Exponential Stability and Stabilization of Linear Time-Varying Singular System

S. Sh. Alaviani and M. Shafiee

Abstract— **In this paper, Lyapunov approach is applied to study exponential stability and stabilization of linear time-varying singular system. Sufficient conditions for exponential stability and stabilization are obtained. For a special case of system, necessary and sufficient conditions for exponential stability and stabilization are obtained. Ultimately, some numerical examples are given to show the theoretical results established.**

Index Terms—**exponential stability, exponential stabilization, linear time-varying system, singular system.**

I. Introduction

In practice one is not only interested in system stability (e.g. in sense of Lyapunov), but also in bounds of system trajectories [1]. These bound properties of system responses i.e. the solution of system models, are very important from the engineering point of view [1]. Realizing this fact, numerous definitions of the so-called technical and practical stability were introduced [1]. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbations of initial conditions and allowable perturbation of system response [1]. In the engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in quantitative manner, possible derivations of system response [1]. Thus, the analysis of these particular bound properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is concerned [1].

Let the linear time-varying singular system be governed by

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t), x(t_0) = x_0, \qquad (1)$$

where $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are time-varying matrices and $E(t)$ is a time-varying singular matrix. It is assumed that system (1) is controllable.

The system defined with (1) are usually known as singular , descriptor, semi-state systems, systems of differential-algebraic equations or generalized state space systems [1]. They arise naturally in many physical applications such as electrical networks, aircraft and robot dynamics, natural delay and large-scale systems, economics and optimization problems, biology, constrained mechanics, as result of partial descritization of partial differential equations etc [1].

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For the special case of system (1) where $A(t) = A$, $E(t) = E$, and $B(t) = B$ are constant matrices the stability analysis has been investigated by [2-5]. The stabilization problem has also been studied by [2,4].

For the special case of system (1), where $A(t + T) = A(t)$ and $B(t + T) = B(t)$ are periodically time-varying matrices with period T and $E(t) = E$ is a constant matrix, stability analysis has been investigated by [6]. Furthermore, stabilization problem for this system has been also investigated by [6] and without periodic parameters by [7-9].

Controllability and observability of the system (1) have been investigated by [10,11]. Impulse observability and impulse controllability of the system (1) have been investigated by [12]. Impulse controllability of system (1) with periodic parameters has been investigated by [13].

For the special case of system (1), where $A(t + T) = A(t)$ and $B(t + T) = B(t)$ are periodically time-varying matrices with period T and $E(t) = E$ is a constant matrix, input-output decoupling problem of the system (1) has been investigated by [14,15]. Robust H_2 control problem has been investigated by [16]. The stability and stabilization problem have been investigated by [17].

State feedback impulse elimination has been applied to control of system (1) in [18]. Optimal control of the system (1) has been investigated in [19]. Optimal control of the system (1) has been investigated in [20].

In this paper, exponential stability and stabilization of system (1) by using Lyapunov approach are studied. Sufficient conditions for exponential stability and stabilization of system (1) are obtained. For the special case of system (1), where $A(t + T) = A(t)$ and $B(t + T) = B(t)$ are periodically time-varying matrices with period T and $E(t) = E$ is a constant matrix, the necessary and sufficient conditions for strong exponential stability and stabilization of the system are obtained. In section 2, exponential stability problem of system (1) is considered. In section 3, exponential stabilization of system (1) is considered. In section 4, some preliminary results related to the special case of system (1) are given. In section 5, exponential stability and stabilization of the special case of system (1) are considered. Eventually, some numerical examples are given in order to present the results established.

II. Exponential Stability

Consider the system (1). As defined in [1], W_k denotes the sub-space of consistent initial conditions generating the smooth solutions. Now, we define the following definition.

Definition 1: System (1) is exponentially stable with decay rate $\gamma > 0$ if exist $x_0 \in W_k$ and a positive constant real number α such that
$\|x_0\|_Q^2 < \alpha$ implies $\|x(t)\|_Q^2 < \alpha e^{-\gamma t}$.

Note [1]: Quadratic form $\|x(t)\|_Q^2$ is defined with

$$\|x(t)\|_Q^2 = x^T(t)Q(t)x(t) \quad ,$$

where $Q(t) = E^T(t)P(t)E(t)$, in which $P(t) = P^T(t) > 0$ is an arbitrarily specified matrix.

Theorem 1: System (1) is exponentially stable with decay rate $\gamma > 0$ if a positive constant real number γ and a positive-definite symmetric matrix P(t) exist such that

$$A^T(t)E^T(t)P(t)E(t) + \dot{E}^T(t)P(t)E(t) + E^T(t)\dot{P}(t)E(t) + E^T(t)P(t)\dot{E}(t) + E^T(t)P(t)A(t) + \gamma E^T(t)P(t)E(t) < 0 \quad . \tag{2}$$

Proof: Consider the following Lyapunov-like function:

$$V(t) = x^T(t)E^T(t)P(t)E(t)x(t) \quad , \tag{3}$$

where $P(t) = P^T(t) > 0$. Taking the derivative $V(.)$ with respect to t along the trajectory (1), we obtain

$$\dot{V}(t) = \frac{d(V(t))}{dt} = x^T(t)(A^T(t)E^T(t)P(t)E(t) + \dot{E}^T(t)P(t)E(t) + E^T(t)\dot{P}(t)E(t) + E^T(t)P(t)\dot{E}(t) + E^T(t)P(t)A(t)) \quad ,$$

using (2), we get

$$\dot{V}(t) < -\gamma V(t) \quad , \tag{4}$$

by integrating (4), we obtain

$$V(t) < V(t_0)\, e^{-\gamma t} \quad ,$$

or

$$\|x(t)\|_Q^2 < \|x_0\|_Q^2\, e^{-\gamma t}$$

and the proof is complete.

III. EXPONENTIAL STABILIZATION

For exponential stabilization of system (1), we obtain the following theorem.

Theorem 2: Consider system (1). The system (1) is exponentially stabilized with decay rate $\gamma > 0$ using controller $u(t) = -K(t)E(t)x(t)$ if a positive constant real number γ and a positive-definite symmetric matrix P(t) exist such that

$$A^T(t)E^T(t)P(t)E(t) + \dot{E}^T(t)P(t)E(t) + E^T(t)\dot{P}(t)E(t) - E^T(t)P(t)B(t)K(t)E(t) + E^T(t)P(t)\dot{E}(t) + E^T(t)P(t)A(t) - E^T(t)K^T(t)B^T(t)P(t)E(t) + \gamma E^T(t)P(t)E(t) < 0 \quad .$$

Proof: Substituting $A(t) - B(t)K(t)$ for $A(t)$ in Theorem 1, Theorem 2 is proven.

Theorem 3: Consider system (1). The system (1) is exponentially stabilized with decay rate $\gamma > 0$ using controller $u(t) = -R^{-1}(t)B^T(t)P(t)E(t)x(t)$, where R(t) is a positive-definite symmetric matrix, if a positive constant real number γ and a positive-definite symmetric matrix P(t) exist such that

$$A^T(t)E^T(t)P(t)E(t) + \dot{E}^T(t)P(t)E(t) + E^T(t)\dot{P}(t)E(t) - E^T(t)P(t)B(t)R^{-1}(t)B^T(t)P(t)E(t) + E^T(t)P(t)\dot{E}(t) + E^T(t)P(t)A(t) - E^T(t)P(t)B(t)R^{-1}B^T(t)P(t)E(t) + \gamma E^T(t)P(t)E(t) < 0 \quad . \tag{5}$$

Proof: Substituting $R^{-1}(t)B^T(t)P(t)$ for K(t) in Theorem 2, Theorem 3 is proven.

IV. SOME PRELIMINARY RESULTS

Consider the special case of system (1) where A(t) and B(t) are continuous T-period matrix functions on $\mathbb{R}^+ \cup \{0\}$ with appropriate dimensions (i.e. $A(t+T) = A(t)$ and $B(t+T) = B(t)$) and $E(t) = E$ is a singular constant matrix. We name this system as system (a).

Definition 2 [17]: For system (a), if there exists a scalar s, such that

$$\det(sE - A(t)) \neq 0 \qquad \forall t$$

then the system (1) is called uniformly regular.

[17]: From Definition 2 the regular of the system (a) is equivalent to the analytical solvability in the sense of Campbell, i.e. in [21].

Put the system into the following decomposition: suppose

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \;,\; PA(t)Q = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \;,$$

$$PB(t) = [B_1(t)/B_2(t)] \;,\quad Q^{-1}x(t) = [x_1(t)/x_2(t)] \;,$$

then the system (1) is restricted equivalent to

$$\dot{x}_1(t) = A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + B_1(t)u(t), \tag{6-a}$$

$$0 = A_{21}(t)x_1(t) + A_{22}(t)x_2(t) + B_2(t)u(t), \tag{6-b}$$

obviously, the necessary and sufficient condition of impulse free for the system (a) is that $A_{22}(t)$ is invertible [17].

Definition 3 [17]: The system (a) is called asymptotically stable, if its subsystem

$$\dot{x}_1(t) = A_{11}(t)x_1(t) + B_1(t)u(t) \;, \tag{7}$$

is asymptotically stable.

Definition 4 [17]: The system (a) is called strong asymptotically stable, if the system (a) is impulse free and asymptotically stable.

Definition 5 [17]: For the system (a), if there exists a state feedback

$$u(t) = -K(t)x(t)$$

such that the close loop system

$$E\dot{x}(t) = (A(t) - B(t)K(t))x(t)$$

is strong asymptotically stable, then the system (a) is called stabilized.

Lemma 1 [22]: The necessary and sufficient condition of asymptotically stable for standard periodically time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

is that there exists a give matrix $Q(t) > 0$, such that Lyapunov equation

$$\dot{P}(t) + P(t)A(t) + A^T(t)P(t) = -Q(t)$$

has a unique positive solution.

Lemma 2 [21]: If the system (a) is analytically solvable, then there exist analytically invertible matrices $P(t) \in \mathbb{R}^{n\times n}, Q(t) \in \mathbb{R}^{n\times n}$, such that the system (a) can be transformed into standard canonical form

$$P(t)E(t)Q(t) = \begin{bmatrix} I & 0 \\ 0 & N(t) \end{bmatrix} \;,$$

$$P(t)A(t)Q(t) = \begin{bmatrix} A_1(t) & 0 \\ 0 & I \end{bmatrix} ,$$
$$P(t)B(t) = [B_1(t)/B_2(t)] ,$$
$$C(t)Q(t) = [C_1(t) \quad C_2(t)] ,$$
$$Q^{-1}(t)x(t) = [x_1(t)/x_2(t)] , D(t) = D_1(t) + D_2(t) ,$$
$$y_1(t) = C_1(t)x_1(t) + D_1(t)u(t) ,$$
$$y_2(t) = C_2(t)x_1(t) + D_2(t)u(t) ,$$

where $N(t)$ is nilpotent matrix, and every block has appropriate size.
From Lemma 2, the necessary and sufficient condition of impulse free for the system (a) is that $N(t) = 0$ [17].

special case of the system

In this section we consider system (a) and the problems of exponential stability and stabilization related to it. For strong exponential stability of system (a) we obtain the following theorem.

Theorem 4: Suppose system (a) is uniformly regular, then system (a) is strong exponentially stable with decay rate γ, iff for a positive constant real number γ the following Lyapunov equation

$$E^T(t)V(t)(A(t) + \gamma E) + (A(t) + \gamma E)^T V(t)E + E^T\dot{V}(t)E = -E^T W(t)E \quad , \qquad (8)$$
where
$W(t) \geq 0 , W(t) \in \mathbb{R}^{n\times n}$,
has positive definite symmetric solution $V(t)$, satisfying

$$\mathrm{rank}(E) = \mathrm{rank}(E^T V(t)E) \ , \ E^T V(t)E \geq 0 \ .$$

Proof: We take the following change of the state variable
$$z(t) = e^{\gamma t}x(t) \quad , \qquad (9)$$
where γ is a positive constant real number. Then system (a) is transformed to the system

$$E\dot{z}(t) = A_0(t)z(t) + B(t)u(t) \quad , \qquad (10)$$
where $A_0(t) = A(t) + \gamma E$.

Remark 1: If system (10) is asymptotically stable then system (a) is exponentially stable with decay rate γ .

Remark 2: If system (10) is strong asymptotically stable then system (a) is strong exponentially stable with decay rate γ.

Remark 3: If system (a) is uniformly regular then system (10) is, too.

Remark 4: If system (a) is impulse free then system (10) is, too.
Now, we define the following definitions.

Definition 6: System (a) is called exponentially stable with decay rate γ, if system (10) is asymptotically stable.

Definition 7: System (a) is called strong exponentially stable with decay rate γ, if system (10) is impulse free and asymptotically stable.

For the asymptotical stability of system (10) we have the following theorem.

Theorem 5 [17]: Suppose the system (10) is uniformly regular, then the system (10) is strong asymptotically stable if and only if the Lyapunov equation

$$E^T V(t)A_0(t) + A_0^T(t)V(t)E + E^T\dot{V}(t)E = -E^T W(t)E$$
where
$W(t) \geq 0 , W(t) \in \mathbb{R}^{n\times n}$,
has positive definite symmetric solution $V(t)$, satisfying

$$\mathrm{rank}(E) = \mathrm{rank}(E^T V(t)E) \ , \ E^T V(t)E \geq 0 \ .$$

From Definition 7, Remark 2, Remark 3, Remark 4, and Theorem 5, Theorem 4 is proven and the proof is complete.

Now, we consider exponential stabilization of system (a). For strong exponential stabilization of system (a) we obtain the following theorems.

Theorem 6: Suppose the system (a) is uniformly regular, then using the controller $u(t) = -K(t)Ex(t)$ the system (a) is strong exponentially stabilized with decay rate γ, iff for a positive constant real number γ the following Lyapunov equation

$$E^T V(t)(A(t) + \gamma E - K(t)E) + (A(t) + \gamma E - K(t)E)^T V(t)E + E^T\dot{V}(t)E = -E^T W(t)E ,$$
where
$W(t) \geq 0 , W(t) \in \mathbb{R}^{n\times n}$,
has positive definite symmetric solution $V(t)$, satisfying
$\mathrm{rank}(E) = \mathrm{rank}(E^T V(t)E)$, $E^T V(t)E \geq 0$.
Proof: Substituting the controller $u(t) = -K(t)Ex(t)$ for system (a) and then using Theorem 4, Theorem 6 is proven and the proof is complete.
We choose the control law as follows

$$u(t) = -R^{-1}(t)B^T(t)V(t)Ex(t) \ , \qquad (11)$$
where
$R(t) > 0$, and $V(t) = V^T(t) > 0$.

Theorem 7: Suppose the system (a) is uniformly regular, then using the control law (11) the system (a) is strong exponentially stabilized with decay rate γ, iff for a positive constant real number γ the following Lyapunov equation

$$E^T V(t)(A(t) + \gamma E - R^{-1}(t)B^T(t)V(t)E) + (A(t) + \gamma E - R^{-1}(t)B^T(t)V(t)E)^T V(t)E + E^T\dot{V}(t)E = -E^T W(t)E , \qquad (12)$$
where
$W(t) \geq 0 , W(t) \in \mathbb{R}^{n\times n}$,
has positive definite symmetric solution $V(t)$, satisfying
$\mathrm{rank}(E) = \mathrm{rank}(E^T V(t)E)$, $E^T V(t)E \geq 0$.

Proof: Substituting the controller (11) for system (a) and then using Theorem 4, Theorem 7 is proven and the proof is complete.

V. NUMERICAL EXAMPLES

Example 1:

Consider the system:
$$\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & \cos(t) \end{pmatrix} x(t) .$$
We choose $\gamma = 1$. Then a solution of the inequality (2) is

$$P(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix},$$

and the system is exponentially stable with decay rate 1.

Example 2:

Consider the system:

$$\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2\sin(t) \end{pmatrix} x(t) + \begin{pmatrix} t \\ 0 \end{pmatrix} u(t) \ .$$

The system is unstable because the solution is

$$x(t) = \begin{pmatrix} t + c_0 \\ 0 \end{pmatrix} \ .$$

where c_0 is a positive constant real number. We choose $R(t) = 1$ and $\gamma = 1$. $P(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix}$ is the solution of the inequality (5). According to Theorem 3, using the controller $u(t) = -R^{-1}(t)B^T(t)P(t)E(t)x(t)$, the system is exponentially stabilized with decay rate $= 1$.

Example 3:

For the system (a) let $T = 10$, and when $1 \leq t \leq 11$, consider the system:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} -2e^t & 0 \\ 0 & t \end{pmatrix} x(t) \,. \qquad (13)$$

We choose $\gamma = 1$. Then the solution of the inequality (8) is

$$P(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

so the system (13) is exponentially stable with decay rate 1 .

Example 4:

For the system (a) let $T = 10$, and when $1 \leq t \leq 11$, consider the following system

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} e^t & 0 \\ 0 & t+1 \end{pmatrix} x(t) + \begin{pmatrix} e^t \\ 0 \end{pmatrix} u(t) \ , \qquad (14)$$

the system (14) is uniformly regular; but, it is unstable because when $u(t) = 0$ the solution of the system is

$$x(t) = \begin{pmatrix} ce^t \\ 0 \end{pmatrix} \ ,$$

where c is a constant real number.

We choose $R(t) = 1$ and $\gamma = 1$; therefore, using the controller (11), the Lyapunov equation (12) has a positive solution

$$V(t) = e^{-t} + 2e^{-t} \ .$$

Then, using the control law (11), the system (14) is exponentially stabilized with decay rate $\gamma = 1$.

VI. Conclusion

In this paper, we consider the exponential stability and stabilization problem of linear time-varying singular system. Sufficient conditions for exponential stability and stabilization of linear time-varying singular system are obtained. For the special case of the system, necessary and sufficient conditions for exponential stability and stabilization are obtained. Finally, some numerical examples are given in order to show the theoretical results established.

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