathbf{V}\phi \\ \text{subject to} & \bar{\boldsymbol{r}}^t\phi\geq f, \\ & \mathbf{1}^t\phi=1\end{array} \tag{2}$$

where f is a target value of total future return. In this problem, introducing a parameter ν, problem (2) is equivalently transformed into the following problem introducing the target value of total variance ν:

$$\begin{array}{ll}\text{Minimize} & \nu \\ \text{subject to} & \phi^t\mathbf{V}\phi\leq\nu, \\ & \bar{\boldsymbol{r}}^t\phi\geq f, \\ & \mathbf{1}^t\phi=1\end{array} \tag{3}$$

In the case that we obtain the strict value of parameters $\bar{\boldsymbol{r}}$ and $\mathbf{V}$, problem (3) is equivalent to a quadratic programming problem and we find an optimal portfolio using standard convex programming approaches. Furthermore, while problem (3) considers minimizing the total variance, the case maximizing the total future return is formulated as the following form:

$$\begin{array}{ll}\text{Maximize} & f \\ \text{subject to} & \bar{\boldsymbol{r}}^t\phi\geq f, \\ & \phi^t\mathbf{V}\phi\leq\nu, \\ & \mathbf{1}^t\phi=1\end{array} \tag{4}$$

This problem is also a quadratic programming problem and so we obtain an optimal portfolio.

However, in real world, it is hard to receive all information and data with respect to future returns and determine the distributions of their random variables. Therefore, in this paper, we consider that parameters $\bar{\boldsymbol{r}}$ and $\mathbf{V}$ have uncertainty and each parameter is included in an uncertainty set. In the case that we consider these uncertainty sets, problems (3) and (4) are not quadratic programming problems. Therefore, we need to construct the solution procedure to solve them. In this paper, we formulate the robust portfolio selection problem Men-tal and Nemirovski [2] have proposed. We formulate the robust portfolio selection problem minimizing the total variance as follows:

$$\begin{array}{ll}\text{Minimize} & \nu \\ \text{subject to} & \max_{\{\mathbf{V}\in S\}}\phi^t\mathbf{V}\phi\leq\nu, \\ & \min_{\{\bar{\boldsymbol{r}}\in M\}}\bar{\boldsymbol{r}}^t\phi\geq f, \\ & \mathbf{1}^t\phi=1\end{array} \tag{5}$$

where $M\subset R^n$ and $S\subset R^{n\times n}$ are uncertainty sets. In a way similar to problem (5), we formulate the robust portfolio selection problem maximizing the total future return as follows:

$$\begin{array}{ll}\text{Maximize} & f \\ \text{subject to} & \min_{\{\bar{\boldsymbol{r}}\in M\}}\bar{\boldsymbol{r}}^t\phi\geq f, \\ & \max_{\{\mathbf{V}\in S\}}\phi^t\mathbf{V}\phi\leq\nu, \\ & \mathbf{1}^t\phi=1\end{array} \tag{6}$$

In these problems, they are not well-defined problems without defining uncertainty sets. Therefore, we first assume the uncertainty set of mean value $\bar{\boldsymbol{r}}$ to be the following ellipsoidal set:

$$M=\left\{\bar{\boldsymbol{r}}\left|(\bar{\boldsymbol{r}}-\bar{\boldsymbol{r}}_0)^t\,\mathbf{G}\,(\bar{\boldsymbol{r}}-\bar{\boldsymbol{r}}_0)\leq 1\right.\right\} \tag{7}$$

where $\mathbf{G}\in R^{n\times n}$ is a symmetric positive definite matrix. In this case, the left part of constraint $\min_{\{\bar{\boldsymbol{r}}\in M\}}\bar{\boldsymbol{r}}^t\phi\geq f$ is transformed into the following form:

$$\inf_{\mu\in M}\boldsymbol{r}^t\phi=\inf_{\left\|\mathbf{G}^{\frac{1}{2}}\hat{\boldsymbol{r}}\right\|\leq 1}\left(\bar{\boldsymbol{r}}_0+\hat{\boldsymbol{r}}\right)^t\phi=\bar{\boldsymbol{r}}_0^t\phi+\inf_{\|\boldsymbol{z}\|\leq 1}\boldsymbol{z}^t\mathbf{G}^{-\frac{1}{2}}\phi \tag{8}$$

where $\left\|\mathbf{G}^{-\frac{1}{2}}\hat{\boldsymbol{r}}\right\|=\sqrt{\hat{\boldsymbol{r}}^t\mathbf{G}\hat{\boldsymbol{r}}}$ and $\|\boldsymbol{z}\|=\sqrt{\boldsymbol{z}^t\boldsymbol{z}}$. Therefore, we obtain the following optimal solution with respect to $\boldsymbol{z}$:

$$\boldsymbol{z}^*=-\frac{\mathbf{G}^{-\frac{1}{2}}\phi}{\left\|\mathbf{G}^{-\frac{1}{2}}\phi\right\|} \tag{9}$$

Using this optimal solution $\boldsymbol{z}^*$, the expression (8) is transformed into the following form:

$$\inf_{\boldsymbol{r}\in M}\boldsymbol{r}^t\phi=\bar{\boldsymbol{r}}_0^t\phi-\left\|\mathbf{G}^{-\frac{1}{2}}\phi\right\| \tag{10}$$

In a way similar to mean value $\bar{\boldsymbol{r}}$, we consider the uncertainty set of variance $\mathbf{V}$ as follows:

$$S=\left\{\mathbf{V}\left|\mathbf{V}\succ 0,\ \mathbf{V}^L\leq\mathbf{V}\leq\mathbf{V}^U\right.\right\} \tag{11}$$

where $\mathbf{V}^L$ and $\mathbf{V}^U$ are symmetric positive definite matrixes. Note that, since $\mathbf{V}$ is restricted to be symmetric, the inequalities $\mathbf{V}^L\leq\mathbf{V}\leq\mathbf{V}^U$ can be represented with $n(n+1)$ componentwise inequalities, say for the upper triangle portions of these symmetric matrices. In other words, $\mathbf{V}^L\leq\mathbf{V}$ is a short-hand notation for $\sigma_{ij}^L\leq\sigma_{ij}$,

$1 \le i \le j \le n$, and similar for $\mathbf{V} \le \mathbf{V}^U$. Therefore, the constraint $\max_{\{\mathbf{V}\in S\}} \phi^t \mathbf{V} \phi \le \nu$ is transformed into

$$\max_{\{\mathbf{V}\in S\}} \phi^t \mathbf{V} \phi \le \nu \quad \Leftrightarrow \quad \phi^t \mathbf{V}^U \phi \le \nu ,$$

and main problem is equivalently transformed into the following problem:

$$\begin{aligned} &\text{Minimize} \quad \nu \\ &\text{subject to} \quad \phi^t \mathbf{V}^U \phi \le \nu, \\ &\qquad\qquad \bar{\mathbf{r}}_0^t \phi - \left\| \mathbf{G}^{-\frac{1}{2}} \phi \right\| \ge f, \\ &\qquad\qquad \mathbf{1}^t \phi = 1 \end{aligned} \tag{12}$$

Then, the problem (12) is also equivalently transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad f \\ &\text{subject to} \quad \bar{\mathbf{r}}_0^t \phi - \left\| \mathbf{G}^{-\frac{1}{2}} \phi \right\| \ge f, \\ &\qquad\qquad \phi^t \mathbf{V}^U \phi \le \nu, \\ &\qquad\qquad \mathbf{1}^t \phi = 1 \end{aligned} \tag{13}$$

These problems are convex programming problems, and so we obtain each optimal solution using the convex programming approach.

III. Fuzzy extension of Robust mean variance optimization problems

In Section 2, we consider that each parameter in the ellipsoidal set is fixed value. However, in real world, there exist various types of efficient and inefficient information, and each investor have an institution with respect to the current market. These factors include ambiguity and so we need to consider a robust portfolio selection problem including ambiguity. In this paper, we assume the $\bar{\boldsymbol{r}}_0$ to include ambiguity and to be a fuzzy number. Therefore, uncertainty set (7) is redefined into the following form:

$$M = \left\{ \bar{\boldsymbol{r}} : \left(\bar{\boldsymbol{r}} - \tilde{\boldsymbol{r}}_0 \right)^t \mathbf{G} \left(\bar{\boldsymbol{r}} - \tilde{\boldsymbol{r}}_0 \right) \le 1 \right\} \tag{14}$$

Then, in this paper, the fuzzy number $\tilde{\boldsymbol{r}}_0$ is assumed to be a following L-shape fuzzy number:

$$\mu_{\tilde{r}_{0j}}(\omega) = \begin{cases} L\left(\dfrac{\omega - \bar{r}_{0j}}{\alpha_j} \right) & \left(\bar{r}_{0j} - \alpha_j \le \omega \le \bar{r}_{0j} + \alpha_j \right) \\ 0 & \left(\omega \le \bar{r}_{0j} - \alpha_j ,\ \bar{r}_{0j} + \alpha_j \le \omega \right) \end{cases} \tag{15}$$

In this paper, we assume the following inequality with respect to each asset:

$$\bar{\mu}_{0j} - L^*(h) \alpha_j \ge 0 \tag{16}$$

The uncertainty set $\tilde{U} = \left(\bar{\boldsymbol{r}} - \tilde{\boldsymbol{r}}_0 \right)^t \mathbf{G} \left(\bar{\boldsymbol{r}} - \tilde{\boldsymbol{r}}_0 \right)$ includes fuzzy numbers $\tilde{\boldsymbol{r}}_0$ and so $\tilde{U}$ is a fuzzy numbers. Therefore, the membership function of $\tilde{U}$ is as follows:

$$\mu_{\tilde{U}}(\omega) = \sup_{\gamma_{0j}} \left\{ \min_{1 \le j \le n} \mu_{\tilde{r}_{0j}}\left(\gamma_{0j} \right) \middle| \omega = \left(\bar{\boldsymbol{r}} - \boldsymbol{\gamma}_0 \right)^t \mathbf{G} \left(\bar{\boldsymbol{r}} - \boldsymbol{\gamma}_0 \right) \right\} \tag{17}$$

Then, the uncertainty set (14) is transformed into the following form in the case introducing the h-cut:

$$M_h = \left\{ \bar{\boldsymbol{r}} \middle| \mu_{\tilde{V}}(\omega) \ge h \right\} \tag{18}$$

Furthermore, taking account of the vagueness of human judgment and flexibility for the execution of a plan, we give a fuzzy goal to the target probability as the fuzzy set characterized by a membership function. In this subsection, we consider the fuzzy goal of probability $\mu_{\tilde{G}}(f)$ which is represented by,

$$\mu_{\tilde{G}_p}(f) = \begin{cases} 0 & f \le f_0 \\ g_F(f) & f_0 \le f \le f_1 \\ 1 & f_1 \le f \end{cases} \tag{19}$$

where $g_F(f)$ is a strictly increasing continuous function. Then, using a concept of possibility measure, we introduce the degree of possibility as follows:

$$\prod_{\tilde{F}} \left(\tilde{G} \right) = \sup_f \min \left\{ \mu_{\tilde{U}}(f),\ \mu_{\tilde{G}_f}(f) \right\} \tag{20}$$

In this possibility measure, in the case that we consider $\mu_{\tilde{U}}(f) \ge h$, we obtain the following transformation:

$$\begin{aligned} &\mu_{\tilde{U}}(\omega) \ge h \\ &\Leftrightarrow \sup_{\gamma_{0j}} \left\{ \min_{1 \le j \le n} \mu_{\tilde{r}_{0j}}\left(\gamma_{0j} \right) \middle| \omega = \left(\bar{\boldsymbol{r}} - \boldsymbol{\gamma}_0 \right)^t \mathbf{G} \left(\bar{\boldsymbol{r}} - \boldsymbol{\gamma}_0 \right) \le 1 \right\} \ge \alpha \\ &\Leftrightarrow \bar{\boldsymbol{r}}^t \mathbf{G} \bar{\boldsymbol{r}} - 2 \bar{\boldsymbol{r}}^t \mathbf{G} \left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right) \\ &\qquad + \left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right)^t \mathbf{G} \left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right) \le 1 \\ &\Leftrightarrow \left(\bar{\boldsymbol{r}} - \left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right) \right)^t \mathbf{G} \left(\bar{\boldsymbol{r}} - \left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right) \right) \le 1 \end{aligned} \tag{21}$$

where $L^*(x)$ is a pseudo inverse function of $L(\omega)$. Using this inequality, the expression (8) is transformed into the following expression:

$$\begin{aligned} \inf_{\bar{\boldsymbol{r}} \in M} \bar{\boldsymbol{r}}^t \phi &= \inf_{\left\| \mathbf{G}^{\frac{1}{2}} \hat{\boldsymbol{r}} \right\| \le 1} \left(\left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right) + \hat{\boldsymbol{r}} \right)^t \phi \\ &= \left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right)^t \phi + \inf_{\|\mathbf{z}\| \le 1} z^t \mathbf{G}^{-\frac{1}{2}} \phi \end{aligned} \tag{22}$$

Then, from the optimal value of (9), this expression is equal to the following form:

$$\inf_{\bar{\boldsymbol{r}} \in M} \bar{\boldsymbol{r}}^t \phi = \left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right)^t \phi - \left\| \mathbf{G}^{-\frac{1}{2}} \phi \right\| \tag{23}$$

Therefore, in the case that we consider the possibility measure constraint $\prod_{\tilde{F}} \left(\tilde{G} \right) \ge h$, this constraint is transformed into the following inequality:

$$\begin{aligned} &\prod_{\tilde{F}} \left(\tilde{G} \right) \ge h \\ &\Leftrightarrow \sup_f \min \left\{ \mu_{\tilde{F}}(f),\ \mu_{\tilde{G}_f}(f) \right\} \ge h \\ &\Leftrightarrow \mu_{\tilde{F}}(f) \ge h,\ \mu_{\tilde{G}_f}(f) \ge h \\ &\Leftrightarrow \sup \left\{ \min_{\mu \in M} \boldsymbol{r}^t \phi \ge f \right\} \ge h,\ f \ge g_F^{-1}(h) \\ &\Leftrightarrow \left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right)^t \phi - \left\| \mathbf{G}^{-\frac{1}{2}} \phi \right\| \ge f,\ f \ge g_F^{-1}(h) \\ &\Leftrightarrow \left(\bar{\boldsymbol{r}}_0 - L^*(h) \boldsymbol{\alpha} \right)^t \phi - \left\| \mathbf{G}^{-\frac{1}{2}} \phi \right\| \ge g_F^{-1}(h) \end{aligned} \tag{24}$$

In a way similar to mean value $\bar{\boldsymbol{r}}$, we consider the uncertainty set of variance $\mathbf{V}$ as follows:

$$S=\left\{\mathbf{V}\middle|\mathbf{V}\succ 0,\ \tilde{\mathbf{V}}^L\le\mathbf{V}\le\tilde{\mathbf{V}}^U\right\} \tag{25}$$

In this paper, we assume this uncertainty set as the following form introducing a L-shape fuzzy number with respect to the each component of $\mathbf{V}$.

$$S=\left\{\mathbf{V}=\left(\tilde{\sigma}_{ij}\right)\middle|\begin{array}{l}\mu_{\tilde{\sigma}_{ij}}(\omega)=L\left(\dfrac{\sigma_{ij}-\omega}{\beta_{ij}}\right),\ \left(\sigma_{ij}-\beta_{ij}\le\omega\le\sigma_{ij}+\beta_{ij}\right)\\ \sigma_{ij}=\sigma_{ji},\ \beta_{ij}=\beta_{ji}\end{array}\right\} \tag{26}$$

Then, we consider the fuzzy goal of total variance $\mu_{\tilde{G}}(\nu)$ which is represented by,

$$\mu_{\tilde{G}_p}(\nu)=\begin{cases}1 & \nu\le\nu_L\\ g_V(\nu) & \nu_L\le\nu\le\nu_U\\ 0 & \nu_U\le\nu\end{cases} \tag{27}$$

where $g_V(\nu)$ is a strictly decreasing continuous function. Then, using a concept of possibility measure, we introduce the degree of possibility as follows:

$$\prod\nolimits_{\tilde{V}}\left(\tilde{G}\right)=\sup_{\nu}\min\left\{\mu_{\tilde{V}}(\nu),\ \mu_{\tilde{G}_p}(\nu)\right\} \tag{28}$$

With respect to this possibility measure, in a way similar to the transformation (24), $\prod_{\tilde{V}}\left(\tilde{G}\right)\ge h$ is transformed into the following inequality:

$$\begin{aligned}&\prod\nolimits_{\tilde{V}}\left(\tilde{G}\right)\ge h\\ \Leftrightarrow\ &\sup_{\nu}\min\left\{\mu_{\tilde{V}}(\nu),\ \mu_{\tilde{G}_p}(\nu)\right\}\ge h\\ \Leftrightarrow\ &\mu_{\tilde{V}}(\nu)\ge h,\ \mu_{\tilde{G}_p}(\nu)\ge h\\ \Leftrightarrow\ &\mathrm{Pos}\left\{\max\nolimits_{\{\mathbf{V}\in S\}}\boldsymbol{\phi}^t\mathbf{V}\boldsymbol{\phi}\ge\nu\right\}\ge h,\ \nu\le g_V^{-1}(h)\\ \Leftrightarrow\ &\boldsymbol{\phi}^t\mathbf{V}_{(h)}^U\boldsymbol{\phi}\le\nu,\ \nu\le g_V^{-1}(h)\\ \Leftrightarrow\ &\boldsymbol{\phi}^t\mathbf{V}_{(h)}^U\boldsymbol{\phi}\le g_V^{-1}(h)\end{aligned} \tag{29}$$

where $\mathbf{V}_{(h)}^U$ is assumed to be a symmetric positive definite matrix whose each component becomes $\sigma_{ij}+L^*(h)\beta_{ij}$. Then, we propose the fuzzy robust portfolio selection problem as the following possibility maximization model:

$$\begin{aligned}&\text{Maximize}\quad h\\ &\text{subject to}\quad \prod\nolimits_{\tilde{V}}\left(\tilde{G}\right)\ge h,\ \prod\nolimits_{\tilde{F}}\left(\tilde{G}\right)\ge h,\\ &\qquad\qquad\ \mathbf{1}^t\boldsymbol{\phi}=1\end{aligned} \tag{30}$$

This problem is equivalently transformed into the following problem using the transformations of possibility constraints (24) and (29):

$$\begin{aligned}&\text{Maximize}\quad h\\ &\text{subject to}\quad \left(\bar{\boldsymbol{r}}_0-L^*(h)\boldsymbol{\alpha}\right)^t\boldsymbol{\phi}-\left\|\mathbf{G}^{-\frac{1}{2}}\boldsymbol{\phi}\right\|\ge g_F^{-1}(h),\\ &\qquad\qquad\ \boldsymbol{\phi}^t\mathbf{V}_{(h)}^U\boldsymbol{\phi}\le g_V^{-1}(h),\\ &\qquad\qquad\ \mathbf{1}^t\boldsymbol{\phi}=1\end{aligned} \tag{31}$$

It should be noted here that problem (31) is a nonconvex programming problem and it is not solved by the linear programming techniques or convex programming techniques. However, since a decision variable h is fixed, this problem is equivalent to the problem to find the feasible solution ϕ_h involving the following set:

$$\phi_h\in S=\left\{\boldsymbol{\phi}\middle|\begin{array}{l}\left(\bar{\boldsymbol{r}}_0-L^*(h)\boldsymbol{\alpha}\right)^t\boldsymbol{\phi}-\left\|\mathbf{G}^{-\frac{1}{2}}\boldsymbol{\phi}\right\|\ge g_F^{-1}(h),\\ \boldsymbol{\phi}^t\mathbf{V}_{(h)}^U\boldsymbol{\phi}\le g_V^{-1}(h),\\ \mathbf{1}^t\boldsymbol{\phi}=1\end{array}\right\} \tag{32}$$

Furthermore, to find feasible solution ϕ_h and optimal solution ϕ^* more efficiently and analytically, we transform problem (31) into the equivalent deterministic problem.

$$\begin{aligned}&\text{Minimize}\quad -\left(\bar{\boldsymbol{r}}_0-L^*(\bar{h})\boldsymbol{\alpha}\right)^t\boldsymbol{\phi}+\left\|\mathbf{G}^{-\frac{1}{2}}\boldsymbol{\phi}\right\|+\boldsymbol{\phi}^t\mathbf{V}_{(h)}^U\boldsymbol{\phi}\\ &\qquad\qquad\ +g_V^{-1}(\bar{h})-g_F^{-1}(\bar{h})\\ &\text{subject to}\quad \mathbf{1}^t\boldsymbol{\phi}=1\end{aligned} \tag{33}$$

With respect to the relation between problems (31) and (33), we obtain the following theorem.

Theorem 1

Let the optimal value of problem (31) be h^*. Furthermore let the optimal solution of problem (33) be $\phi_{\bar{h}}$ and its optimal value be $Z_{\bar{h}}$. Then the following relationship holds.

$$\begin{aligned}h^*>\bar{h}\quad&\Leftrightarrow\quad Z_{\bar{h}}<0\\ h^*=\bar{h}\quad&\Leftrightarrow\quad Z_{\bar{h}}=0\\ h^*<\bar{h}\quad&\Leftrightarrow\quad Z_{\bar{h}}>0\end{aligned} \tag{34}$$

Then, the optimal solution of this problem is equivalent to that of the following problem:

$$\begin{aligned}&\text{Minimize}\quad -\left(\bar{\boldsymbol{r}}_0-L^*(h)\boldsymbol{\alpha}\right)^t\boldsymbol{\phi}+\left\|\mathbf{G}^{-\frac{1}{2}}\boldsymbol{\phi}\right\|+\boldsymbol{\phi}^t\mathbf{V}_{(h)}^U\boldsymbol{\phi}\\ &\text{subject to}\quad \mathbf{1}^t\boldsymbol{\phi}=1\end{aligned} \tag{35}$$

i.e.

$$\begin{aligned}&\text{Minimize}\quad -\left(\bar{\boldsymbol{r}}_0-L^*(h)\boldsymbol{\alpha}\right)^t\boldsymbol{\phi}+\sqrt{\boldsymbol{\phi}^t\mathbf{G}\boldsymbol{\phi}}+\boldsymbol{\phi}^t\mathbf{V}_{(h)}^U\boldsymbol{\phi}\\ &\text{subject to}\quad \mathbf{1}^t\boldsymbol{\phi}=1\end{aligned} \tag{36}$$

This problem is also a convex programming problem. Therefore, to solve it more efficiently and analytically, we introduce the following auxiliary problem including a parameter R:

$$\begin{aligned}&\text{Minimize}\quad R\left\{-\left(\bar{\boldsymbol{r}}_0-L^*(h)\boldsymbol{\alpha}\right)^t\boldsymbol{\phi}+\boldsymbol{\phi}^t\mathbf{V}_{(h)}^U\boldsymbol{\phi}\right\}+\frac{1}{2}\left(\boldsymbol{\phi}^t\mathbf{G}\boldsymbol{\phi}\right)\\ &\text{subject to}\quad \mathbf{1}^t\boldsymbol{\phi}=1\end{aligned} \tag{37}$$

Since this problem is a parametric quadratic programming problem, we obtain an optimal portfolio more efficiently and analytically than problem (37). Furthermore, with respect to the relation between problem (36) and (37), the following theorem hold.

Theorem 2

Let ϕ^* be an optimal solution of problem (37). If $R = \sqrt{\phi^{*t}\mathbf{G}\phi^*}$ is satisfied, ϕ^* is also an optimal solution of problem (36).

Proof

Comparing KKT condition of problem (36) with that of problem (37), this theorem holds obviously. □

From this theorem, we consider solving the auxiliary problem (37). Lagrange function and KKT condition of problem (37) is asa follows:

(Lagrange function)

$$L = R\left\{-\left(\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha}\right)^t \phi + \phi^t \mathbf{V}_{(h)}^U \phi\right\} + \frac{1}{2}\left(\phi^t \mathbf{G}\phi\right) + \lambda\left(\mathbf{1}^t \phi - 1\right) \quad (38)$$

(KKT condition)

$$\begin{aligned} &\frac{\partial L}{\partial \phi} = R\left\{-\left(\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha}\right) + 2\mathbf{V}_{(h)}^U \phi\right\} + \mathbf{G}\phi + \lambda \cdot \mathbf{1} = \mathbf{0} \\ &\lambda\left(\mathbf{1}^t \phi - 1\right) = 0 \end{aligned} \quad (39)$$

Therefore, we obtain an optimal portfolio involving the parameter R to solve the following simultaneous linear equations:

$$\begin{cases} \left(2R\mathbf{V}_{(h)}^U + \mathbf{G}\right)\phi + \lambda \cdot \mathbf{1} - R\left(\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha}\right) = \mathbf{0} \\ \mathbf{1}^t \phi = 1 \end{cases} \quad (40)$$

In the simultaneous linear equations, since the number of linear equations is $n+1$ and that of decision variables is $n+1$, the optimal portfolio is obtained uniquely in the case that the parameter R is assumed to be a fixed value. However, we cannot obtain the strict optimal solution without deciding parameter R. Therefore, we consider the decision method with respect to R.

First, with respect to the objective function of problem (37), the following lemmas holds:

Lemma 1

$-\left(\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha}\right)^t \phi + \phi^t \mathbf{V}_{(h)}^U \phi$ is an increasing function of R.

Lemma 2

$\frac{1}{2}\left(\phi^t \mathbf{G}\phi\right)$ is an increasing function of R.

Proof

In a way similar to the proof in the previous studies such as [16, 17], these lemmas holds. □

Let $g(R) = R - \sqrt{\phi^{*t}\mathbf{G}\phi^*}$. From these lemmas, the following theorem holds.

Theorem 3

Let the optimal solution to auxiliary problem be ϕ^*. Then the following relationship holds.

$$\begin{aligned} R^* > R &\iff g(R) > 0 \\ R^* = R &\iff g(R) = 0 \\ R^* < R &\iff g(R) < 0 \end{aligned} \quad (41)$$

From this theorem, ϕ^* satisfying $R^2 = \phi^{*t}\mathbf{G}\phi^*$ is an optimal solution of problem (37). Consequently, using a bisection algorithm with respect to R and Theorem 2, we obtain the strict optimal solution of main problem. Thus, we construct the following solution procedure.

Solution procedure

STEP1: Elicit the membership function of a fuzzy goal with respect to the total expected return and variance.

STEP2: Set $h \leftarrow 1$ and solve problem (33). If the optimal objective value Z_h of problem (33) satisfies $Z_h < 0$, then terminate. In this case, the obtained current solution is an optimal solution of main problem.

STEP3: Set $h \leftarrow 0$ and solve problem (33). If the optimal objective value Z_h of problem (33) satisfies $Z_h > 0$, then terminate. In this case, there is no feasible solution and it is necessary to reset a fuzzy goal with respect to the total expected return and variance.

STEP4: Set $U_h \leftarrow 1$ and $L_h \leftarrow 0$.

STEP5: Set $h \leftarrow \frac{U_h + L_h}{2}$

STEP6: Solve problem (33) and calculate the optimal objective value Z_h of problem (33). If $Z_h > 0$, then set $U_h \leftarrow h$ and return to Step 5. If $Z_h < 0$, then set $L_h \leftarrow h$ and return to Step 5. If $Z_h = 0$, then terminate the algorithm. In this case, $\phi^*(h)$ is equal to a global optimal solution of main problem.

IV. Conclusion

In this paper, we have proposed extension models of robust portfolio selection problems considering uncertainty conditions. Since these problems are not well-defined problems due to fuzzy numbers, we have introduced the chance constraints and transformed them into the deterministic equivalent problems. Furthermore, to solve them analytically, we have constructed the solution method. Our proposed models include the basic robust portfolio selection problems and so we may apply our models to the more flexible and complex portfolio selection problems in real investment markets than the previous models.

As the future studies, we need to consider not only mean-variance portfolio selection problem but also other portfolio selection models. Then, we are now attacking the cases that optimal solutions are restricted to be integers and multi-period portfolio selection problem.

REFERENCES

[1] V.S. Bawa and E.B. Lindenberg, “Capital market equilibrium in a mean-lower partial moment framework”, *Journal of Financial Economics*, 5, pp.189-200, 1977.

[2] A. Ben-Tal and A. Nemirovski, “Robust solutions of uncertain linear programs”, *Operations Research Letters*, 25(1), pp.1-13, 1999.

[3] A. Ben-Tal and A. Nemirovski, “Lectures on modern convex optimization”, SIAM, Philadelphia, PA, 2001.

[4] A. Bilbao-Terol, B. Perez-Gladish, M. Arenas-Parra and M.V. Rodriguez-Uria,“Fuzzy compromise programming for portfolio selection”, *Applied Mathematics and Computation*, 173, pp.251-264, 2006.

[5] C. Carlsson, R. Fuller and P. Majlender, “A possibilistic approach to selecting portfolios with highest utility score”, *Fuzzy Sets and Systems*, 131, pp.12-21, 2002.

[6] E.J. Elton, M.J. Gruber, *Modern Portfolio Theory and Investment Analysis*, Wiley, New York, 1995.

[7] L.E. Ghaoui and H. Lebret, “Robust solutions to least squares problems with uncertain data”, *Journal of Matrix Analysis and Applications*, 18, pp.1035-1064, 1997.

[8] D. Goldfarb and G. Iyengar, “Robust portfolio selection problems”, *Mathematics of Operations Research*, 28, pp.1-38, 2003

[9] P. Guo and H. Tanaka, “Possibility data analysis and its application to portfolio sekection problems”, *Fuzzy Economic Rev*., 3, pp.3-23, 1998.

[10] B.V. Halldórsson and R.H. Tütüncü, “An interior-point method for a class of saddle-point problems”, Technical report, Carnegie Mellon University, 2000.

[11] X.Huang, “Fuzzy chance-constrained portfolio selection”, *Applied Mathematics and Computation*, 177, pp.500-5007, 2006.

[12] X. Hung, “Two new models for portfolio selection with stochastic returns taking fuzzy information”, *European Journal of Operational Research*, 180, pp.396-405, 2007.

[13] H. Inaba, S. Mizuno and K. Nakata, “Robust tracking error optimization problems by second-order cone programming”, *Transactions of the Operations Research Society of Japan* (in Japanese), 48, pp.12-25, 2005.

[14] M. Inuiguchi, J. Ramik, “Possibilisitc linear programming: A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem”, *Fuzzy Sets and Systems*, 111, pp.3-28, 2000.

[15] M.Inuiguchi and T. Tanino, “Portfolio selection under independent possibilistic information”, *Fuzzy Sets and Systems*, 115, pp.83-92, 2000.

[16] H. Ishii, T. Nishida, “Stochastic linear knapsack problem: probability maximization model”, *Mathmatica Japonicae*, 29, pp.273-281.1984.

[17] H. Katagiri, H. Ishii and M. Sakawa, “On fuzzy random linear knapsack problems”, *Central European Journal of Operations Research*, Vol.12 No.1, pp.59-70, 2004.

[18] H. Katagiri, M. Sakawa and H. Ishii, “A study on fuzzy random portfolio selection problems using possibility and necessity measures”, *Scientiae Mathematicae Japonicae*, Vol.65 No.2, pp.361-369, 2005.

[19] H. Konno, “Piecewise linear risk functions and portfolio optimization”, *Journal of Operations Research Society of Japan*, 33, pp.139-159, 1990.

[20] H. Konno, H. Shirakawa and H. Yamazaki, “A mean-absolute deviation-skewness portfolio optimization model”, *Annals of Operations Research*, 45, pp.205-220, 1993.

[21] B J. Lintner, “Valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets”, Rev. Econom. Statist. 47, pp. 13-37, 1965.

[22] M.S. Lobo, *Robust and convex optimization with applications in finance*, Doctor thesis of the Department of Electrical Engineering and the Committee on Graduate Studies, Stanford University, 2000

[23] David G. Luenberger, *Investment Science*, Oxford Univ. Press, 1997.

[24] H. Markowitz, *Portfolio Selection*, Wiley, New York, 1952.

[25] J. Mossin, “Equilibrium in capital asset markets”, *Econometrica* 34(4), pp. 768-783, 1966.

[26] R.T. Rockfellar and S. Uryasev, “Optimization of conditional value-at-risk”, *Journal of Risk*, 2(3), pp.1-21, 2000.

[27] W.F. Sharpe, “Capital asset prices: A theory of market equivalent under conditions of risk”, *Journal of Finance* 19(3), pp. 425-442, 1964.

[28] Marc C. Steinbach, “Markowitz revisited: Mean-variance models in financial portfolio analysis”, *SIAM Review*, Vol. 43, No. 1, pp.31-85, 2001.

[29] H. Tanaka, P. Guo, “Portfolio selection based on upper and lower exponential possibility distributions”, *European Journal of Operational Researches*, 114, pp. 115-126, 1999.

[30] H. Tanaka, P. Guo and I.B. Turksen, “Portfolio selection based on fuzzy probabilities and possibility distributions”, *Fuzzy Sets and Systems*, 111, pp.387-397, 2000.

[31] J. Watada, “Fuzzy portfolio selection and its applications to decision making”, Tatra Mountains Math. Pub. 13, pp.219-248, 1997.