Interactive Multiobjective Fuzzy Random Programming through Level Set Optimization

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Abstract— **This paper focuses on multiobjective linear programming problems involving fuzzy random variable coefficients. A new decision making model and Pareto optimal solution concept are proposed using α-level cuts of membership function. It is shown that the problem including both randomness and fuzziness is equivalently transformed into a deterministic problem. An interactive algorithm is proposed in order to obtain a satisficing solution for a decision maker through interaction.**

Keywords: fuzzy random variable, multiobjective programming, Pareto optimal solution, interactive algorithm, level set

1 Introduction

In the real world, there are a number of cases where one should make a decision based on data with uncertainty. Many researchers considered stochastic programming and fuzzy programming to deal with such uncertainty in decision-making. In stochastic programming, Dantzig [3] considered two-stage problems, and Charnes and Cooper [2] proposed chance constrained programming and several decision making models based on various optimization criteria. One of those models is a probability maximization model which is to maximize the probability that the objective function. On the other hand, in fuzzy mathematical programming, Zimmerman [20] considered flexible programming, and Dubois and Prade [5] developed possibilistic programming.

Most of studies on mathematical programming take account of either fuzziness or randomness. However, in practice, decision makers are faced with decision making systems where parameters include both fuzziness and randomness. In such a case, the parameters are not always estimated using a well-known concept such as a random variable or a fuzzy set. For example, in a production planning problem, the demand of some commodity is often dependent on weathers, i.e., fine, cloudy and rainy. If each of weathers occurs randomly and an expert estimates the demand for each of weathers as an ambiguous value such as a fuzzy number, then the demand is represented with a fuzzy random variable. As another example in real-world decision making, a profit per unit comes under the influence of the economic conditions which changes randomly is expressed by a fuzzy random variable because a profit per unit under each scenario of the economic conditions is often estimated with a fuzzy set. A fuzzy random variable was first defined by Kwakernaak [11]. The mathematical basis of fuzzy random variables was developed by Puri and Ralescu [15]. Some literatures considered linear programming problems including fuzzy random variables [7, 12, 13, 14, 17]. Recently, fuzzy random variables were applied to various topics such as stopping games [19], inventory problems [4], 0-1 programming [8] and spanning tree problems [9].

In this article, for a multi-objective linear programming problem including fuzzy random variable coefficients, we shall define a new Pareto optimal solution concept which is an extended version of Pareto optimal solutions in fuzzy programming [16], and construct an interactive algorithm based on the reference point method proposed by Wierzbicki [18] to obtain a satisficing solution for a decision maker through the interaction.

This paper is organized as follows: In the next section, we formulate a multiobjective linear programming problem where the coefficients of each objective function are fuzzy random variable. Section 3 transforms the original problem including fuzzy random variable into a deterministic equivalent problem. In Section 4, we define a new solution concept and provide an interactive algorithm to find a satisficing solution for a decision maker. In Section 5, we provide a numerical example to illustrate the usefulness of the proposed model and algorithm. Finally, in Section 6, we conclude this paper and discuss future studies.

2 Problem formulation

In this paper, we consider the following multiobjective linear programming problem:

$$\left.\begin{array}{ll} \min & \tilde{\bar{\boldsymbol{C}}}_i \boldsymbol{x},\ i=1,\ldots,k \\ \text{s. t.} & \tilde{\bar{\boldsymbol{A}}}_l \boldsymbol{x} \leq \tilde{\bar{B}}_l,\ l=1,\ldots,m \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}\right\} \quad (1)$$

where $\boldsymbol{x}$ is an n-dimensional decision variable column vector, and $\tilde{\bar{\boldsymbol{C}}}_i = (\tilde{\bar{C}}_{i1}, \ldots, \tilde{\bar{C}}_{in}),\ i=1,\ldots,k,\ \tilde{\bar{\boldsymbol{A}}}_l =$

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$(\tilde{\bar{A}}_{l1},\ldots,\tilde{\bar{A}}_{ln}),\ l=1,\ldots,m.$

Let us assume that each of $\tilde{\bar{C}}_{ij}$ is a fuzzy random variable that takes a fuzzy number under the occurrence of an elementary event ω, which is characterized by the following membership function:

$$\mu_{\tilde{\bar{C}}_{ij}(\omega)}(\tau)=\begin{cases} L\left(\dfrac{\bar{d}^c_{ij}(\omega)-\tau}{\beta^c_{ij}}\right) & (\tau\le \bar{d}^c_{ij}(\omega)) \\ R\left(\dfrac{\tau-\bar{d}^c_{ij}(\omega)}{\delta^c_{ij}}\right) & (\tau> \bar{d}^c_{ij}(\omega)) \end{cases} \tag{2}$$

where L and R are reference functions satisfying the following conditions:

1. $L(t)\stackrel{\triangle}{=}\max\{0,l(t)\}$ and $R(t)\stackrel{\triangle}{=}\max\{0,r(t)\}$.
2. $l(t)$ and $r(t)$ are strictly decreasing continuous functions on $[0,\infty)$.
3. $l(0)=r(0)=1$.

We denote by $\tilde{\bar{C}}_{ij}=(\bar{d}^c_{ij},\ \beta^c_{ij},\ \delta^c_{ij})_{LR}$ the fuzzy random variable characterized by (2). In a similar manner, let us denote $\tilde{\bar{A}}_{lj}$ and $\tilde{\bar{B}}_l$ by $\tilde{\bar{A}}_{lj}=(\bar{d}^a_{lj},\ \beta^a_{lj},\ \delta^a_{lj})_{LR}$ and $\tilde{\bar{B}}_l=(\bar{d}^b_l,\ \beta^b_l,\ \delta^b_l)_{LR}$, respectively. We assume that $\bar{\boldsymbol{d}}^c_i=(\bar{d}^c_{i1},\ldots,\bar{d}^c_{in})$ and $\bar{\boldsymbol{d}}^a_l=(\bar{d}^a_{l1},\ldots,\bar{d}^a_{ln})$ are n-dimensional normal random variables with mean vectors $\boldsymbol{m}^c_i=(m^c_{i1},\ldots,m^c_{in})$, $\boldsymbol{m}^a_l=(m^a_{l1},\ldots,m^a_{ln})$ and variance-covariance matrices $V^c_i,\ V^a_l$, respectively, and that $\bar{d}^b_l$ is a normal random variable with m^b_l and variance v^b_l. Parameters $\boldsymbol{\beta}^c_i=(\beta^c_{i1},\ldots,\beta^c_{in})$, $\boldsymbol{\delta}^c_i=(\delta^c_{i1},\ldots,\delta^c_{in})$, $\boldsymbol{\beta}^a_l=(\beta^a_l,\ldots,\beta^a_l)$ and $\boldsymbol{\delta}^a_l=(\delta^a_{l1},\ldots,\delta^a_{ln})$ are positive constant vectors, and β^b_l and δ^b_l are positive constants. In this paper, we denote randomness and fuzziness included in coefficients or goals by "$\bar{\ }$" and "$\sim$", respectively.

Since all the coefficients of each objective function are L-R type fuzzy random variables, we can apply the operation on fuzzy numbers induced by Zadeh's extension principle. As a result, the objective functions become the fuzzy random variables characterized by following membership functions:

$$\mu_{\tilde{\bar{\boldsymbol{C}}}_i(\omega)\boldsymbol{x}}(y)=\begin{cases} L\left(\dfrac{\bar{\boldsymbol{d}}^c_i(\omega)\boldsymbol{x}-y}{\boldsymbol{\beta}^c_i\boldsymbol{x}}\right) & (y\le \bar{\boldsymbol{d}}^c_i(\omega)\boldsymbol{x}) \\ R\left(\dfrac{y-\bar{\boldsymbol{d}}^c_i(\omega)\boldsymbol{x}}{\boldsymbol{\delta}^c_i\boldsymbol{x}}\right) & (y\ge \bar{\boldsymbol{d}}^c_i(\omega)\boldsymbol{x}). \end{cases} \tag{3}$$

3 Level set optimization model for fuzzy random programming problems

Since problem (1) is not a well-defined problem, there are a lot of possible decision making approaches in terms of various optimization criteria. In this paper, we consider a case where the decision maker attaches importance to the fact that all grades of membership functions are greater than or equal to some satisficing level α, called "admissible level." In other words, we focus on "α-level sets" of coefficient vectors. Then, the problem to be considered is formulated as

$$\left.\begin{array}{ll} \min & \bar{\boldsymbol{c}}_i\boldsymbol{x},\ i=1,\ldots,k \\ \text{s. t.} & \bar{\boldsymbol{a}}_l\boldsymbol{x}\le \bar{b}_l,\ l=1,\ldots,m,\ \ \boldsymbol{x}\ge\boldsymbol{0} \\ & (\bar{\boldsymbol{a}}_l,\bar{b}_l,\bar{\boldsymbol{c}}_i)\in(\tilde{\bar{\boldsymbol{A}}}_{l\alpha},\tilde{\bar{B}}_{l\alpha},\tilde{\bar{\boldsymbol{C}}}_{i\alpha}), \\ & \quad l=1,\ldots,m,\ i=1,\ldots,k. \end{array}\right\} \tag{4}$$

Considering the imprecise nature of the decision maker's judgment, it is natural to assume that the decision maker may have fuzzy goals for each of the objective functions in problem (4). In minimization problems, a goal stated by the decision maker may be to achieve "substantially less than or equal to some value." This type of statement is represented with a fuzzy goal characterizing the following linear membership function:

$$\mu_i(y)=\left\{\begin{array}{cl} 0, & y>g^0_i \\ \dfrac{y-g^0_i}{g^1_i-g^0_i}, & g^1_i\le y\le g^0_i \\ 1, & y<g^1_i, \qquad i=1,\ldots,k \end{array}\right. \tag{5}$$

where g^0_i and g^1_i are positive constants. We apply Zimmermann's method to determine the parameters g^0_i and g^1_i. First, we solve the following individual minimization problems under the given constraints:

$$\left.\begin{array}{ll} \min & z_i(\boldsymbol{x})=\boldsymbol{m}^c_i\boldsymbol{x} \\ \text{s. t.} & \boldsymbol{m}^a_l\boldsymbol{x}\le m^b_l,\ l=1,\ldots,m \\ & \boldsymbol{x}\ge\boldsymbol{0}. \end{array}\right\} \tag{6}$$

Let $\boldsymbol{x}^{io}$ be an optimal solution to the ith minimization problem and calculate the optimal values

$$z^{\min}_i=z_i(\boldsymbol{x}^{io}),\quad i=1,\ldots,k. \tag{7}$$

The next step to set up the parameters is to calculate

$$z^{\mathrm{m}}_i=\max(z_i(\boldsymbol{x}^{1o}),\ldots,z_i(\boldsymbol{x}^{i-1,o}),z_i(\boldsymbol{x}^{i+1,o}),\ldots,z_i(\boldsymbol{x}^{ko})) \tag{8}$$

and set $g^1_i=z^{\min}_i$ and $g^0_i=z^{\mathrm{m}}_i$. Then (4) is rewritten as the following problem:

$$\left.\begin{array}{ll} \max & \mu_i(\bar{\boldsymbol{c}}_i\boldsymbol{x}),\ i=1,\ldots,k \\ \text{s. t.} & \bar{\boldsymbol{a}}_l\boldsymbol{x}\le \bar{b}_l,\ l=1,\ldots,m,\ \ \boldsymbol{x}\ge\boldsymbol{0} \\ & (\bar{\boldsymbol{a}}_l,\bar{b}_l,\bar{\boldsymbol{c}}_i)\in(\tilde{\bar{\boldsymbol{A}}}_{l\alpha},\tilde{\bar{B}}_{l\alpha},\tilde{\bar{\boldsymbol{C}}}_{i\alpha}), \\ & l=1,\ldots,m,\ i=1,\ldots,k. \end{array}\right\} \tag{9}$$

It should be emphasized here that the above problem is regarded as a kind of stochastic programming problems because $\bar{\boldsymbol{c}}_i$ and $\bar{\boldsymbol{a}}_l$ are random set vectors, and $\bar{b}_l$ are a random set. Especially, since these random set are defined on one-dimensional space as shown in (2), all of coefficients are so-called "interval random variables."

There are several stochastic programming models such as an expectation optimization model, a variance minimization model, a probability maximization model by Charnes and Cooper [2] and a fractile criterion optimization model by Kataoka [10] and Geoffrion [6]. In this article, we consider the following problem based on the fractile criterion optimization model:

$$\left.\begin{array}{ll} \max & h_i,\ i=1,\dots,k \\ \text{s. t.} & \Pr[\mu_i(\bar{\boldsymbol{c}}_i\boldsymbol{x}) \geq h_i] \geq \theta_i,\ i=1,\dots,k \\ & \Pr[\bar{\boldsymbol{a}}_l\boldsymbol{x} \leq \bar{b}_l] \geq \eta_l,\ l=1,\dots,m,\quad \boldsymbol{x}\geq \boldsymbol{0} \\ & (\bar{\boldsymbol{a}}_l,\bar{b}_l,\bar{\boldsymbol{c}}_i) \in (\tilde{\bar{\boldsymbol{A}}}_{l\alpha},\tilde{\bar{B}}_{l\alpha},\tilde{\bar{\boldsymbol{C}}}_{i\alpha}), \\ & l=1,\dots,m,\ i=1,\dots,k \end{array}\right\} \tag{10}$$

where θ_i and η_l are constants called "confidence levels," and they are determined by a decision maker. We assume that $\theta_i \geq 1/2$ and $\eta_l \geq 1/2$.

In the case of multiobjective programming problems, a complete optimal solution that simultaneously optimizes all of the multiple objective functions does not always exist when the objective functions conflict with each other. Thus, instead of a complete optimal solution, a Pareto optimal solution is well known as one of the reasonable solutions.

Since fuzzy random programming problems include not only fuzziness but also randomness, it is reasonable to define a new solution concept reflecting both fuzzy information and stochastic information. In multiobjective fuzzy programming, Sakawa [16] defined M-α-Pareto optimal solutions by extending a classical Pareto optimal solution. Since we have constructed our model using the fractile optimization model, we call the new Pareto optimal solution "FM-α-Pareto optimal solution" after the Fractile optimization model and M-α-Pareto optimal solution. The FM-α-Pareto optimal solution is defined as follows:

Definition 1 (FM-α-Pareto optimal solution) $\boldsymbol{x}^* \in X(\bar{\boldsymbol{a}}_l^*,\bar{b}_l^*,\bar{\boldsymbol{c}}_i^*)$ *is said to be an FM-α-Pareto optimal solution if and only if there does not exist another* $\boldsymbol{x} \in X(\bar{\boldsymbol{a}}_l,\bar{b}_l,\bar{\boldsymbol{c}}_i) C(\bar{\boldsymbol{a}}_l,\bar{b}_l,\bar{\boldsymbol{c}}_i) \in (\tilde{\bar{\boldsymbol{A}}}_{l\alpha},\tilde{\bar{B}}_{l\alpha},\tilde{\bar{\boldsymbol{C}}}_{i\alpha}) Ci = 1,\dots,k,\ l=1,\dots,m$ *such that* $h_i \geq h_i^* Ci = 1,\dots,k$ *with a strict inequality holding for at least one i for* $\boldsymbol{x}^* \in X(\bar{\boldsymbol{a}}_l^*,\bar{b}_l^*,\bar{\boldsymbol{c}}_i^*) \stackrel{\triangle}{=} \{\boldsymbol{x}^* \in R^n \mid Pr[\mu_i(\bar{\boldsymbol{c}}_i^*\boldsymbol{x}^*) \geq h_i^*] \geq \theta_i,\ \Pr[\bar{\boldsymbol{a}}_l^*\boldsymbol{x}^* \leq \bar{b}_l^*] \geq \eta_l\ ;\ \boldsymbol{x}^* \geq \boldsymbol{0}\} C(\bar{\boldsymbol{a}}_l^*,\bar{b}_l^*,\bar{\boldsymbol{c}}_i^*) \in (\tilde{\bar{\boldsymbol{A}}}_{l\alpha},\tilde{\bar{B}}_{l\alpha},\tilde{\bar{\boldsymbol{C}}}_{i\alpha}) Ci = 1,\dots,k,\ l=1,\dots,m,$ *where the corresponding values of parameters* $(\bar{\boldsymbol{a}}_l^*,\bar{b}_l^*,\bar{\boldsymbol{c}}_i^*) \in (\tilde{\bar{\boldsymbol{A}}}_{l\alpha},\tilde{\bar{B}}_{l\alpha},\tilde{\bar{\boldsymbol{C}}}_{i\alpha})$ *are said to be α-optimal parameters.*

Now we shall transform problem (10) into a deterministic equivalent problem.

First of all, in order to find α-level sets of fuzzy random variable coefficients, we have

$$\alpha = L\left(\frac{\bar{\boldsymbol{d}}_i^c - \bar{\boldsymbol{c}}_{i\alpha}^L}{\boldsymbol{\beta}_i^c}\right) = R\left(\frac{\bar{\boldsymbol{c}}_{i\alpha}^R - \bar{\boldsymbol{d}}_i^c}{\boldsymbol{\delta}_i^c}\right),\ i=1,\dots,k$$

$$\alpha = L\left(\frac{\bar{\boldsymbol{d}}_l^a - \bar{\boldsymbol{a}}_{l\alpha}^L}{\boldsymbol{\beta}_l^a}\right) = R\left(\frac{\bar{\boldsymbol{a}}_{l\alpha}^R - \bar{\boldsymbol{d}}_l^a}{\boldsymbol{\delta}_l^a}\right),\ l=1,\dots,m$$

$$\alpha = L\left(\frac{\bar{d}_l^b - \bar{b}_{l\alpha}^L}{\beta_l^b}\right) = R\left(\frac{\bar{b}_{l\alpha}^R - \bar{d}_l^b}{\delta_l^b}\right),\ l=1,\dots,m.$$

Consequently, we obtain

$$\left.\begin{array}{l} \bar{\boldsymbol{c}}_{i\alpha}^L = (\bar{\boldsymbol{d}}_i^c - L^*(\alpha)\boldsymbol{\beta}_i^c),\ i=1,\dots,k \\ \bar{\boldsymbol{c}}_{i\alpha}^R = (\bar{\boldsymbol{d}}_i^c + R^*(\alpha)\boldsymbol{\delta}_i^c),\ i=1,\dots,k \\ \bar{\boldsymbol{a}}_{l\alpha}^L = (\bar{\boldsymbol{d}}_l^a - L^*(\alpha)\boldsymbol{\beta}_l^a),\ l=1,\dots,m \\ \bar{\boldsymbol{a}}_{l\alpha}^R = (\bar{\boldsymbol{d}}_l^a + R^*(\alpha)\boldsymbol{\delta}_l^a),\ l=1,\dots,m \\ \bar{b}_{l\alpha}^L = (\bar{d}_l^b - L^*(\alpha)\beta_l^b),\ l=1,\dots,m \\ \bar{b}_{l\alpha}^R = (\bar{d}_l^b + R^*(\alpha)\delta_l^b),\ l=1,\dots,m \end{array}\right\} \tag{11}$$

where $L^*(\cdot)$ and $R^*(\cdot)$ are pseudo inverse functions of $L(\cdot)$ and $R(\cdot)$, respectively. This means that $\tilde{\bar{\boldsymbol{C}}}_{i\alpha}(\omega)$, $\tilde{\bar{\boldsymbol{A}}}_{l\alpha}(\omega)$ and $\tilde{\bar{B}}_{l\alpha}(\omega)$ are denoted by $[\bar{\boldsymbol{c}}_{i\alpha}^L(\omega),\bar{\boldsymbol{c}}_{i\alpha}^R(\omega)]$, $[\bar{\boldsymbol{a}}_{l\alpha}^L(\omega),\bar{\boldsymbol{a}}_{l\alpha}^R(\omega)]$ and $[\bar{b}_{l\alpha}^L(\omega),\bar{b}_{l\alpha}^R(\omega)]$, respectively. Therefore, we obtain the following problem which is equivalent to problem (10):

$$\left.\begin{array}{ll} \max & h_i,\ i=1,\dots,k \\ \text{s. t.} & \Pr[\bar{\boldsymbol{c}}_{i\alpha}^L\boldsymbol{x} \leq \mu_i^*(h_i)] \geq \theta_i,\ i=1,\dots,k \\ & \Pr[\bar{\boldsymbol{a}}_{l\alpha}^L\boldsymbol{x} \leq \bar{b}_{l\alpha}^R] \geq \eta_l,\ l=1,\dots,m \\ & \boldsymbol{x}\geq\boldsymbol{0} \end{array}\right\} \tag{12}$$

where $\mu_i^*(\cdot)$ denote a pseudo-inverse function of $\mu_i(\cdot)$.

From the result of (11), problem (12) is equivalently transformed into the following problem:

$$\left.\begin{array}{ll} \max & h_i,\ i=1,\dots,k \\ \text{s. t.} & \Pr\left[\dfrac{\bar{\boldsymbol{d}}_i^c\boldsymbol{x} - \boldsymbol{m}_i^c\boldsymbol{x}}{\sqrt{\boldsymbol{x}^T V_i^c \boldsymbol{x}}} \leq \dfrac{\mu_i^*(h_i) - \{\boldsymbol{m}_i^c - L^*(\alpha)\boldsymbol{\beta}_i^c\}\boldsymbol{x}}{\sqrt{\boldsymbol{x}^T V_i^c \boldsymbol{x}}}\right] \\ & \quad \geq \theta_i,\ i=1,\dots,k \\ & \Pr\left[\dfrac{(\bar{\boldsymbol{d}}_l^a\boldsymbol{x} - \bar{d}_l^b) - (\boldsymbol{m}_l^a\boldsymbol{x} - m_l^b)}{\sqrt{\boldsymbol{x}^T V_l^a \boldsymbol{x} + v_l^b}} \leq \right. \\ & \quad \left. \dfrac{-(\{\boldsymbol{m}_l^a - L^*(\alpha)\boldsymbol{\beta}_l^a\}\boldsymbol{x} - m_l^b - R^*(\alpha)\delta_l^b)}{\sqrt{\boldsymbol{x}^T V_l^a \boldsymbol{x} + v_l^b}}\right] \geq \eta_l, \\ & l=1,\dots,m \\ & \boldsymbol{x}\geq\boldsymbol{0}. \end{array}\right\} \tag{13}$$

Let $\Phi(\cdot)$ denote a distribution function of the normal random variable. Then problem (13) is transformed into the following deterministic equivalent problem:

$$\left.\begin{array}{ll}\max & h_i,\ i=1,\ldots,k \\ \text{s. t.} & \Phi\left(\dfrac{\mu_i^*(h_i)-\{\boldsymbol{m}_i^c-L^*(\alpha)\boldsymbol{\beta}_i^c\}\boldsymbol{x}}{\sqrt{\boldsymbol{x}^T V_i^c \boldsymbol{x}}}\right)\geq\theta_i, \\ & i=1,\ldots,k \\ & \Phi\left(\dfrac{-(\{\boldsymbol{m}_l^a-L^*(\alpha)\boldsymbol{\beta}_l^a\}\boldsymbol{x}-m_l^b-R^*(\alpha)\delta_l^b)}{\sqrt{\boldsymbol{x}^T V_l^a\boldsymbol{x}+v_l^b}}\right) \\ & \geq\eta_l,\ l=1,\ldots,m \\ & \boldsymbol{x}\geq\boldsymbol{0}.\end{array}\right\}\quad(14)$$

Since $\Phi(\cdot)$ is a increasing continuous function, we transform problem (13) equivalently into the following deterministic problem:

$$\left.\begin{array}{ll}\max & h_i,\ i=1,\ldots,k \\ \text{s. t.} & h_i\leq\mu_i\left(z_i^c(\boldsymbol{x})\right),\ i=1,\ldots,k \\ & \{\boldsymbol{m}_l^a-L^*(\alpha)\boldsymbol{\beta}_l^a\}\boldsymbol{x}-(m_l^b+R^*(\alpha)\delta_l^b) \\ & \quad+\Phi^{-1}(\eta_l)\sqrt{\boldsymbol{x}^T V_l^a\boldsymbol{x}+v_l^b}\leq 0,\ l=1,\ldots,m \\ & \boldsymbol{x}\geq\boldsymbol{0}\end{array}\right\}\quad(15)$$

where

$$z_i^c(\boldsymbol{x})=\{\boldsymbol{m}_i^c-L^*(\alpha)\boldsymbol{\beta}_i^c\}\boldsymbol{x}+\Phi^{-1}(\theta_i)\sqrt{\boldsymbol{x}^T V_i^c\boldsymbol{x}}$$

and $\Phi^{-1}(\cdot)$ is an inverse function of $\Phi(\cdot)$. Since the maximum value of h_i is equal to $\mu_i\left(\{\boldsymbol{m}_i^c-L^*(\alpha)\boldsymbol{\beta}_i^c\}\boldsymbol{x}+\Phi^{-1}(\theta_i)\sqrt{\boldsymbol{x}^T V_i^c\boldsymbol{x}}\right)$, $i=1,\ldots,k$, we can transform problem (15) into the following simpler problem:

$$\left.\begin{array}{ll}\max & \mu_i\left(z_i^c(\boldsymbol{x})\right),\ i=1,\ldots,k \\ \text{s. t.} & \{\boldsymbol{m}_l^a-L^*(\alpha)\boldsymbol{\beta}_l^a\}\boldsymbol{x}-(m_l^b+R^*(\alpha)\delta_l^b) \\ & \quad+\Phi^{-1}(\eta_l)\sqrt{\boldsymbol{x}^T V_l^a\boldsymbol{x}+v_l^b}\leq 0,\ l=1,\ldots,m \\ & \boldsymbol{x}\geq\boldsymbol{0}.\end{array}\right\}\quad(16)$$

It should be noted here that problem (16) is a deterministic multiobjective nonlinear programming problem, which is equivalent to problem (10).

4 Interactive algorithm

If we introduce a general aggregation function $\mu_D(\boldsymbol{x})=G(\mu_1(z_1^c(\boldsymbol{x})),\ldots,\mu_k(z_k^c(\boldsymbol{x})))$ using some nonlinear function G, problem (16) is rewritten as the following problem:

$$\left.\begin{array}{ll}\max & \mu_D(\boldsymbol{x}) \\ \text{s. t.} & \{\boldsymbol{m}_l^a-L^*(\alpha)\boldsymbol{\beta}_l^a\}\boldsymbol{x}-(m_l^b+R^*(\alpha)\delta_l^b) \\ & \quad+\Phi^{-1}(\eta_l)\sqrt{\boldsymbol{x}^T V_l^a\boldsymbol{x}+v_l^b}\leq 0,\ l=1,\ldots,m \\ & \boldsymbol{x}\geq\boldsymbol{0}.\end{array}\right\}\quad(17)$$

Such an aggregation function $\mu_D(\boldsymbol{x})$ represents a degree of satisfaction or preference of decision makers for k objective functions. A minimum operator and a product operator are major examples, but they are special cases of aggregation functions. Since it is difficult to identify an aggregation function explicitly such that the function reflects the precise preference of decision maker, we need to use an interactive algorithm which derives a satisficing solution for a decision maker through interaction.

To generate a candidate for the satisficing solution which is an FM-α-Pareto optimal solution, the decision maker is asked to specify reference levels of achievement of the objective functions, called reference membership levels. The idea of the reference point first was introduced by Wierzbicki [18]. To be more explicit, the corresponding Pareto optimal solution for a given reference membership levels is obtained by solving the following minimax problem:

$$\left.\begin{array}{ll}\min & \max\limits_{i=1,\ldots,k}\left\{\bar{\mu}_i-\mu_i\left(z_i^c(\boldsymbol{x})\right)\right\} \\ \text{s. t.} & \{\boldsymbol{m}_l^a-L^*(\alpha)\boldsymbol{\beta}_l^a\}\boldsymbol{x}-(m_l^b+R^*(\alpha)\delta_l^b) \\ & \quad+\Phi^{-1}(\eta_l)\sqrt{\boldsymbol{x}^T V_l^a\boldsymbol{x}+v_l^b}\leq 0,\ l=1,\ldots,m \\ & \boldsymbol{x}\geq\boldsymbol{0}\end{array}\right\}\quad(18)$$

or equivalently,

$$\left.\begin{array}{ll}\min & v \\ \text{s. t.} & \bar{\mu}_i-\mu_i\left(z_i^c(\boldsymbol{x})\right)\leq v,\ i=1,\ldots,k \\ & \{\boldsymbol{m}_l^a-L^*(\alpha)\boldsymbol{\beta}_l^a\}\boldsymbol{x}-(m_l^b+R^*(\alpha)\delta_l^b) \\ & \quad+\Phi^{-1}(\eta_l)\sqrt{\boldsymbol{x}^T V_l^a\boldsymbol{x}+v_l^b}\leq 0,\ l=1,\ldots,m \\ & \boldsymbol{x}\geq\boldsymbol{0}.\end{array}\right\}\quad(19)$$

Consequently, problem (19) is transformed into the following problem:

$$\left.\begin{array}{ll}\min & v \\ \text{s. t.} & \{\boldsymbol{m}_i^c-L^*(\alpha)\boldsymbol{\beta}_i^c\}\boldsymbol{x}+\Phi^{-1}(\theta_i)\sqrt{\boldsymbol{x}^T V_i^c\boldsymbol{x}} \\ & \qquad\leq\mu_i^*(\bar{\mu}_i-v),\ i=1,\ldots,k \\ & \{\boldsymbol{m}_l^a-L^*(\alpha)\boldsymbol{\beta}_l^a\}\boldsymbol{x}+\Phi^{-1}(\eta_l)\sqrt{\boldsymbol{x}^T V_l^a\boldsymbol{x}+v_l^b} \\ & \qquad\leq(m_l^b+R^*(\alpha)\delta_l^b),\ l=1,\ldots,m \\ & \boldsymbol{x}\geq\boldsymbol{0}.\end{array}\right\}\quad(20)$$

Since the constraints are nonlinear in (20), it is apparent that the problem is not a usual convex programming problem. It is important to note here that when the value of v is fixed, all of the constraints are convex. Obtaining the optimal solution v^* to the above problem is equivalent to determining the minimum value of v so that there exists an admissible set satisfying the constraints of the above problem. The problem of finding a point in the intersection of a finite family of closed convex sets is called a "convex feasible problem." The minimum value of v is searched by combined use of the bisection method and an algorithm for solving convex feasibility problems [1]. Now we are ready to construct the interactive algorithm in order to derive the satisficing solution for the decision maker from the Pareto solution set.

[**Interactive algorithm for multiobjective fuzzy random programming**]

Step 1: Calculate the individual maximum $z_{i,\min},\ i = 1,\ldots,k$ under the given constraints. Based on the obtained individual minimum value, elicit the membership functions $\mu_i,\ i = 1,\ldots,k$ of fuzzy goals from the decision maker.

Step 2: Set the confidence levels $\theta_i(\geq 1/2)$ and $\eta_l(\geq 1/2)$. Set an initial admissible level α and initial reference membership levels $\bar{\mu}_i,\ i = 1,\ldots,k$.

Step 3: Solve problem (20) to obtain an FM-α-Pareto optimal solution which is a candidate of satisficing solution for a decision maker.

Step 4: If the decision maker is satisfied with the current solution, terminate the algorithm. Then the current FM-α-Pareto optimal solution is a satisficing solution of the decision maker. Otherwise, ask the decision maker to update the current reference membership levels $\bar{\mu}_i$ and/or the current admissible level α by considering the current objective function values, and return to step 3.

5 Numerical example

To demonstrate the effectiveness of the proposed model and algorithm, we provide a numerical example of agriculture production planning problems. There are three kinds of crops to be planted, Crops 1, 2 and 3. We assume that Crop 2 will be planted with less working time than Crop 1 and Crop 3 although Crop 2 will earn less profit than others, and that Crop 3 will earns the most profit although it will need the most working time for planting. Working time and profit are represented with fuzzy random variables. The decision variables x_1, x_2, x_3 represents cropping acreages of Crops 1, 2 and 3, respectively. The objectives are to maximize total profit and to minimize total working time. Then, the problem is formulated as

$$\left.\begin{array}{ll}
\min & \tilde{\bar{C}}_{11}x_1 + \tilde{\bar{C}}_{12}x_2 + \tilde{\bar{C}}_{13}x_1 \\
\min & \tilde{\bar{C}}_{21}x_1 + \tilde{\bar{C}}_{22}x_2 + \tilde{\bar{C}}_{23}x_1 \\
\text{s. t.} & \tilde{\bar{A}}_{11}x_1 + \tilde{\bar{A}}_{12}x_2 + \tilde{\bar{A}}_{13}x_3 \leq \tilde{\bar{B}}_1 \\
& \tilde{\bar{A}}_{21}x_1 + \tilde{\bar{A}}_{22}x_2 + \tilde{\bar{A}}_{23}x_3 \leq \tilde{\bar{B}}_2 \\
& 7x_1 + 6x_2 + 4x_3 \leq 100 \\
& x_j \geq 0,\ j = 1,\ldots,3
\end{array}\right\} \tag{21}$$

where

$$\alpha = 0.7, \quad (m^c_{ij}) = \begin{pmatrix} -5.0 & -3.0 & -6.0 \\ 4.0 & 2.0 & 7.0 \end{pmatrix},$$

$$V^c_1 = \begin{pmatrix} 1.4 & 0.2 & 0.9 \\ 0.2 & 1.3 & 0.4 \\ 0.9 & 0.4 & 1.6 \end{pmatrix}, \quad V^c_2 = \begin{pmatrix} 1.4 & 0.2 & 0.9 \\ 0.2 & 1.7 & 0.4 \\ 0.9 & 0.4 & 1.6 \end{pmatrix},$$

$$(\beta^c_{ij}) = \begin{pmatrix} 1.0 & 1.5 & 1.0 \\ 1.5 & 1.0 & 1.5 \end{pmatrix}, \ (\delta^c_{ij}) = \begin{pmatrix} 1.0 & 1.5 & 1.0 \\ 1.5 & 1.0 & 1.5 \end{pmatrix},$$

$$(m^a_{lj}) = \begin{pmatrix} 4.0 & 6.0 & 3.0 \\ 5.0 & 3.0 & 2.0 \end{pmatrix},$$

$$V^a_1 = \begin{pmatrix} 1.4 & 0.2 & -0.9 \\ 0.2 & 1.5 & -0.4 \\ -0.9 & -0.4 & 1.3 \end{pmatrix},$$

$$V^a_2 = \begin{pmatrix} 1.5 & 0.3 & -0.7 \\ 0.3 & 1.3 & 0.4 \\ -0.7 & 0.4 & 1.2 \end{pmatrix},$$

$$(\beta^a_{lj}) = \begin{pmatrix} 1.5 & 1.0 & 1.5 \\ 1.0 & 1.5 & 1.0 \end{pmatrix}, \ (\delta^a_{lj}) = \begin{pmatrix} 1.5 & 1.0 & 1.5 \\ 1.0 & 1.5 & 1.0 \end{pmatrix},$$

$$(m^b_l) = \begin{pmatrix} 140 \\ 135 \end{pmatrix}, \ (v^b_l) = \begin{pmatrix} 8 \\ 10 \end{pmatrix}, \ (\beta^b_l) = \begin{pmatrix} 12 \\ 15 \end{pmatrix},$$

$$(\delta^b_l) = \begin{pmatrix} 12 \\ 15 \end{pmatrix}, \ L(t) = R(t) = 1 - t.$$

First, we calculate the individual minimums $z_{i,\min}$ of each objective functions $\bar{\boldsymbol{c}}_i\boldsymbol{x}$ and obtain $z_{1,\min} = -150$ and $z_{2,\min} = 0$. Secondly, considering these obtained values, the decision maker determines the parameters of linear membership functions as $g^0_1 = 0$, $g^1_1 = -150$, $g^0_2 = 175$ and $g^1_2 = 0$.

Suppose that a decision maker gives the confidence levels, admissible level and the reference membership levels as $\theta_1 = \theta_2 = \eta_1 = \eta_2 = 0.7$, $\alpha = 0.7$ and $(\bar{\mu}_1, \bar{\mu}_2) = (1.0, 1.0)$, respectively. Then, solving the minimax problem yields the Pareto optimal solutions and the objective function values, as shown in the second column of Table 1.

On the basis of such information, the decision maker updates the reference confidence levels in order to improve the membership function value $\mu_1(\boldsymbol{x})$ of the first objective function at the expense of $\mu_2(\boldsymbol{x})$ of the second objective function value, as shown in the third column of the table. In this stage, the decision maker considers that the satisfaction level $\mu_2(\boldsymbol{x})$ is a little too small and that he/she prefers to make the value $\mu_2(\boldsymbol{x})$ a little larger at the expense of $\mu_1(\boldsymbol{x})$. Therefore, the decision maker updates the reference membership levels and obtain the satisfaction levels as shown in the forth column of the table.

Since the decision maker is not still satisfied with the obtained solution, he/she turns to lower the admissible level α as shown in the fifth column of the table. Finally, since the decision maker is satisfied with the obtained solution, the interactive process is terminated.

Table 1: Interactive process

	First	Second	Third	Forth
$\bar{\mu}_1$	1.00	1.00	0.90	0.90
$\bar{\mu}_2$	1.00	0.80	0.80	0.80
α	0.700	0.70	0.70	0.60
$\mu_1(\boldsymbol{x})$	0.544	0.628	0.586	0.600
$\mu_2(\boldsymbol{x})$	0.544	0.428	0.486	0.500
x_1	6.66	6.04	6.32	6.27
x_2	4.90	3.16	4.05	4.09
x_3	6.00	9.69	7.85	7.88

6 Conclusion

In the present work, we have proposed a fuzzy random multiobjective linear programming model and defined the FM-α-Pareto optimal solution concept. We have shown that fuzzy random programming problem is transformed into the deterministic equivalent problem. In order to derive a satisficing solution from a set of FM-α-Pareto optimal solutions, we have established an interactive algorithm. It should be remarkable that a minimax problem to be iteratively solved in the interactive algorithm is a convex feasibility problem. Although it is generally difficult to obtain an optimal solution of the problem including both fuzziness and randomness due to its complexity, our model has an advantage that the resulting problem is solved by some algorithm for solving convex feasibility problems.

In the future, it is interesting to consider decision making models based on other stochastic programming models such as the expectation optimization model and the variance minimization model.

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