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Triangular Factors of the Inverse of Vandermonde Matrices

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Abstract— **This paper is concerned with the decomposition of the inverses of Vandermonde matrices as a product of one lower and one upper triangular matrices. The algorithm proposed here is suitable for both hand and machine computation.**

Keywords: Vandermonde matrix, triangular decomposition, partial fractions

1 Introduction

Vandermonde matrices arise in many applications such as polynomial interpolation [1], digital signal processing [2], and control theory [3].

For a set of n distinct numbers $\mu_1, \ldots, \mu_n$, the $n \times n$ matrix

$$\mathbf{V}(\mu_1, \ldots, \mu_n) := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & & \vdots \\ \mu_1^{n-1} & \mu_2^{n-1} & \cdots & \mu_n^{n-1} \end{bmatrix}$$

is called a Vandermonde matrix, and its determinant is given by the formula

$$\det \mathbf{V}(\mu_1, \ldots, \mu_n) = \prod_{1 \le i < j \le n} (\mu_i - \mu_j)$$

(see [4]). The μ_i's being distinct, it follows that $\mathbf{V}(\mu_1, \ldots, \mu_n)$ is invertible.

As far as the inverse of Vandermonde matrix $\mathbf{V}(\mu_1, \ldots, \mu_n)$ is concerned, a number of explicit formulas and computational schemes for the entries of the inverse have been given in [4], [5] and [6]. Recently, a recursive algorithm for inverting Vandermonde matrix as well as its confluent type has also been given in [7].

In this note we present a decomposition formula expressing the inverse $\mathbf{V}^{-1}(\mu_1, \ldots, \mu_n)$ as a product of two triangular matrices whose elements are easily computed by means of recursive algorithms.

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2 Triangular decomposition

Consider the polynomials:

$$\begin{aligned} \psi_1(s) &:= 1, \\ \psi_j(s) &:= (s - \mu_{j-1})\, \psi_{j-1}(s) \\ &= \prod_{k=1}^{j-1} (s - \mu_k), \qquad j = 2, \ldots, n. \end{aligned}$$

Let $\mathbf{L}$ be the $n \times n$ matrix whose rows are associated with the coefficients of the polynomials $\psi_1, \ldots, \psi_n$. Notationally,

$$\mathbf{L} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} = \begin{bmatrix} \psi_1(s) \\ \psi_2(s) \\ \vdots \\ \psi_n(s) \end{bmatrix}.$$

It is clear from the construction of the polynomials $\psi_1, \ldots, \psi_n$ that the rows of $\mathbf{L}$ can be recursively computed (see Appendix), and that $\mathbf{L}$ is lower triangular with 1's on the main diagonal since the leading coefficient of each of the polynomials $\psi_1, \ldots, \psi_n$ is 1.

Then the inverse $\mathbf{V}^{-1}(\mu_1, \ldots, \mu_n)$ factors into two $n \times n$ matrices as

$$\mathbf{V}^{-1}(\mu_1, \ldots, \mu_n) = \mathbf{HL}.$$

The $n \times n$ matrix $\mathbf{H}$ appearing in this decomposition turns out to be upper triangular and is characterized by the theorem below (see [8]).

Theorem 1 *The $n \times n$ matrix*

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \cdots & \mathbf{h}_n \end{bmatrix}$$

is upper triangular, and its column vectors $\mathbf{h}_1, \ldots, \mathbf{h}_n$ can be recursively computed by the following scheme:
Let

$$\mathbf{d}(s) := \begin{bmatrix} \mu_1 - s & \mu_2 - s & \cdots & \mu_n - s \end{bmatrix}^T.$$

Then

$$\mathbf{h}_{i-1} := \mathbf{h}_i . * \mathbf{d}(\mu_i), \qquad i = n, n-1, \ldots, 2,$$

ending at

$$\mathbf{h}_1 = \begin{bmatrix} 1, & 0 & \cdots & 0 \end{bmatrix}^T.$$

The initial vector

$$\mathbf{h}_n := \left[\begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array}\right]^T$$

is determined from the partial fraction expansion

$$\frac{1}{(s-\mu_1)\cdots(s-\mu_n)} = \frac{c_1}{s-\mu_1} + \frac{c_2}{s-\mu_2} + \cdots + \frac{c_n}{s-\mu_n}.$$

Remark. (a) Here the symbol '.∗' denotes array multiplication as defined by

$$\left[\begin{array}{c} u_1 \\ \vdots \\ u_n \end{array}\right] .* \left[\begin{array}{c} v_1 \\ \vdots \\ v_n \end{array}\right] := \left[\begin{array}{c} u_1 v_1 \\ \vdots \\ u_n v_n \end{array}\right].$$

(b) Let

$$\psi_{n+1}(s) := (s-\mu_1)\cdots(s-\mu_n).$$

Then

$$c_i = \frac{1}{\psi'_{n+1}(\mu_i)} = \left(\prod_{k=1,\,k\neq i}^{n} (\mu_i - \mu_k)\right)^{-1}.$$

Corollary 1 *The upper triangular matrix* $\mathbf{H} = [h_{ij}]_{n\times n}$ *is given by*

$$h_{ij} = \begin{cases} \frac{1}{\psi'_{j+1}(\mu_i)} & \text{if} \quad i \le j, \\ 0 & \text{if} \quad i > j. \end{cases}$$

3 Example

Let us consider the 4×4 Vandermonde matrix

$$\mathbf{V}(1,-2,\ 3,-1) = \left[\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & -1 \\ 1 & 4 & 9 & 1 \\ 1 & -8 & 27 & -1 \end{array}\right]$$

which has inverse

$$\mathbf{V}^{-1}(1,-2,\ 3,-1) = \left[\begin{array}{rrrr} \frac{1}{2} & \frac{7}{12} & 0 & -\frac{1}{12} \\ -\frac{1}{5} & \frac{1}{15} & \frac{1}{5} & -\frac{1}{15} \\ -\frac{1}{20} & -\frac{1}{40} & \frac{1}{20} & \frac{1}{40} \\ \frac{3}{4} & -\frac{5}{8} & -\frac{1}{4} & \frac{1}{8} \end{array}\right].$$

In this case the polynomials $\psi_1, \ldots, \psi_4$ are easily computed recursively to be

$$\begin{array}{rcl} \psi_1(s) & = & 1 \\ \psi_2(s) & = & (s-1)\,\psi_1(s) = -1+s \\ \psi_3(s) & = & (s+2)\,\psi_2(s) = -2+s+s^2 \\ \psi_4(s) & = & (s-3)\,\psi_3(s) = \ 6-5s-2s^2+s^3 \end{array}$$

so that

$$\mathbf{L} = \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 6 & -5 & -2 & 1 \end{array}\right].$$

To determine the upper triangular matrix $\mathbf{H}$ in the triangular decomposition

$$\mathbf{V}^{-1}(1,-2,\ 3,-1) = \mathbf{HL},$$

we proceed first by constructing the vector

$$\mathbf{d}(s) = \left[\begin{array}{cccc} 1-s & -2-s & 3-s & -1-s \end{array}\right]^T$$

and expanding

$$\frac{1}{(s-1)(s+2)(s-3)(s+1)} = \frac{-\frac{1}{12}}{s-1} + \frac{-\frac{1}{15}}{s+2} + \frac{\frac{1}{40}}{s-3} + \frac{\frac{1}{8}}{s+1}$$

to get the initial column vector

$$\mathbf{h}_4 = \left[\begin{array}{cccc} -\frac{1}{12} & -\frac{1}{15} & \frac{1}{40} & \frac{1}{8} \end{array}\right]^T.$$

Then according to the recursive scheme given in Theorem 1, we have

$$\begin{array}{rcl} \mathbf{h}_3 & = & \mathbf{h}_4 .* \mathbf{d}(-1) \\ & = & \left[\begin{array}{cccc} -\frac{1}{12} & -\frac{1}{15} & \frac{1}{40} & \frac{1}{8} \end{array}\right]^T .* \left[\begin{array}{cccc} 2 & -1 & 4 & 0 \end{array}\right]^T \\ & = & \left[\begin{array}{cccc} -\frac{1}{6} & \frac{1}{15} & \frac{1}{10} & 0 \end{array}\right]^T \end{array}$$

$$\begin{array}{rcl} \mathbf{h}_2 & = & \mathbf{h}_3 .* \mathbf{d}(3) \\ & = & \left[\begin{array}{cccc} -\frac{1}{6} & \frac{1}{15} & \frac{1}{10} & 0 \end{array}\right]^T .* \left[\begin{array}{cccc} -2 & -5 & 0 & -4 \end{array}\right]^T \\ & = & \left[\begin{array}{cccc} \frac{1}{3} & -\frac{1}{3} & 0 & 0 \end{array}\right]^T \end{array}$$

$$\begin{array}{rcl} \mathbf{h}_1 & = & \mathbf{h}_2 .* \mathbf{d}(-2) \\ & = & \left[\begin{array}{cccc} \frac{1}{3} & -\frac{1}{3} & 0 & 0 \end{array}\right]^T .* \left[\begin{array}{cccc} 3 & 0 & 5 & 1 \end{array}\right]^T \\ & = & \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]^T \end{array}$$

Hence

$$\mathbf{H} = \left[\begin{array}{cccc} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 & \mathbf{h}_4 \end{array}\right] = \left[\begin{array}{cccc} 1 & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{12} \\ 0 & -\frac{1}{3} & \frac{1}{15} & -\frac{1}{15} \\ 0 & 0 & \frac{1}{10} & \frac{1}{40} \\ 0 & 0 & 0 & \frac{1}{8} \end{array}\right].$$

It is easy to check that

$$\mathbf{HL} = \mathbf{V}^{-1}(1,-2,\ 3,-1),$$

that is

$$\begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{12} \\ 0 & -\frac{1}{3} & \frac{1}{15} & -\frac{1}{15} \\ 0 & 0 & \frac{1}{10} & \frac{1}{40} \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 6 & -5 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{7}{12} & 0 & -\frac{1}{12} \\ -\frac{1}{5} & \frac{1}{15} & \frac{1}{5} & -\frac{1}{15} \\ -\frac{1}{20} & -\frac{1}{40} & \frac{1}{20} & \frac{1}{40} \\ \frac{3}{4} & -\frac{5}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}.$$

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5 Appendix

The element l_{ij} of the $n \times n$ lower triangular matrix $\mathbf{L} = [l_{ij}]$ may be recursively computed by the Matlab routine:

$\mu = [\mu_1, \ldots, \mu_n]$;
n =length(μ);
$\mathbf{L}$ =eye(n);
for $i = 1 : n - 1$
  $\mathbf{L}(i+1, 1) = -\mu(i) * \mathbf{L}(i, 1)$;
  for $j = 2 : i$
    $\mathbf{L}(i+1, j) = \mathbf{L}(i, j-1) - \mu(i) * \mathbf{L}(i, j)$;
  end
end
disp($\mathbf{L}$)

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