On the Periodic Motion of Nonlinear Discrete Dynamical System

Jiemin Zhao

***Abstract*—We give an existence-uniqueness result of periodic motion for nonlinear discrete dynamical system**

$$X(n+1)=F(n,\ X(n)),\ (n,X)\in Z\times R^m$$

by means of the analysis and computing method.

***Index Terms*—discrete system, initial value, integer, periodic motion.**

I. INTRODUCTION

We do research on the nonlinear discrete dynamical system

$$X(n+1)=F(n,\ X(n)) \qquad (1)$$

where $F:Z\times R^m\to R^m$ is a given continuous function, Z is a set of integer, $R=(-\infty,+\infty)$, R^m is an m-dimensional linear vector space over the reals with norm $\|\cdot\|$. Suppose there is an integer $k>1$ such that

$$F(n+k,\ X)=F(n,\ X)$$

for all $n\in Z$ and $X\in R^m$. There is a unique solution of the system (1) through $(n_0,\ X_0)$.

The existence-uniqueness problem of periodic motion for discrete dynamical system (1) is very important [1—4]. In this paper, a result about the existence-uniqueness problem of $k-$periodic motion of system (1) is given.

II. ANALYSIS

Suppose $G:Z_+\times R^m\times R^m\to R_+^1$ is a given mapping satisfying the following hypothesis:

There is a natural number M such that, for $n\geq M$, we have

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Jiemin Zhao is with the Department of Applied Mathematics and Physics, Beijing Union University, Beijing 100101, China.
E-mail: jieminzhao@sina.com

$$\begin{aligned}&\|X(n,\ X_0)-Y(n,Y_0)\|\\&\leq G(n,\ X_0,Y_0)\|X_0-Y_0\|\end{aligned}$$

for arbitrary motions

$$X(n,X_0)\ \text{and}\ Y(n,Y_0)$$

of system (1), then

$$\begin{aligned}&\|X(kN,\ X_0)-Y(kN,Y_0)\|\\&\leq G(kN,\ X_0,Y_0)\|X_0-Y_0\|.\end{aligned}$$

for natural number $N\geq M$.

If there is a natural number $N\geq M$ such that

$$G(kN,X_0,Y_0)\leq C=\text{const.}<1,$$

then

$$\begin{aligned}&\|X(kN,\ X_0)-Y(kN,Y_0)\|\\&\leq G(kN,\ X_0,Y_0)\|X_0-Y_0\|\\&\leq C\|X_0-Y_0\|.\end{aligned}$$

When $Y_0=X(kN,X_0)$, we have

$$\begin{aligned}&\|X(kN,\ X_0)-Y(kN,Y_0)\|\\&=\|X(kN,\ X_0)-X(2kN,Y_0)\|\\&\leq G(kN,\ X_0,Y_0)\|X_0-Y_0\|\\&\leq C\|X_0-Y_0\|\\&=C\|X_0-X(kN,\ X_0)\|.\end{aligned}$$

Using

$$\begin{aligned}&\|X(kN,\ X_0)-X(2kN,Y_0)\|\\&\leq C\|X_0-X(kN,\ X_0)\|,\end{aligned}$$

we obtain

$$\begin{aligned}&-C\|X_0-X(kN,\ X_0)\|\\&\leq-\|X(kN,\ X_0)-X(2kN,Y_0)\|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\| X_0 - X(kN, X_0)\| - C\| X_0 - X(kN, X_0)\| \\ &\le \| X_0 - X(kN, X_0)\| - \| X(kN, X_0) - X(2kN, Y_0)\|,\end{aligned}$$

we obtain

$$\begin{aligned}&\| X_0 - X(kN, X_0)\| - C\| X_0 - X(kN, X_0)\| \\ &= (1-C)\| X_0 - X(kN, X_0)\| \\ &\le \| X_0 - X(kN, X_0)\| - \| X(kN, X_0) - X(2kN, Y_0)\|,\end{aligned}$$

Thus, for any $X_0 \in R^m$, we have

$$\begin{aligned}&\| X_0 - X(kN, X_0)\| \\ &\le \frac{1}{1-C}\| X_0 - X(kN, X_0)\| \\ &\quad - \frac{1}{1-C}\| X(kN, X_0) - X(2kN, X_0)\|.\end{aligned}$$

We choose

$$\varphi(X_0) = \frac{1}{1-C}\| X_0 - X(kN, X_0)\|.$$

Let T denote the Poicaić mapping $TX_0 = X(k, X_0)$, we have

$$\rho(X_0, T^N X_0) \le \varphi(X_0) - \varphi(T^N X_0)$$

for all $X_0 \in R^m$. Therefore, the T^N has a fixed-point $\overline{X}$.

Let $\overline{X}$ and $\overline{Y}$ be two fixed-point of the T^N, then

$$\begin{aligned}\| \overline{X} - \overline{Y} \| &= \| T^N \overline{X} - T^N \overline{Y} \| \\ &= \| X(kN, \overline{X}) - Y(kN, \overline{Y})\|.\end{aligned}$$

Using the inequality

$$\begin{aligned}&\| X(kN, X_0) - Y(kN, Y_0)\| \\ &\le G(kN, X_0, Y_0)\| X_0 - Y_0\| \\ &\le C\| X_0 - Y_0\|,\end{aligned}$$

we have

$$\| \overline{X} - \overline{Y} \| \le C\| \overline{X} - \overline{Y} \|.$$

But $C < 1$ and this last inequality can only be satisfied if

$$\| \overline{X} - \overline{Y} \| = 0.$$

Thus, $\overline{X} = \overline{Y}$. Therefore, the T^N has a unique fixed-point $\overline{X}$.

From

$$T^N(T\overline{X}) = T(T^N \overline{X}) = T\overline{X},$$

That is the T^N has a fixed-point $T\overline{X}$. According to the fixed-point uniqueness of the T^N, (1) has a unique k-periodic motion.

III. Main Result

From analysis above, we have result as follow:

Suppose $G: Z_+ \times R^m \times R^m \to R_+^1$ is a given mapping satisfying the following hypotheses:

(i) There is a natural number M such that, for $n \ge M$, we have

$$\begin{aligned}&\| X(n, X_0) - Y(n, Y_0)\| \\ &\le G(n, X_0, Y_0)\| X_0 - Y_0\|\end{aligned}$$

for arbitrary motions

$$X(n, X_0) \text{ and } Y(n, Y_0)$$

of the system (1).

(ii) There is a natural number $N \ge M$ such that

$$G(kN, X_0, Y_0) \le C = \text{const.} < 1.$$

then there is a unique k-periodic motion of the system (1).

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