Stability of Fuzzy Elman Neural Network Using Joint Spectral Radius Spectral Radius of Matrix Dynamic Systems

Leila Fallah Araghi, Farbod Setoudeh

Abstract— **in this paper, a new method is derived for the existence of a common quadratic Lyapunov function for Robust Stability Analysis of Fuzzy Elman Neural Network using joint spectral radius spectral radius of Matrix.**

Index Terms— **Robust Stability, Fuzzy Elman Neural Network. Joint spectral radius spectral radius**

I. INTRODUCTION

For many real-world systems, a mathematical description in the form of differential/difference equations or similar conventional model is either infeasible or impracticable; due to the complexities involved, and the intrinsic nature of information incompleteness. The fuzzy modeling is generally presented to overcome these difficulties. Neural Networks have emerged as one important enabling technology in many scientific disciplines in solving previously unsolvable problems and in improving system performance. Stability is very important property of Neural Network system, in one of the method for stability analysis a common positive definite matrix P_i should be found to satisfy a set of Lyapunov equations [1]. Recently many researchers have worked on this area.

C. Lee Giles, et al proposed the stability of fuzzy finite state dynamics of the constructed neural networks for finite values of network weight and, through simulations, give empirical validation of the proofs [2].

Corresponding Author Leila Fallah Araghi, Assistant professor in, Electrical Engineering, Engineering Faculty Tehran University. (E-mail: l_fallah@srbau.ac.ir)

Farbod Setoudeh Electrical Engineering, Engineering Faculty Tehran University. (Email: farbodsetoudeh@srbau.ac.ir)

Zhigang Zeng, worked on the global asymptotic stability and the global exponential stability of neural networks with unbounded time-varying delays and with bounded and Lipchitz continuous activation functions [3].

Nikita, et al worked on Stability Analysis of Discrete-Time Recurrent Neural Network [4]. Jinde Cao proposed two related problems, global asymptotic stability (GAS) and global robust stability (GRS) of neural networks with time delays [5]. J. J. Rubio, et al worked on Stability Analysis of Nonlinear System Identification via Delayed Neural Networks [6]. Nikita E, et al proposed the problem of global Lyapunov stability of discrete-time recurrent neural networks in the unforced setting [7].

This paper proposed the existences of a common quadratic Lyapunov function for stability of Fuzzy Elman Neural Network.

II. A REVIEW ON STABILITY ANALYSIS ON SWITCHING SYSTEM [15, 16, 17, 18, 19, 20, 21, 22]

In switched linear systems, the subsystems of which are continuous-time linear time-invariant (LTI) systems

$$\dot{x} = A_i x, i = \{1,\ldots,n\}$$

(1)

or a collection of discrete-time LTI systems

$$x[k+1] = A_i x[k], k \in Z^+, i \in \{1,2,\ldots,n\}$$

(2)

Where $A_i \in R^{n\times n}$.

The existence of a common quadratic Lyapunov function (CQLF) for all its subsystems assures the quadratic stability of the switched system. Quadratic stability is a special class of exponential stability, which implies asymptotic stability, and has attracted a lot of research efforts due to its importance in practice. It is known that the conditions for the existence of a CQLF can be expressed as linear matrix inequalities (LMIs) [15, 16]. Namely, there exists a positive definite symmetric matrix $P, P \in R^{n\times n}$ *, such that*

$$A_i^{\,T} P + PA_i \prec 0, i \in \{1,...,n\}$$

(3)

for the continuous-time case, or

$$A_i^{\,T} PA_i - P_i \prec 0$$

(4)

for the discrete-time case, hold simultaneously. However, the standard interior point methods for LMIs may become ineffective have the number of modes increases [16].

If we consider the discrete time dynamical system as

$$x[k+1] = A_i x[k], k \in Z^+, i \in \{1,2,...,n\}$$

(5)

Let us first recall a robust stability result for linear time variant Systems with polytypic uncertainty

$$x[k+1] = A(k)x[k]$$

(6)

Where $A(k) \in \mathrm{A} = conv\{A_1, A_2, A_3,..., A_K\}$ .Here, $conv\{.\}$ stands for convex combination. In other words, the state matrix $A(k)$ of the above linear time-variant system (6) is constructed by convex combinations (with time-variant coefficients) of all the subsystems' state matrices of the switched linear system (2) [15, 16].

Proposition 1[18]: The following statements are equivalent:

1) The switched linear system $x[k+1] = A_\delta x[k]$ where $A_\delta \in \{A_1, A_2, A_3,..., A_K\}$ is asymptotically stable under arbitrary switching;

2) The linear time-variant system $x[k+1] = A(k)x[k]$ where $A(k) \in \mathrm{A} = conv\{A_1, A_2, A_3,..., A_K\}$, is robustly asymptotically stable.

3) There exists a finite integer n such that

$$\|A_{i1}A_{i2}...A_{in}\| \prec 1$$

For all **n**-tuple: $A_{ij} \in \{A_{i1}, A_{i2},..., A_{in}\}$ where $j = 1,...,n$.

It is quite interesting that the study of robust stability of a polytopic uncertain linear time-variant system, which has infinite number of possible dynamics (modes), is equivalent to considering only a finite number of its vertex dynamics in an arbitrary switching system.

In discrete time, the concept of joint spectral radius [19, 20] gives a necessary and sufficient condition for the stability of difference inclusions [20].

The joint spectral radius is the maximal growing rate which may be obtained using long products of matrices from a given set. Consider the notation $\bar{A} = \{A_1, A_2, A_3,..., A_N\}$ the joint spectral radius of the set A is formally defined as [20]:

$$\rho\ (\bar{A}) \equiv \lim_{P\to\infty} \sup\ \rho_P(\bar{A})$$

(7)

Where

$$\rho_P(\bar{A}) \equiv \underset{A_{I1},A_{I2},...,A_{IP}\in A}{Sup} \|A_{I1}, A_{I2},..., A_{IP}\|^{1/P}$$

(8) The linear difference inclusion [20]

$$x[k+1] \in F(x) = \{y : y = Ax, A \in \bar{A}\}$$

(9)

Is asymptotically stable if and only if the joint spectral radius satisfies the inequality [20]:

$$\rho\ (\bar{A}) \prec 1$$

(10)

This condition can be directly applied to the case of discrete time switched linear Systems [20]:

$$x[k+1] = A_\delta x[k], A_\delta \in \bar{A}$$

(11)

The main difficulty of this approach is the practical computation of the joint spectral radius [21]. An approximation procedure is given in [19]. When ellipsoidal norms are used for computing the approximation, it is possible to find a relation between the joint spectral radius approach and the existence of a common quadratic Lyapunov function. However this approximation implies some conservatism. Less conservative approximations are given in [19, 22].

III. A REVIEW ON ELMAN NEURAL NETWORK [8, 9, 10, 11, 12]

The classes of Neural Networks which contain cycles or feedback connections are called Recurrent Neural Networks (RNNs). While the set of topologies of a feed forward networks is fairly constrained, an RNN can take on any arbitrary topology as any node in the network may be linked with any other node (including itself). The recurrent network developed by Elman has a simple architecture; this network has been proved to be effective for modeling linear systems not higher than the first order [8, 9].

Elman proposes a simple recurrent neural network model.

Elman networks are two-layer back propagation networks, with the addition of a feedback connection from the output of the hidden layer to its input. This feedback path allows Elman networks to learn to recognize and generate temporal patterns, as well as spatial patterns [11].

In fact Elman neural network belongs to special type of feed forward neural network with additional memory neurons and local feedback; it comprises four layers, namely the input layer, hidden layer, output layer, and context layer. can store internal states [10] Fig. 1 the architecture of an Elman neural network is shown,

Where *Si (t)* indicates the states of the input layer, *Sh(t)* the states of the hidden layer and *So(t)* the states of the output layer [12].

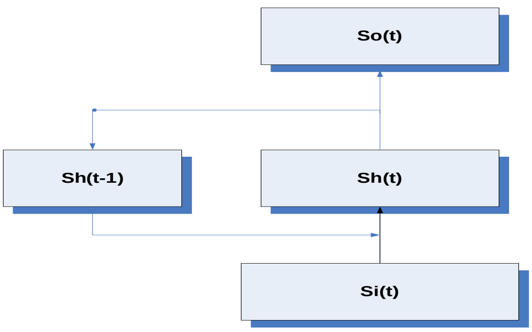

Figure 1. Structure of Elman Neural Network

The Elman network is embedded with feedback connections that offer a convenient way to accumulate previous knowledge as "experiences" and perform future predictions based on these "experiences".

IV. FUZZY ELMAN NEURAL NETWORK [30]

Figure 2 shows the seven-layer network structure of FENN, with the basic concepts taken from the Elman networks and fuzzy neural networks [30].

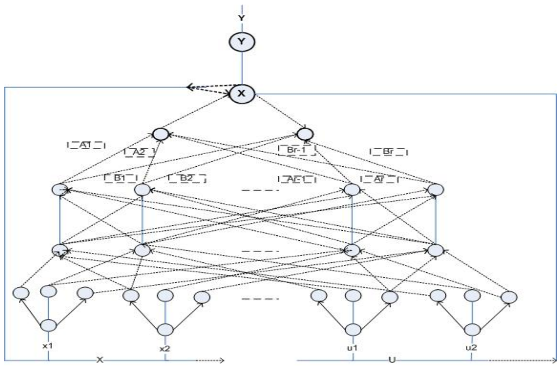

Figure2: the structure of FENN

In this network, input nodes which accept the environment inputs and context nodes which copy the value of the state space vector from layer 5 are all at layer 1 (the Input Layer). They represent the linguistic Variables known as u_j and x_i in the fuzzy rules. Nodes at layer 2 act as the membership Functions, translating the linguistic variables from layer 1 into their membership degrees [30].

Since there may exist several terms for one linguistic variable, one node in layer 1 may have links to several nodes in layer 2, which is accordingly named as the term nodes. The number of nodes in the Rule Layer (layer 3) and the one of the fuzzy rules are the same - each node represents one fuzzy rule and calculates the firing strength of the rule using membership degrees from layer 2. The connections between layer 2 and layer 3 correspond with the antecedent of each fuzzy rule. Layer 4, as the Normalization Layer, simply does the normalization of the firing strengths. Then with the normalized firing strengths h_r , rules are combined at layer 5, the Parameter Layer, where ***A*** and ***B*** become available [27]. In the Linear System Layer, the 6^{th} layer, current state vector $X(t)$ and input vector $U(t)$ are used to get the next state $X(t+1)$, which is also fed back to the context nodes for fuzzy inference at time $t+1$. The last layer is the Output Layer, multiplying $X(t+1)$ with C to get $Y(t+1)$ and outputting it [30].

Next we shall describe the feed forward procedure of FENN by giving the detailed

Node functions of each layer, taking one node per layer as example. We shall use notations like $u_i^{[k]}$ to denote the ith input to the node in layer k, and $o^{[k]}$ the output of the node in layer k. Another issue to mention here is the initial values of the context nodes. Since FENN is a recurrent network, the initial values are essential to the temporal output of the network. Usually they are preset to 0, as zero-state, but non-zero initial state is also needed for some particular case [30].

Layer 1: each node in this layer has only one input, either from the environment or

the Parameter Layer. Function of nodes is to transmit the input values to the next layer, i.e. [30]

$$o^{[1]} = u^{[1]}$$

(12)

Layer 2: there is only one input to each node at layer 2. That is, each term node can link to only one node at layer 1, though each node at layer 1 can link to several nodes at layer 2 (as described before). The Gaussian function is adopted here as the membership function [30]:

$$o^{[1]} = e^{-\frac{(u^{[2]}-c^r)^2}{2(s^r)^2}}$$

(13)

Where c^r and s^r give the center (mean) and width (variation) of the corresponding

Linguistic term of input u^2 in Rule r, i.e., one of $T_{x_i}^r$ or $T_{u_j}^r$.

Layer 3: in the Rule Layer, the firing strength of each rule is determined [30].

Each node in this layer represents a rule and accepts the outputs of all the term nodes associated with the rule as inputs. The function of node is fuzzy operator AND :(multiplication here)

$$o^{[3]} = \prod_i u^{[3]}$$

(14)

Layer 4: the Normalization Layer also has the same number of nodes as the rules,

and is fully connected with the Rule Layer. Nodes here do the function of (21), i.e.[30],

$$o^{[4]} = u^{[4]} \Big/ \sum_i u_i^{[4]}$$

(15)

In (28) we use $u^{[4]}$ to denote the specific input corresponding to the same rule with the node.

Layer 5: this layer has two nodes, one for figuring matrix A and the other for B.

Though we can use many nodes to represent the components of A and B separately, it is more convenient to use matrices. So with a little specialty, its weights of links from layer 4 are matrices A_r (to node for A) and B_r (to node for B). It is also fully connected with the previous layer. The functions of nodes for A and B are [30]:

$$o_{forA}^{[5]} = \sum_{r=1}^{R} u_r^{[5]} A^r, o_{forB}^{[5]} = \sum_{r=1}^{R} u_r^{[5]} B^r$$

(16)

Respectively.

Layer 6: the Linear System Layer has only one node, which has all the outputs of layer 1 and layer 5 connected to it as inputs. Using matrix form of inputs and output, we have [see (24)] [30]

$$o^{[6]} = AX + BU = o_{forA}^{[5]} o_{context}^{[1]} + o_{forB}^{[5]} o_{input}^{[1]}$$

(17)

So the output of layer 6 is $X(t+1)$ in (17).

Layer 7: simply as layer 1, the unique node in the Output Layer passes the input value from layer 6 to output. The only difference is that the weight of the link is matrix ***C***, not unity [30],

$$Y = o^{[7]} = Cu^{[7]}$$

(18)

This proposed network structure implements the dynamic system combined by our discrete fuzzy rules and the structure of recurrent networks. With preset human knowledge, the network can do some tasks well [30].

If we consider (17), (18) then we can say that:

$$X(t+1) = o^{[6]}$$
$$A = o^{[5]}_{forA}$$
$$B = o^{[5]}_{forB}$$
$$X = o^{[1]}_{context}$$
$$U = o^{[1]}_{input}$$

(19)

Let us consider the switched systems (17)

Where u (t) is the control and the switching signal is available in real-time. The stabilizing state feedback control problem is to find [31]:

$$u(t) = kx(t)$$

(20)

Such that the corresponding closed-loop switched system is:

$$o^{[6]} = (A + Bk)X$$

(21)

V, ROBUST STABILITY OF FUZZY ELMAN NEURAL NETWORK

we can say that $x[k+1] = \bar{A}_\delta \, x[k]$ where $A_\delta \in \{A\ , kB\}$ is asymptotically stable if and only if $\rho(A_\delta) \prec 1$ then using Proposition 1 we can proposed that system (21) is robustly asymptotically stable.

VI. CONCLUSION

In this study a new method for Robust Stability Analysis of Elman Neural Network has been used using joint spectral radius spectral radius of Matrix.

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