Proceedings of the International MultiConference of Engineers and Computer Scientists 2009 Vol II
IMECS 2009, March 18 - 20, 2009, Hong Kong

Application of Higher Order Derivatives of Lyapunov Functions in Stability Analysis of Nonlinear Homogeneous Systems

Vahid Meigoli, and S. K. Y. Nikravesh

***Abstract*--The Lyapunov stability analysis method for nonlinear dynamic systems needs a positive definite function whose time derivative is at least negative semi-definite in the direction of the system's solutions. However merging the both properties in a single function is a challenging task. In this paper some linear combination of higher order derivatives of the Lyapunov function with non-negative coefficients is resulted. If the resultant summation is negative definite and all the derivatives are decrescent then the zero equilibrium state of the nonlinear system is asymptotically stable. If the higher order time derivatives of the Lyapunov function are not well-defined, then some well-defined smooth functions may be used instead. In this case a linear combination of time derivatives of all functions, with non-negative coefficients, must be negative definite. The new conditions are then reformed to be applied for stability analysis of nonlinear homogeneous systems. Some examples are presented to describe the approach.**

***Index Terms*--nonlinear systems, stability analysis, Lyapunov functions, higher order derivatives, homogeneous systems.**

NOMENCLATURE

$\|\cdot\|$	A given norm on $\mathbb{R}^n$
$x(t,t_0,x_0)$	A trajectory starting at $x(t_0)=x_0$
$\underline{u}$	The underline variable means a vector quantity
$\underline{V}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$	vector function of dimension m (VF)
$\phi\in K$ ($\phi\in K_\infty$)	ϕ is a function of class K(K infinity)[2]
$v^{(i)}(t,\underline{x})$	The *i*-th total time derivative of $v(t,x)$ func.
$\underline{a}\le\underline{b}$	component-wise inequality

I. INTRODUCTION

Consider the following n-dimensional nonautonomous dynamic system with a *zero equilibrium state* (ZES):

$$\dot{\underline{x}}=\underline{f}(t,\underline{x})\quad t\ge 0,\quad \underline{x}\in\mathbb{R}^n \tag{1}$$

The advantage of the Lyapunov method is the use of *Lyapunov functions* (LF) or energy like functions. The Lyapunov stability analysis method for *uniform asymptotic stability* (u.a.s.) of ZES of nonlinear dynamic systems needs a *locally decrescent* (LD)*locally positive definite function* (LPDF) $v(t,\underline{x})$ whose time derivative $\dot{v}(t,\underline{x})$ is negative definite in the direction of the system's solutions. When the derivative is negative semidefinite, stability rather than asymptotic stability (a.s.) follows. When the complexity of a nonlinear system is increased, selecting a suitable LF having at least negative semi-definite derivative is an involving task, See [1] and [2].

Gunderson [3] considered the stability analysis of (1), using a LF $v(t,\underline{x})$ with the inequality $v^{(m)}(t,\underline{x})\le g_m(t,v,\dot{v},\cdots,v^{(m-1)})$, for some positive integer m, where all the higher order derivatives $v^{(i)}(t,\underline{x})$ were computed with respect to time t along the trajectories of (1). S/he compared this inequality by a nonlinear co-system $u^{(m)}(t)=g_m(t,u,\dot{u},\cdots,u^{(m-1)})$. If the map $g_m(\cdot)$ is of class W (non-decreasing) and the co-system has an a.s. ZES then the ZES of (1) is also a.s. The method uses a special *Vector Lyapunov functions* (VLF)

$$\underline{V}(t,\underline{x})=[v_1(t,\underline{x}),v_2(t,\underline{x}),\ldots,v_m(t,\underline{x})]^T \tag{2}$$

defining $v_i(t,\underline{x})\triangleq v^{(i-1)}(t,\underline{x})$, for $i=1,\ldots,m$, but only the first component of $\underline{V}(t,\underline{x})$ is *positive definite function* (PDF) and the other components might be indefinite. This is different from ordinary VLFs with all positive semi-definite components and generating a linear combination $\sum_{i=1}^m k_i v_i(t,\underline{x})$, $k_i>0$ which is PDF, [4] and [5].

We call the VLF used by Gunderson [3], *derivatives vector Lyapunov function* (DVLF). Then we generalize the definition and refer to any vector function $\underline{V}(t,\underline{x})$ a DVLF as far as having a first component which is a PDF $v_1(t,\underline{x})$ and the remainder components are possibly indefinite functions.

Butz [6] considered the autonomous system $\dot{x}=f(\underline{x})$ together with a LPDF $v(\underline{x})$ satisfying $a_3\dddot{v}(\underline{x})+a_2\ddot{v}(\underline{x})+\dot{v}(\underline{x})<0$, $\forall\underline{x}\ne 0$ ($a_i\ge 0$ for $i=2,3$) and concluded a.s. of the ZES.

The previous researches [7] and [8] used a differential inequality $\dot{\underline{V}}\le\mathbf{A}\underline{V}$ for a DVLF to analyse the stability of ZES of (1). The $\mathbf{A}$ matrix was in controllable canonical form with a Hurwitz characteristic equation, i.e. $\det(sI-\mathbf{A})=0$.

In [9] we extended the result of Butz [6] to analyse the u.a.s. of ZES of (1) using the higher order time derivatives of a time varying LPDF $v_1(t,x)$, i.e. if $\sum_{i=1}^m a_{mi}(d^i/dt^i)v_1(t,x)$ is negative definite when all $a_{mi}\ge 0$, then the ZES of (1) is u.a.s.. The new method could take care of the cases where the LPDF $v_1(t,x)$ and/or the systems are not smooth enough and the higher order

Manuscript received August 10, 2008. Both authors are with the Electrical Engineering Department, Aboureyhan Building, Amirkabir Univ. of Tech. (Tehran Polytechnic), Tehran, Iran.

Vahid Meigoli (corresponding author, phone: (+9821) 64543398; fax: (+9821) 66406469; e-mail: vahid_mey@yahoo.com, vmeigoli@aut.ac.ir).

S. K. Y. Nikravesh (phone: 98-21-64543316; e-mail: nikravsh@aut.ac.ir).

time derivatives of the LF are not well-defined. We considered a more general form of differential inequality for a DVLF $\underline{V}(t,x)$, and the ZES of (1) would be u.a.s. as far as $\sum_{i=1}^{m} a_{mi}\dot{v}_i(t,x)$ is negative definite and $v_i(t,x) < \alpha_i(\|x\|)$ for $\alpha_i \in K$ and $i = 1,2,\ldots,m$.

Now let us consider the homogeneous nonlinear systems which contain a wide group of nonlinear systems, and have been popular during the last three decade. A nice property about homogeneous systems is that, they act some how in between linear and nonlinear systems. Also a lot of subjects concerning the nonlinear systems first have been applied to homogeneous systems or are most related to them, such as: controllability and local approximation [10], exponential stabilization [11], control by adding power integrator technique [12], and finite time stabilization [13].

The first theorem of this paper summarizes the main results of [9] about the higher order time derivatives of LF without proof. Then we focus on the applications of this theorem to stability analysis of homogeneous nonlinear systems. This theorem is shown to be useful only for stability analysis of nonlinear zero degree homogeneous systems, hence a new theorem for general nonlinear homogeneous systems is developed. We assume the reader is familiar with the Lyapunov stability methods [1-2].

This paper is organized as follows. The preliminary definitions and results about homogeneous systems are given in section II. The main theorem on stability analysis of homogeneous systems is presented in Section III. Some examples are given in IV. Concluding remarks are given in Section V.

II. The Preliminary Definitions and Results

A. The Higher Order Time Derivatives of LF

If a function $v(t,x)$ and the nonlinear system (1) are smooth enough, then the higher order total time derivatives $v^{(i)}(t,x)$, for $i = 1,2,\ldots$ along the solutions of (1) are computed iteratively, using ($v^{(0)}(t,x) = v(t,x)$)

$$v^{(i)}(t,x) \triangleq [\partial v^{(i-1)}/\partial x]^T f(t,x) + \partial v^{(i-1)}/\partial t \tag{3}$$

Definition 1 [9]: An arbitrary scalar function $v(t,x)$ (may be indefinite)

i. is called *locally decrescent* (LD) if there exist $r > 0$ and $\alpha \in K$ such that for every $\|x\| < r$

$$v(t,x) < \alpha(\|x\|) \tag{4}$$

ii. is called *globally decrescent* (GD) if (4) satisfies globally. ■

In the following a general theorem for analyzing the stability of (1) is introduced.

Theorem 1 [9]: Consider the m-vector C^1 function $\underline{V}(t,x)$ of the form (2) with the following properties:

i. The first component $v_1(t,x)$ of $\underline{V}(t,x)$ is *radially unbounded* (RU) and PDF, i.e. $v_1(t,0) = 0$, $\forall t \geq 0$ and there exists some $\phi_1 \in K_\infty$, such that:

$$v_1(t,x) \geq \phi_1(\|x\|) \quad \forall x \in \mathbb{R}^n, \forall t \geq 0 \tag{5}$$

ii. All the $v_i(t,x)$ components are GD, i.e. there exist $\alpha_i \in K$ for $i = 1,\ldots,m$ such that

$$v_i(t,x) \leq \alpha_i(\|x\|) \quad \forall x \in \mathbb{R}^n, \forall t \geq 0 \tag{6}$$

a) If the following differential inequality satisfies for all $v_i(t,x)$ along the solutions of (1):

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 & \vdots \\ \vdots & a_{ij} & \ddots & 0 & 0 \\ a_{m-1,1} & \cdots & & a_{m-1,m-1} & 0 \\ a_{m1} & \cdots & & a_{m,m-1} & a_{mm} \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{v}_{m-1} \\ \dot{v}_m \end{bmatrix} \leq \begin{bmatrix} v_2 \\ v_3 \\ \vdots \\ v_m \\ -\phi_2(\|x\|) \end{bmatrix} \tag{7}$$

where $\phi_2 \in K$ with the domain of $D_{\phi_2} = [0,+\infty)$ and $A = [a_{ij}]_{m\times m}$ is a lower triangular matrix with the following properties:

$$a_{ij} \begin{cases} = 0 & , \text{ if } i < j \\ > 0 & , \text{ if } i = j \\ \geq 0 & , \text{ if } i > j \end{cases} \tag{8}$$

then the ZES of (1) is globally uniformly asymptotically stable (g.u.a.s.).

b) If the above conditions hold only locally, i.e. for $\|x\| < r$ for a given $r > 0$ then the ZES of (1) is u.a.s. ■

Corollary 1 [9]: Consider the smooth enough time varying system (1) and a smooth enough RU and PDF $v(t,\underline{x})$. If the higher order derivatives $v^{(i)}(t,\underline{x})$ for all $i = 0,1,\ldots,m-1$ are GD and there exist $a_i \geq 0$ for $i = 1,\ldots,m$ and $\phi_2 \in K$ such that

$$\sum_{i=1}^{m} a_i v^{(i)}(t,\underline{x}) \leq -\phi_2(\|\underline{x}\|), \quad \forall \underline{x} \in \mathbb{R}^n \tag{9}$$

hen the ZES of (1) is g.u.a.s. ■

Proof: use the Theorem 1 with $v_i(t,\underline{x}) \triangleq v^{(i-1)}(t,\underline{x})$ for $i = 1,\ldots,m$. ■

Remark 1: For $m = 1$ the above corollary is reduced to the Lyapunov direct method for the stability analysis of the ZES of (1).

B. The Homogeneous Systems

Consider a function $v : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ and the vector field $\underline{f}(t,\underline{x})$ of the nonlinear system (1), we briefly recall the notion of homogeneity for v and $\underline{f}$ from [14]:

For a sequence of positive weights $r = (r_1,\ldots,r_n)$, $r_i \geq 1$ and a non-negative variable $\lambda \geq 0$, a dilation is defined as a linear map $\Delta_\lambda^r(\underline{x}) \triangleq (\lambda^{r_1}x_1,\ldots,\lambda^{r_n}x_n)$. Then the $v(t,\underline{x})$ function and the $\underline{f}(t,\underline{x})$ vector field are defined to be homogeneous of order p *with respect to* (w.r.t.) the dilation Δ_λ^r, if $v(t,\Delta_\lambda^r\underline{x}) = \lambda^p v(t,\underline{x})$ and $\underline{f}(t,\Delta_\lambda^r\underline{x}) = \lambda^p \Delta_\lambda^r \underline{f}(t,\underline{x})$ respectively. In this case we briefly define v and $\underline{f}$ are Δ-homogeneous of order p and symbolize with $v \in H_p$ and $\underline{f} \in \underline{n}_p$.

The special weights $r = (1,1,\ldots,1)$ are referred as standard weights, hence $v(t,\underline{x})$ and $\underline{f}(t,\underline{x})$ are said to be standard homogeneous of order p if $v(t,\lambda\underline{x}) = \lambda^p v(t,\underline{x})$ and $\underline{f}(t,\lambda\underline{x}) = \lambda^{p+1}\underline{f}(t,\underline{x})$ respectively.

For $p \geq 2\max r_i$ the Δ-homogeneous *p*-norm is defined by $\|\cdot\|_{\Delta,p} \triangleq (\sum_{i=1}^{n}|x_i|^{p/r_i})^{1/p}$. It is clear that $\|\cdot\|_{\Delta,p} \in H_1$, while this is not a true norm, because it doesn't satisfy the triangular inequality.

Considering a Δ-homogeneous LF in the Lyapunov direct method for the stability analysis of a given Δ-homogeneous vector field is a usual task in the literature [15]. In the following we concentrate on the applications of higher order time derivatives of Δ-homogeneous LFs to Δ-homogeneous systems.

Example 1 [9]: Consider the following nonlinear dynamic system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_1 - x_2\left(b + x_2 \Big/ \sqrt{x_1^2 + x_2^2}\right) \end{cases} \tag{10}$$

with the following parameters

$$a = 0.1 \quad , \quad b = 1.2 \tag{11}$$

which is continuous at $x_1 = x_2 = 0$ and has a ZES. This system is obviously of the standard zero order homogeneous form. Let us rewrite the dynamic equation (10) in the polar coordinate form $R(t)$ and $\theta(t)$ using $x_1 = R\cos\theta \triangleq RC_\theta$ and $x_2 = R\sin\theta \triangleq RS_\theta$:

$$\begin{cases} \dot{R} = -RS_\theta[S_\theta^2 + bS_\theta + C_\theta(a-1)] \\ \dot{\theta} = -S_\theta^2 - aC_\theta^2 - (b + S_\theta)S_\theta C_\theta \end{cases} \tag{12}$$

Using the LF candidate $v(\underline{x}) = x_1^2 + x_2^2 = R^2$, one has

$$\dot{v}(\underline{x}) = 2R\dot{R} = -2R^2 S_\theta[S_\theta^2 + bS_\theta + C_\theta(a-1)] \tag{13}$$

which is indefinite for the parameter values (11), and thus the Lyapunov direct method fails in proving g.u.a.s. of ZES using this LF. The higher order time derivatives of $v(\underline{x})$ function would be as follows:

$$\begin{aligned} \ddot{v}(\underline{x}) &= \dot{R}\,\partial\dot{v}(\underline{x})/\partial R + \dot{\theta}\,\partial\dot{v}(\underline{x})/\partial\theta = R^2[11.56 + 0.6C_\theta - 13.74C_\theta^2 \\ &\quad -2.4C_\theta^3 + 1.8C_\theta^5 + 2C_\theta^6 + S_\theta(9.6 - 1.68C_\theta - 7.2C_\theta^2 - 2.4C_\theta^4)] \end{aligned} \tag{14}$$

$$\begin{aligned} \dddot{v}(\underline{x}) &= R^2[-48.624 - 5.76C_\theta + 58.392C_\theta^2 - 0.48C_\theta^3 - 7.2C_\theta^4 \\ &\quad +19.2C_\theta^5 + 14.4C_\theta^6 - 12.96C_\theta^7 - 16.8C_\theta^8 + S_\theta(-45.56 - 5.256C_\theta \\ &\quad +27.98C_\theta^2 - 1.8C_\theta^3 + 14.04C_\theta^4 + 19.8C_\theta^5 + 11.78C_\theta^6 - 12.6C_\theta^7 - 8C_\theta^8)] \end{aligned} \tag{15}$$

These derivatives will be used in the stability analysis. Note that $v^{(i)}(\underline{x})/R^2$ for each $i = 0,1,2,3$ is a periodic function only of θ. It was shown in [9] for $a_1 = 1$, $a_2 = 2.43$, $a_3 = 2$ that

$$\sum_{i=1}^{3} a_i v^{(i)}(\underline{x})/R^2 < 0 \quad \forall\theta \tag{16}$$

Hence $\sum_{i=1}^{3} a_i v^{(i)}(\underline{x})$ is negative definite and the conditions of Corollary 1 are satisfied for the nonlinear autonomous system (10) and thus the ZES is g.u.a.s. ■

In the above example the nonlinear system was homogeneous of zero order, and all the higher order derivatives of LF $v(\underline{x})$ (see eq. (13)-(15)) were homogeneous of order two. The phenomenon of same order of homogeneity for $v^{(i)}(\underline{x})$ is not accidental, and it is a consequence of the following important fact about the Δ-homogeneity:

Lemma 1 [10]: If the function $v(t,\underline{x}) \in H_p$ and the vector field $\underline{f}(t,\underline{x}) \in \underline{n}_k$ w.r.t. some dilation Δ_λ^r, then the scalar multiplication $v \cdot \underline{f} \in \underline{n}_{p+k}$, and the total time derivative of v along the solutions of $\underline{f}$, i.e. $\dot{v}(t,\underline{x}) \in H_{p+k}$. Therefore by induction $[v(t,\underline{x})]^i \cdot \underline{f}(t,\underline{x}) \in \underline{n}_{pi+k}$ and $v^{(i)}(t,\underline{x}) \in H_{p+ki}$ for $i = 1,2,\dots$. ■

In the previous example $v(\underline{x}) \in H_2$ and $\underline{f}(\underline{x}) \in \underline{n}_0$, and thus $v^{(i)}(\underline{x}) \in H_{2+0\cdot i}$ for $i = 1,2,\dots$. Therefore any linear combinations of $v^{(i)}(\underline{x})$ for several i are homogeneous functions of order two, and we could easily obtain this sign using the polar coordinate. However the following remark shows some difficulties for homogeneous nonlinear systems of order $k > 0$.

Remark 2: If the nonlinear system is homogeneous of order $k > 0$, then the higher order derivatives of a homogeneous LF $v(\underline{x})$ are homogeneous of different order and we can not easily determine the sign of their linear combinations.

Moreover in this case it could be shown that if $\sum_{i=1}^{m} a_i v^{(i)}(\underline{x}) < 0 \quad \forall \underline{x} \neq 0$ and $a_1 \neq 0$ then $\dot{v}(\underline{x}) \leq 0$ very near the origin, because the first derivative dominates the other derivatives in a very small neighborhood of zero (see Lemma 1). Thus the Lyapunov direct method is useful for this case, and the Theorem 1 is meaningless for homogeneous nonlinear systems of order $k > 0$. ■

III. The Main Results

It was shown in the previous section that the Theorem 1 is not useful for stability analysis of nonlinear homogeneous systems of order $k > 0$. Hear we do some small changes in Theorem 1 and make it useful for stability analysis of nonlinear homogeneous systems of arbitrary order. For simplicity we consider only autonomous case, i.e. the following nonlinear system:

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \quad , \quad \underline{x} \in \mathbb{R}^n \tag{17}$$

Let $\underline{f} \in \underline{n}_k$ for some $k > 0$ in (17) and $\|\underline{x}\|_\Delta$ is a given homogeneous norm w.r.t. a given dilation Δ_λ^r, define the following nonlinear system:

$$\dot{\underline{x}} = \tilde{f}(\underline{x}) = \begin{cases} \underline{f}(\underline{x})/\|\underline{x}\|_\Delta^k & , \quad \underline{x} \neq \underline{0} \\ \underline{0} & , \quad \underline{x} = \underline{0} \end{cases} \tag{18}$$

It is clear that $\tilde{f} \in \underline{n}_0$ and $\tilde{f}(\underline{x})$ is continuous at zero. The described mapping from nonlinear system (17) to the nonlinear system (18) was first used in [14] for implementing the invariant homogeneous cones in stability analysis, but we use this mapping for a different purpose.

Lemma 2: The ZES of (17) is g.u.a.s. iff the ZES of (18) is g.u.a.s.

Proof: It is clear from the definition that any nonlinear homogeneous system of non-negative order such as (17) and (18) has a ZES. Moreover since $\|\underline{x}\|_\Delta \neq 0$ for $\underline{x} \neq \underline{0}$, then the nonlinear system (17) has not any non-zero equilibrium point iff the other system (18) has not either. Moreover the solution curves of both systems coincide with each other, but with different velocities at each point. Hence we consider the solution curves of both systems as reparameterizations of each other. The phase portraits of the two systems are equivalent and some qualitative performances such as g.u.a.s. of ZES are equivalent. ■

Now consider a given C^1 function $g(\underline{x})$, we want to compare the time derivative $\dot{g}(\underline{x})$ along the solutions of (17) and (18) at each point $\underline{x}$. This is simply done, by using (3), (17) and (18). Let t and $\tilde{t}$ be the time variables in (17) and (18) respectively, and thus

$$dg(\underline{x})/d\tilde{t} = [\partial g(\underline{x})/\partial \underline{x}]^T \tilde{f}(\underline{x}) = \tag{19}$$

$$[\partial g(\underline{x})/\partial \underline{x}]^T \underline{f}(\underline{x})/\|\underline{x}\|_\Delta^k = dg(\underline{x})/\|\underline{x}\|_\Delta^k dt$$

Let us view that both systems (17) and (18) are equivalent using the same state vector $\underline{x}$ and the variable time scaling *(depending on state)*, because

$$d\underline{x}/d\tilde{t} = \tilde{f}(\underline{x}) = \underline{f}(\underline{x})/\|\underline{x}\|_\Delta^k = (d\underline{x}/dt)/\|\underline{x}\|_\Delta^k \Rightarrow$$

$$d\tilde{t} = \|\underline{x}\|_\Delta^k dt \tag{20}$$

The relationship (20) shows the relativity of time scaling in two systems. It depends on the homogeneous norm of the state vector $\underline{x}$. Also (20) gives a new interpretation of (19). Since $\tilde{f} \in \underline{n}_0$, the Theorem 1 may be helpful in proving the g.u.a.s. of ZES of (18).

A. The Main Theorem

The following theorem concerns the stability analysis of (17), and uses (20) and Lemma 2 in its proof.

Theorem 2: Consider the nonlinear homogeneous system (17) ($\underline{f} \in \underline{n}_k$) and a m-vector C^1 function $\underline{V}(\underline{x}) = [v_1(\underline{x}), v_2(\underline{x}), \ldots, v_m(\underline{x})]^T$. If the following conditions are satisfied:

i. $v_1(\underline{x})$ is RU, PDF.
ii. All $v_i(\underline{x})$ are GD and $v_i \in H_p$ for $i = 1,\ldots,m$.
iii. the following differential inequality satisfies for derivatives along the solutions of (17):

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 & \vdots \\ \vdots & a_{ij} & \ddots & 0 & 0 \\ a_{m-1,1} & \cdots & & a_{m-1,m-1} & 0 \\ a_{m1} & \cdots & & a_{m,m-1} & a_{mm} \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{v}_{m-1} \\ \dot{v}_m \end{bmatrix} \le \|\underline{x}\|_\Delta^k \begin{bmatrix} v_2 \\ v_3 \\ \vdots \\ v_m \\ -v_{m+1} \end{bmatrix} \tag{21}$$

where $v_{m+1}(\underline{x}) \in H_p$ is a PDF and $A = [a_{ij}]_{m\times m}$ is a matrix with the property (8), then the ZES of (17) is g.u.a.s. ■

Proof: Using (19) yields $(dv_i/dt)/\|\underline{x}\|_\Delta^k = dv_i/d\tilde{t}$ for the time derivatives of each $v_i(\underline{x})$ along the solutions of (17) and (18). Moreover implementing $v_i \in H_p$ and $\underline{f} \in \underline{n}_k$ yields $(dv_i/dt) \in H_{p+k}$ for $i = 1,\ldots,m$. Each term in (21) is a homogeneous function of order $p+k$. Dividing (21) by $\|\underline{x}\|_\Delta^k$ and using $(dv_i/dt)/\|\underline{x}\|_\Delta^k = dv_i/d\tilde{t}$ results the following relationship, component vise:

$$A\, d\underline{V}(\underline{x})/d\tilde{t} \le [v_2(\underline{x}),\ldots,v_m(\underline{x}),-v_{m+1}(\underline{x})]^T \tag{22}$$

Hence the conditions of Theorem 1 are satisfied for g.u.a.s. of ZES of (18). Using Lemma 2 results the g.u.a.s. of ZES of (17). ■

B. The homogeneous polar coordinate

Although the Theorem 2 is applicable for arbitrary order homogeneous systems, but we need some designing tools to find the useful $v_i(\underline{x})$ functions for a given nonlinear system. In the previous example we used the polar coordinate. The usual polar coordinate is useful only for standard nonlinear homogeneous systems, but not for general homogeneity. Here a new polar coordinate w.r.t. given weights $r = (r_1, r_2)$ for $n = 2$ is introduced. We designate to each point $\underline{x} = [x_1, x_2]^T$ a pair (R,θ) as *Δ-polar* coordinate. Considering a given Δ-homogeneous norm $\|\cdot\|_{\Delta,p}$ let

$$\begin{cases} x_1 = R^{r_1} C_\theta^{2r_1/p} \\ x_2 = R^{r_2} S_\theta^{2r_2/p} \end{cases} \tag{23}$$

Defining $\underline{u}_\theta \triangleq [C_\theta^{2r_1/p}, S_\theta^{2r_2/p}]^T$ we have $\|\underline{u}_\theta\|_{\Delta,p} = \sqrt[p]{(C_\theta^{2r_1/p})^{p/r_1} + (S_\theta^{2r_2/p})^{p/r_2}} = 1$ and $\underline{x} = \Delta_R^r \underline{u}_\theta$, and thus $\|\underline{x}\|_{\Delta,p} = R$ and $R \in H_1$. Moreover each $v(\underline{x}) \in H_p$ and $\underline{f}(\underline{x}) \in \underline{n}_k$ could be decomposed as:

$$v(\underline{x}) = R^p v(\underline{u}_\theta) \tag{24}$$

$$\underline{f}(\underline{x}) = R^k \Delta_R^r \underline{f}(\underline{u}_\theta)$$

The decomposition of R and θ is very important and will be used in this paper. Differentiating (23) w.r.t. time and solving for $\dot{R}$ and $\dot{\theta}$ we obtain:

$$\begin{bmatrix} \dot{R} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{r_1} c_\theta & \frac{1}{r_2} s_\theta \\ -\frac{p}{2r_1} s_\theta & \frac{p}{2r_2} c_\theta \end{bmatrix} \begin{bmatrix} c_\theta^{(1-2r_1/p)} & 0 \\ 0 & s_\theta^{(1-2r_2/p)} \end{bmatrix} \Delta_{R^{-1}}^r \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \tag{25}$$

Using (17) and (24) we have $\Delta_{R^{-1}}^r \dot{\underline{x}} = \Delta_{R^{-1}}^r \underline{f}(\underline{x}) = R^k \Delta_{RR^{-1}}^r \underline{f}(\underline{u}_\theta) = R^k \underline{f}(\underline{u}_\theta)$. Substituting this into (25) yields:

$$\begin{bmatrix} \dot{R} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} R^{k+1} & 0 \\ 0 & R^k \end{bmatrix} \begin{bmatrix} \frac{1}{r_1} c_\theta & \frac{1}{r_2} s_\theta \\ -\frac{p}{2r_1} s_\theta & \frac{p}{2r_2} c_\theta \end{bmatrix} \begin{bmatrix} c_\theta^{(1-2r_1/p)} & 0 \\ 0 & s_\theta^{(1-2r_2/p)} \end{bmatrix} \underline{f}(\underline{u}_\theta) \tag{26}$$

The last equation is the Δ-polar differential equation of the nonlinear system with $\underline{f}(\underline{x}) \in \underline{n}_k$. The extension of Δ-polar coordinates to $n > 2$ is straightforward; Just set $x_i = R^{r_i} C_{\theta_i}^{2r_i/p}$ for $i = 1,\ldots,n$ where $\sum_{i=1}^n C_{\theta_i}^2 = 1$.

C. Theorem 2 Implementation

An important question is : *How to use the Δ-polar coordinates to implement the Theorem 2?*

Answer: Using the assumption $\underline{f}(\underline{x}) \in \underline{n}_k$ and the Δ-polar coordinates we have $\|\underline{x}\|_{\Delta,p}^k = R^k$. In our procedure of implementing the Theorem 2, we Construct (21) one row after another. Let us be at the *i'th* iteration, i.e. $v_j(\underline{x})$ for $j = 1,2,\ldots,i$ are previously defined and we aim to find $v_{i+1}(\underline{x})$ and a_{ij} for $j = 1,\ldots,i$ and construct the *i'th* row of (21), i.e. $\sum_{j=1}^i a_{ij}\dot{v}_j(\underline{x}) \le R^k v_{i+1}(\underline{x})$ or equivalently

$$\sum_{j=1}^{i} a_{ij}\left(\frac{\dot{v}_j(\underline{x})}{R^{k+p}}\right) \le \frac{v_{i+1}(\underline{x})}{R^p} \tag{27}$$

According to the assumption $v_j(\underline{x}) \in H_p$, $\dot{v}_j(\underline{x}) \in H_{k+p}$, therefore for $j = 1,\ldots,i$ the functions $(\dot{v}_j(\underline{x})/R^{k+p}) \in H_0$ in (27) are independent of R, i.e. they are known periodic functions only of θ. Hence using numerical methods such as plotting $\dot{v}_j(\underline{x})/R^{k+p}$ versus θ, one can find a linear combination of them and a new function $(v_{i+1}(\underline{x})/R^p) \in H_0$ such that (27) satisfies.

IV. Some Examples

Example 2: The nonlinear system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1^3\\ x_2^3\end{bmatrix} \tag{28}$$

is standard homogeneous of order two, i.e. $\underline{f} \in \underline{n}_2$. Similarly to Example 1 we change (28) to the polar differential equations. When $r_1 = r_2 = 1$ and $p = 2$ are used, the Δ-polar coordinates for standard homogeneity, coincide with the usual polar coordinates. Substituting $\underline{u}_\theta = [C_\theta\ , S_\theta]^T$, $k = 2$, $r_1 = r_2 = 1$, and $p = 2$ for (26) we have:

$$\begin{bmatrix}\dot{R}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}R^3 & 0\\ 0 & R^2\end{bmatrix}\begin{bmatrix}C_\theta & S_\theta\\ -S_\theta & C_\theta\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}C_\theta^3\\ S_\theta^3\end{bmatrix} \tag{29}$$

Let $\|\underline{x}\|_\Delta^k = R^2$ to apply Theorem 2 for this example. Starting with $v_1(\underline{x}) = x_1^2 + x_2^2 = R^2$, we use the following special form of (21) for stability analysis:

$$\begin{bmatrix}1 & & & & \\ 0 & 1 & & & \\ \vdots & 0 & \ddots & & \\ 0 & \cdots & 0 & 1 & \\ a_{m1} & \cdots & & a_{m,m-1} & a_{mm}\end{bmatrix}\begin{bmatrix}\dot{v}_1\\ \dot{v}_2\\ \vdots\\ \dot{v}_{m-1}\\ \dot{v}_m\end{bmatrix} = R^2\begin{bmatrix}v_2\\ v_3\\ \vdots\\ v_m\\ -v_{m+1}\end{bmatrix} \tag{30}$$

Hence for $i = 1, 2, \ldots, m-1$

$$v_{i+1}(\underline{x}) \triangleq \frac{\dot{v}_i(\underline{x})}{R^2} = \begin{bmatrix}\dfrac{\partial v_i(\underline{x})}{\partial R} & \dfrac{\partial v_i(\underline{x})}{\partial \theta}\end{bmatrix}\begin{bmatrix}\dot{R}\\ \dot{\theta}\end{bmatrix}\frac{1}{R^2} \tag{31}$$

Substituting (29) into (31) we have for $i = 1, 2, \ldots, m-1$

$$v_{i+1}(\underline{x}) = \begin{bmatrix}R\dfrac{\partial v_i}{\partial R} & \dfrac{\partial v_i}{\partial \theta}\end{bmatrix}\begin{bmatrix}C_\theta & S_\theta\\ -S_\theta & C_\theta\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}C_\theta^3\\ S_\theta^3\end{bmatrix} \tag{32}$$

All the higher order derivatives are well-defined, C^1 everywhere and belong to H_2, e.g.

$$v_2(\underline{x}) = \frac{\dot{v}_1(\underline{x})}{R^2} = 2R^2\begin{bmatrix}C_\theta & S_\theta\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}C_\theta^3\\ S_\theta^3\end{bmatrix} \tag{33}$$

We considered the parameter values $A = \begin{bmatrix}-0.2 & -1\\ 1 & -1\end{bmatrix}$, and computed $v_3(\underline{x})$ using (32) as well. Although $v_1(\underline{x})$ is PDF, but for this parameters $\dot{v}_1(\underline{x})$ is not negative definite, and thus the Lyapunov direct method fails to prove g.u.a.s. of ZES of (28). We have found numerically that $\frac{\dot{v}_1(\underline{x})}{R^4} + 0.1\frac{\dot{v}_3(\underline{x})}{R^4} \triangleq -\frac{v_4(\underline{x})}{R^2}$ is a negative function only of θ (see (27) for our method). Letting $m = 3$, the relationship (30) (and thus (21)) is satisfied. Moreover all $v_i(\underline{x})$ are GD, and thus the ZES of (28) is g.u.a.s. ■

Example 3: The nonlinear system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & x_1^2\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1^3\\ x_2\end{bmatrix} = \begin{bmatrix}a_{11}x_1^3 + a_{12}x_2\\ a_{21}x_1^5 + a_{22}x_1^2x_2\end{bmatrix} \tag{34}$$

is Δ-homogeneous of order two w.r.t. weights $r = (r_1, r_2) = (1,3)$, i.e. $\underline{f} \in \underline{n}_2$, because

$$\underline{f}(\Delta_\lambda^r \underline{x}) = \begin{bmatrix}a_{11}(\lambda x_1)^3 + a_{12}(\lambda^3 x_2)\\ a_{21}(\lambda x_1)^5 + a_{22}(\lambda x_1)^2(\lambda^3 x_2)\end{bmatrix} = \lambda^2\begin{bmatrix}\lambda & 0\\ 0 & \lambda^3\end{bmatrix}\begin{bmatrix}a_{11}x_1^3 + a_{12}x_2\\ a_{21}x_1^5 + a_{22}x_1^2x_2\end{bmatrix} = \lambda^2\Delta_\lambda^r \underline{f}(\underline{x})$$

We use the Δ-polar coordination $(x_1, x_2) = (R\sqrt[3]{\cos\theta}, R^3\sin\theta)$ and the Δ-homogeneous norm $\|\underline{x}\|_{\Delta,6} = \sqrt[6]{x_1^6 + x_2^2} = R$ for this system. Substituting $(r_1, r_2) = (1,3)$, $p = 6$, $\underline{u}_\theta = [\sqrt[3]{C_\theta}\ , S_\theta]^T$ and $k = 2$ into (26) then we obtain the Δ-polar differential equation as follows:

$$\begin{bmatrix}\dot{R}\\ \dot{\theta}\end{bmatrix} = C_\theta^{2/3}\begin{bmatrix}R^3 & 0\\ 0 & R^2\end{bmatrix}\begin{bmatrix}C_\theta & \frac{1}{3}S_\theta\\ -3S_\theta & C_\theta\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}C_\theta\\ S_\theta\end{bmatrix} \tag{35}$$

The PDF $v_1(\underline{x}) \triangleq \|\underline{x}\|_{\Delta,6}^6 = x_1^6 + x_2^2 = R^6$ and the equation (30) will be used to stability analysis of ZES of (34) using Theorem 2. substituting (35) into (31) we will have for $i = 1, 2, \ldots, m-1$

$$v_{i+1}(\underline{x}) = C_\theta^{2/3}\begin{bmatrix}R\dfrac{\partial v_i}{\partial R} & \dfrac{\partial v_i}{\partial \theta}\end{bmatrix}\begin{bmatrix}C_\theta & \frac{1}{3}S_\theta\\ -3S_\theta & C_\theta\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}C_\theta\\ S_\theta\end{bmatrix} \tag{36}$$

All the higher order derivatives are well-defined, C^1 everywhere and belong to H_6. We have considered the parameter values $A = \begin{bmatrix}0 & -1\\ 2 & -1\end{bmatrix}$, and for this parameters $\dot{v}_1(\underline{x})$ is not negative definite, therefore the Lyapunov direct method fails to prove g.u.a.s. of ZES of (34). Letting $m = 3$, $v_i(\underline{x}) = \dot{v}_{i-1}(\underline{x})/R^2$ for $i = 2,3$ are computed. Then we have found numerically that $\frac{\dot{v}_1(\underline{x})}{R^8} + \frac{\dot{v}_3(\underline{x})}{60R^8} \triangleq -\frac{v_4(\underline{x})}{R^6}$ is a negative function only of θ (see (27) for our method), and thus $\dot{v}_1(\underline{x}) + \frac{1}{60}\dot{v}_3(\underline{x}) \triangleq -R^2 v_4(\underline{x})$ is negative definite. Moreover all $v_i(\underline{x})$ are GD, and thus the ZES of (34) is g.u.a.s. ■

V. Conclusion

The new method introduced in this paper is briefly summarized as follows: Suppose the a.s. of ZES of a given homogeneous dynamic system using the Lyapunov direct method is under consideration. First one tries to guess the correct homogeneous LF candidate with negative definite first order derivative. If the first order LF derivative was not negative definite, then the Lyapunov direct method is failed using the given LF, even if the LF candidate is chosen very expertly.

By the use of Theorem 2, some approximations of the higher order time derivatives of the LF are used to compensate the role of non-negative definiteness of the LF first order derivative in the stability analysis. Some examples are given to show the validity of the approach.

REFERENCES

[1] M. Vidyasagar, “Nonlinear Systems Analysis”, Prentice Hall, 2’nd Ed, 1993.
[2] H. K. Khalil, “Nonlinear systems”, Third ed., 2002.
[3] R. W. Gunderson, “A Comparison Lemma for Higher Order Trajectory Derivatives”, Proceedings of the American Mathematical Society, Vol. 27, No. 3, pp. 543-548, 1971.
[4] V. M. Matrosov, “Method of vector Liapunov functions of interconnected systems with distributed parameters (survey)” (in Russian), Avtomatika i Telemekhanika, vol. 33, pp. 63–75, 1972.
[5] S. G. Nersesov, W.M. Haddad, “On the stability and control of nonlinear dynamical systems via vector Lyapunov functions”, IEEE Transaction on Automatic Control, vol. 51, no. 2, 2006.
[6] A. Butz, “Higher order derivatives of Liapunov functions”, IEEE Transactions on Automatic Control (Correspondence), Vol. 14, pp. 111-112, 1969.
[7] V. Meigoli, and S. K. Y. Nikravesh, “Extension of Higher Order Derivatives of Lyapunov Functions in Stability Analysis of Nonlinear Systems”, Accepted to the Amirkabir Journal of Science and Technology, 2008.
[8] V. Meigoli, and S. K. Y. Nikravesh, “A New Theorem on Higher Order Derivatives of Lyapunov Functions”, Accepted to the ISA Transactions, Elsevier, 2008.
[9] V. Meigoli, and S. K. Y. Nikravesh, “Higher Order derivatives of Lyapunov Function Approach for Stability Analysis of Nonlinear Systems”, submitted to Systems and Control Letters, 2008.
[10] H. Hermes, “Nilpotent approximations of control systems and distributions”, SIAM Journal Control Optimization, vol. 24, no. 4, pp. 731-736, July 1986.
[11] R.T. M’Closkey, R.M. Murray, “Nonholonomic systems and exponential convergence: Some Analysis tools”, Proceeding of the 32nd Conf. on Decision and Control, pp. 943-948, San Antonio, Texas, Dec 1993.
[12] C. Qian, W. Lin, “A continuous feedback approach to global strong stabilization of nonlinear systems”, IEEE Trans. On Automatic Control, Vol. 46, No. 7, pp. 1061-1079, July 2001
[13] S.P. Bhat, D.S. Bernstein, “Finite-time stability of homogeneous systems”, Proc. American Control Conf., pp. 2513-2514, June 1997.
[14] M. Kawski, “Homogeneous feedback stabilization”, in: New Trends in Systems Theory, G. Conte, A. M. Perdon and B. Wyman eds., Progress in Systems and Control Theory, vol. 7 (1991) pp. 464-471.
[15] L. Rosier, “Homogeneous Lyapunov function for homogeneous continuous vector field”, Systems and Control Letters 19, pp. 467-473, North-Holland, 1992.