Proceedings of the International MultiConference of Engineers and Computer Scientists 2009 Vol II
IMECS 2009, March 18 - 20, 2009, Hong Kong

Stability Analysis of Homogeneous Nonlinear Systems Using Homogeneous Eigenvalues

Vahid Meigoli and S. K. Y. Nikravesh

***Abstract*— The Lyapunov's second method for the stability analysis of nonlinear dynamic systems requires finding Lyapunov functions. Unfortunately, finding a suitable Lyapunov function is a tedious process for a given complex nonlinear system if it is not impossible. On the other hand there are several algebraic approaches like the eigenvalue method for analyzing or designing linear time invariant systems. In this paper, we develop the eigenvalue method for the stability analysis of extended homogeneous nonlinear systems. In the case of polynomial homogeneous systems of zero degree it is shown that the zero equilibrium state of system is globally asymptotically stable if and only if all the homogeneous eigenvalues have negative real parts. Ultimately, an example is presented to describe the approach.**

***Index Terms*-- homogeneous nonlinear system, asymptotical stability, nonlinear eigenvalue.**

NOMENCLATURE

$\|\cdot\|$	A given norm on $\mathbb{R}^n$
$x(t, t_0, x_0)$	A trajectory starting at $x(t_0) = x_0$
$\underline{u}$	The underline variable means a vector quantity
ZES	Zero Equilibrium State
(G)AS	(Globally) Asymptotically Stable
$\mathbb{K}$	$\mathbb{R}$ or $\mathbb{C}$

I. INTRODUCTION

Consider the following n-dimensional autonomous dynamical system with a ZES (see the nomenclature):

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \quad , \quad \underline{x} \in \mathbb{R}^n \tag{1}$$

The advantage of the Lyapunov's second method is to use *Lyapunov functions* or energy like functions. When the complexity of a nonlinear system is increased, selecting a suitable Lyapunov function which has at least negative semi-definite derivative is an involving task, see [1].

The lifetime of eigenvalue tool for the stability analysis of linear time invariant (LTI) systems reaches to the history of linear differential equations, and almost all designing methods for LTI systems are based on eigenvalues. Finding a similar tool for the stability analysis of nonlinear systems is an ideal of each control engineer. Several researchers have followed the concept of *nonlinear eigenvalue problem*. Let us denote a pair of nonlinear eigenvalue λ and nonlinear eigenvector $\underline{v} \neq 0$ by $(\lambda, \underline{v}) \in \mathbb{C} \times \mathbb{C}^n$. There are generally two types of definitions for nonlinear eigenvalues: 1) Most researchers define a pair $(\lambda, \underline{v})$ as a nontrivial solution of $T(\lambda)\underline{v} = \underline{0}$, for some family $T(\lambda) \in \mathbb{C}^{n\times n}, \lambda \in \mathbb{C}$ of complex matrices [2] and [3]. 2) However, some other papers consider a nonlinear operator $\underline{f}: \mathbb{C}^n \to \mathbb{C}^n$ with an extra condition $\underline{f}(\underline{0}) = \underline{0}$, and define a pair $(\lambda, \underline{v})$ as a nontrivial solution of $\underline{f}(\underline{v}) = \lambda \underline{v}$ [4].

Samardzija introduced the nonlinear eigenvalues solving $\underline{f}(\underline{v}) = \lambda \underline{v}$ for the *standard homogeneous systems* of form (1) [5]. He used the real nonlinear eigenvalues in the stability analysis of ZES of these systems. Then he used real nonlinear eigenvalues for stabilization of standard homogeneous systems in [6].

The *standard homogeneous system* used by Samardzija [5] has a natural generalization, called *extended homogeneous system* with respect to a family of dilations, or briefly *homogeneous system* (the difference with the terminology *standard homogeneous system* is obvious). Many subjects, which are concern with the nonlinear systems, first have been applied to extended homogeneous systems or are most related to them, such as: controllability and local approximation [8], exponential stabilization [9], control by adding power integrator technique [10], and finite time stabilization [11]. The nonlinear extended homogeneous systems have some similarities to LTI systems; e.g. it is well-known that the ZES of a nonlinear time invariant extended homogeneous system is AS iff it is GAS [12].

Nakamura *et al* [13] defined real nonlinear eigenvalues for extended homogeneous systems. The real eigenvalues were used for the stability analysis of planar homogeneous systems. However, their definition has several drawbacks: The extension of it to complex nonlinear eigenvalues for homogeneous systems is not straightforward. Also it does not contain the Samardzija's definition of nonlinear eigenvalues for standard homogeneous systems as a special case.

In this paper, we modify the definition of nonlinear eigenvalues for extended homogeneous systems. The new definition includes complex nonlinear eigenvalues as well as the special eigenvalues for standard homogeneous systems. Also, using the new definition, the role of complex nonlinear eigenvalues in the stability analysis of extended homogeneous systems is analyzed. It is shown for polynomial extended homogeneous systems with zero degree of homogeneity, the ZES of system is GAS iff all the nonlinear eigenvalues (complex or real) have negative real parts. We assume that the reader is familiar with the Lyapunov stability definitions and theorems [1].

This paper is organized as follows. In section 2, the preliminary definitions and results about homogeneous systems are given. In section 3, the main results are presented on the stability analysis of homogeneous systems. Eventually, an example is given to present the established results.

Manuscript received January 7, 2009. Both authors are with the Electrical Engineering Department, Aboureyhan Building, Amirkabir Univ. of Tech. (Tehran Polytechnic), Tehran, Iran.

Vahid Meigoli (corresponding author, phone: (+9821) 64543398; fax: (+9821) 66406469; e-mail: vahid_mey@yahoo.com, vmeigoli@aut.ac.ir).

S.K.Y. Nikravesh (phone: 98-21-64543316; e-mail: nikravsh@aut.ac.ir).

II. The Preliminary Definitions and Results

A. The Homogeneous Systems

Let $\mathbb{K}$ denote anyone of $\mathbb{R}$ or $\mathbb{C}$ and consider a function $v(\underline{x}):\mathbb{K}^n \to \mathbb{K}$ and a nonlinear dynamic system with ZES:

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \quad , \quad \underline{x} \in \mathbb{K}^n \tag{2}$$

We briefly recall the notion of homogeneity [12]: For a sequence of positive weights $r=(r_1,\dots,r_n)$, $r_i \ge 1$ and a variable $\alpha \in \mathbb{K}$ a dilation is defined as a family of linear maps $\Delta_\alpha^r(\underline{x}) \triangleq (\alpha^{r_1}x_1,\dots,\alpha^{r_n}x_n)$. The $v(\underline{x})$ function is homogeneous of degree p with respect to the dilation Δ_α^r if $v(\Delta_\alpha^r\underline{x}) = \alpha^p v(\underline{x})$. Briefly it is called Δ-homogeneous of degree p and is denoted by $v \in H_p$. The $\underline{f}(\underline{x})$ vector field is homogeneous of degree k with respect to the dilation Δ_α^r if $\underline{f}(\Delta_\alpha^r\underline{x}) = \alpha^k \Delta_\alpha^r \underline{f}(\underline{x})$. Briefly it is called Δ-homogeneous of degree k and is denoted by $\underline{f} \in \underline{n}_k$.

The special weights $r=(1,1,\dots,1)$ are referred as standard weights, thus $v(\underline{x})$ and $\underline{f}(\underline{x})$ are said to be standard homogeneous of degree p if $v(\alpha\underline{x}) = \alpha^p v(\underline{x})$ and $\underline{f}(\alpha\underline{x}) = \alpha^{p+1}\underline{f}(\underline{x})$ respectively.

For $p \ge 2\max r_i$ the Δ-homogeneous *p*-norm is defined by $\|\cdot\|_{\{\Delta,p\}} \triangleq (\sum_{i=1}^n |x_i|^{p/r_i})^{1/p}$. It is clear that $\|\cdot\|_{\Delta,p} \in H_1$, while this is not a true norm, because it doesn't satisfy the triangular inequality. The Δ-homogeneous Euler vector field $\upsilon(\underline{x}) = \sum_{j=1}^n r_j x_j \frac{\partial}{\partial x_j}$ ($\upsilon(x) = [r_1x_1,\dots,r_nx_n]^t$ simply) on $\mathbb{K}^n$ is the infinitesimal generator of the group $\Delta = \{\Delta_\alpha^r\}_{\alpha>0}$ acting on $\mathbb{K}^n$. A trajectory of $\dot{x} = \upsilon(x)$ started from each $\underline{x}_0 \in \mathbb{K}^n$ is geometrically an orbit $\{\Delta_\alpha^r(\underline{x}_0):\alpha>0\}$ of $\{\Delta_\alpha^r\}_{\alpha>0}$ group ($\alpha = e^t$), and it is called a Δ-*homogeneous ray*. A Δ-homogeneous ray is shown in Fig. 1. It is clear that $\upsilon \in \underline{n}_0$. The following lemma demonstrates the radial symmetry of Δ-homogeneous systems (shown in Fig. 1.).

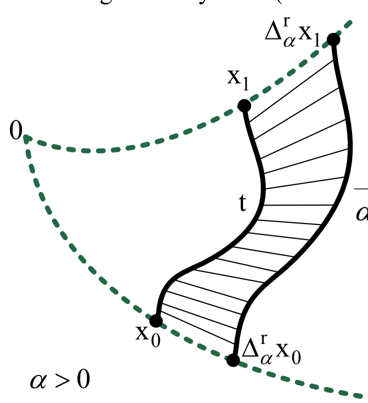

Fig. 1. The radial symmetry of Δ-homogeneous systems.

Lemma 1 [14]: Consider the Δ-homogeneous nonlinear system (2) of degree k and let $\phi(t,\underline{x}_0)$ denote a trajectory of this system starting from any initial state $\underline{x}_0 \in \mathbb{K}^n$ and parametrized by t. Then applying Δ_α^r for $\alpha>0$ to $\phi(t,\underline{x}_0)$ leads to a new trajectory of system, i.e.

$$\Delta_\alpha^r\phi(t,\underline{x}_0) = \phi(t/\alpha^k, \Delta_\alpha^r\underline{x}_0) \quad ,\forall\alpha>0, \forall\underline{x}_0 \in \mathbb{K}^n \tag{3}$$

■

A Δ-homogeneous ray is $\{\Delta_\alpha^r(\underline{x}_0):\alpha>0\}$ for some $\underline{x}_0 \ne 0$. Someone may consider an arbitrary $\alpha \in \mathbb{K}$ to define the set $\{\Delta_\alpha^r(\underline{x}_0):\alpha \in \mathbb{K}\}$ as a KΔ-*homogeneous ray* (K means real or complex depending on $\mathbb{K}$).

B. The Pervious Results on Homogeneous Eigenvalues

Samardzija introduced the standard homogeneous eigenvalues for standard homogeneous systems and used real eigenvalues in the stability analysis of ZES [5].

Definition 1 [5]: Considering a standard homogeneous system (2), a standard homogeneous eigenvalue $\lambda \in \mathbb{K}$ and a standard homogeneous eigenvector $\underline{v} \in \mathbb{K}^n$ are defined as a pair of nontrivial solutions for

$$\underline{f}(\underline{v}) = \lambda\underline{v} \tag{4}$$

Theorem 1 [5]: Consider a standard homogeneous system (1) on $\mathbb{R}^n$ and its real standard homogeneous eigenvalues corresponding to real eigenvectors defined by (4):

i. A necessary condition for GAS of ZES of (1) is that every real λ defined by (4) is negative.
ii. If a planar standard homogeneous system ($n=2$) has at least one real eigenvector $\underline{v} \in \mathbb{R}^2$ then the ZES of it is GAS iff every real λ defined by (4) is negative. ■

Nakamura *et al* [13] defined real homogeneous eigenvalues and eigenvectors for a Δ-homogeneous system (1) of degree k ($\underline{f} \in \underline{n}_k$) as pairs of nontrivial solutions of

$$\underline{f}(\underline{v}) = \lambda \|\underline{v}\|_{\{\Delta,2\}}^k \upsilon(\underline{v}) \quad , \quad \underline{v} \in \mathbb{R}^n \tag{5}$$

where $\|\underline{v}\|_{\{\Delta,2\}}$ denotes the Δ-homogeneous second norm and $\upsilon(\cdot)$ is the Δ-homogeneous Euler vector field. A simplified version of their theorem (Theorem 2 in [13]) is the following:

Theorem 2 [13]: Consider a Δ-homogeneous system (1) of degree k ($\underline{f} \in \underline{n}_k$) and let all the trajectories of (1) approach to real homogeneous eigenvectors defined by (5) and every real homogeneous eigenvalue λ defined by (5) is negative then the ZES of this system is GAS. ■

III. The New Results

A. The Δ-Homogeneous Eigenvalues

Nakamura *et al* [13] defined real homogeneous eigenvalues for Δ-homogeneous nonlinear systems using (5). We introduce complex eigenvalues for such systems and implement them in the stability analysis of ZES. However (5) is not useful for complex eigenvalues, because the homogeneous norm used in (5) destroys the usefulness of complex solutions of (5) for the stability analysis. Thus, a new idea will be used to define the homogeneous eigenvalues of Δ-homogeneous systems.

The radial symmetry of the Δ-homogeneous systems was shown in Fig. 1. It is somehow similar to the radial symmetry of the LTI systems. Therefore a Δ-homogeneous ray is a generalization of a homogeneous ray in the LTI systems to the Δ-homogeneous systems. On the other hand, a trajectory of a given LTI system starting from an eigenvector of it remains in the direction of that eigenvector. Thus the similarities between the Δ-homogeneous systems and the LTI systems lead us to the following definition:

Definition 2: A Δ-homogeneous eigenvector (or briefly a Δ-eigenvector) of a given Δ-homogeneous system (2) is defined as a vector $\underline{v} \in \mathbb{K}^n$ such that the trajectory $\phi(t,\underline{v})$ of the system lies in a KΔ-homogeneous ray. In other words, a continuous function $g(t):\mathbb{R} \to \mathbb{K}$ exists such that

$$\phi(t,\underline{v}) = \Delta^r_{g(t)}\underline{v} \tag{6}$$

The following useful lemma will lead to the definition of Δ-homogeneous eigenvalues.

Lemma 2: Using the Definition 2, a vector $\underline{v} \in \mathbb{K}^n$ is a Δ-eigenvector of a given Δ-homogeneous system (2) of degree k ($\underline{f} \in \underline{n}_k$) iff there exists a $\lambda \in \mathbb{K}$ such that

$$\underline{f}(\underline{v}) = \lambda \upsilon(\underline{v}) \quad , \quad \underline{v} \in \mathbb{K}^n \ , \lambda \in \mathbb{K} \tag{7}$$

Proof: Necessity: Let $\underline{v} \in \mathbb{K}^n$ be a **Δ-**eigenvector and (6) be satisfied. First replacing $t=0$ in (6) leads to $\Delta^r_1\underline{v} = \underline{v} = \phi(0,\underline{v}) = \Delta^r_{g(0)}\underline{v}$, thus $g(0)=1$ in (6). Then differentiation of (6) with respect to t, some manipulation and substitution $\dot{\phi}(t,\underline{v}) = \underline{f}[\phi(t,\underline{v})]$ yields:

$$\dot{\phi}(t,\underline{v}) = \tfrac{d}{dt}\Delta^r_{g(t)}\underline{v} = \tfrac{d}{dt}[g^{r_1}v_1,\ldots,g^{r_n}v_n] =$$
$$\left(\dot{g}(t)/g(t)\right)[r_1 g^{r_1}v_1,\ldots,r_n g^{r_n}v_n] = \left(\dot{g}(t)/g(t)\right)\upsilon[\Delta^r_{g(t)}\underline{v}] \quad \Rightarrow$$
$$\underline{f}[\Delta^r_{g(t)}\underline{v}] = \left(\dot{g}(t)/g(t)\right)\upsilon[\Delta^r_{g(t)}\underline{v}] \tag{8}$$

Then substituting $t=0$ and $g(0)=1$ in (8) yields $\underline{f}(\underline{v}) = \dot{g}(0)\upsilon(\underline{v})$. Defining $\dot{g}(0) \triangleq \lambda$ leads to (7).

Sufficiency: Let a pair $(\lambda,\underline{v})$ be a nontrivial solution for (7). Then we find a continuous function $g(t)$ satisfying (6) or equivalently (8). Substitute the homogeneity definitions for $\underline{f} \in \underline{n}_k$ and $\upsilon \in \underline{n}_0$ in (8) and then substitute (7) and use the linearity property of the dilation operator:

$$[g(t)]^k \Delta^r_{g(t)}\underline{f}(\underline{v}) = \left(\dot{g}(t)/g(t)\right)\Delta^r_{g(t)}\upsilon(\underline{v}) \quad \Rightarrow$$
$$[g(t)]^{k+1}\lambda\Delta^r_{g(t)}\upsilon(\underline{v}) = \dot{g}(t)\Delta^r_{g(t)}\upsilon(\underline{v}) \quad \Rightarrow$$

Hence it is necessary to have

$$\dot{g}(t) = [g(t)]^{k+1}\lambda \tag{9}$$

The differential equation (9) could be integrated using $g(0)=1$ to obtain the g(t) function:

$$\int_1^{g(t)} \frac{dg}{g^{k+1}} = \int_0^t \lambda dt' \Rightarrow \quad g(t) = \begin{cases} \dfrac{1}{\sqrt[k]{1-(k+1)\lambda t}} & , \quad k \neq 0 \\ e^{\lambda t} & , \quad k=0 \end{cases}$$

Finding the g(t) function completes the proof, and the trajectory started from a Δ-eigenvector $\underline{v}$ is given by

$$\phi(t,\underline{v}) = \Delta^r_{g(t)}\underline{v} = \begin{cases} \Delta^r_{\frac{1}{\sqrt[k]{1-(k+1)\lambda t}}}\underline{v} & , \quad k \neq 0 \\ \Delta_{e^{\lambda t}}\underline{v} & , \quad k=0 \end{cases} \tag{10}$$

■

The above lemma introduces a $\lambda \in \mathbb{K}$ corresponding to each Δ-eigenvector $\underline{v} \in \mathbb{K}^n$. Considering the literature of eigenvectors leads to the following definition:

Definition 3: A Δ-homogeneous eigenvector (or briefly Δ-eigenvector) and corresponding Δ-homogeneous eigenvalue (or briefly Δ-eigenvalue) for a given Δ-homogeneous system (2) are defined as a $(\lambda,\underline{v})$ pair of nontrivial solutions of (7). ■

Remark 1: The Definition 3 is reduced to the Definition 1 in the case of standard homogeneity. Comparing with the result of [13], the new definition is useful for both real and complex Δ-eigenvalues, while the previous definition was useful only for real case. Also in the real case the Δ-homogeneous eigenvectors in both definitions are equal, while the Δ-homogeneous eigenvalues are multiple with a positive factor $\|\underline{v}\|^k_{\{\Delta,2\}}$. ■

Note that each Δ-eigenvector has a unique Δ-eigenvalue because (7) could be used to obtain the following equation:

$$\lambda = [\upsilon(\underline{v})^T f(\underline{v})] / \|\upsilon(\underline{v})\|^2 \tag{11}$$

It is known that a multiple of each eigenvector in a LTI system is again an eigenvector. The following lemma state a similar property for the Δ-homogeneous systems.

Lemma 3: If $(\lambda,\underline{v})$ is a pair of Δ-eigenvalues and Δ-eigenvectors for a given Δ-homogeneous system (2) of degree k ($\underline{f} \in \underline{n}_k$) then $(\lambda_e,\underline{v}_e) = (\alpha^k\lambda, \Delta^r_\alpha\underline{v})$ is another such a pair for an arbitrary $\alpha \in \mathbb{K}$.

Proof: Let (7) be held for the $(\lambda,\underline{v})$ pair. Then some manipulations yields:

$$\underline{f}(\underline{v}_e) = \underline{f}(\Delta^r_\alpha\underline{v}) = \alpha^k\Delta^r_\alpha\underline{f}(\underline{v}) = \alpha^k\Delta^r_\alpha[\lambda\upsilon(\underline{v})] = \alpha^k\lambda\Delta^r_\alpha\upsilon(\underline{v})$$
$$= (\alpha^k\lambda)\upsilon(\Delta^r_\alpha\underline{v}) = \lambda_e\upsilon(\underline{v}_e)$$

■

The lemma means that the set of Δ-eigenvectors of a given Δ-homogeneous system is the union of several KΔ-homogeneous rays. We call such rays the KΔ-*eigenvector rays* (K means real or complex depending on $\mathbb{K}$).

B. The Stability Analysis Using Eigenvalues

Using (10) a solution started from a given Δ-eigenvector lies in a KΔ-homogeneous ray. Such a solution which is analytically given by (10) is called a *characteristic solution* of a Δ-homogeneous system. This terminology was first used in [5] for the case of standard homogeneous systems.

The asymptotical behavior of trajectories of a Δ-homogeneous system is related to the asymptotical behavior of its characteristic solutions, and the later one depends on the Δ-eigenvalues (Let $t \to +\infty$ into (10)). Table 1 summarizes the correspondence between the asymptotical behavior and the Δ-eigenvalues. This table shows that Δ-homogeneous systems with zero degree of homogeneity behave similar to LTI systems, while the behavior of other Δ-homogeneous systems is more complex.

TABLE 1. THE ASYMPTOTICAL BEHAVIOR OF CHARACTERISTIC SOLUTIONS

Asymptotical Behavior	k > 0	k = 0
$\phi(t,\underline{v}) \to \infty$ (Divergent)	$\lambda\in\mathbb{R}$, $\lambda > 0$	Re $\lambda > 0$
Marginally Stable (Oscillating)	$\lambda = 0$	Re $\lambda = 0$
$\phi(t,\underline{v}) \to 0$ (Convergent)	Otherwise	Re $\lambda < 0$

For example consider the case where $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and $k > 0$ in (10): The denominator of fraction in this equation never vanishes, thus letting $t \to +\infty$ implies that $\phi(t,\underline{v}) \to 0$; this is an unfamiliar behavior comparing with the LTI systems. Table 1 divides the λ-plane for the positive degree Δ-homogeneous systems into three regions depending on the asymptotical behavior of characteristic solutions. This is shown in Fig. 2.

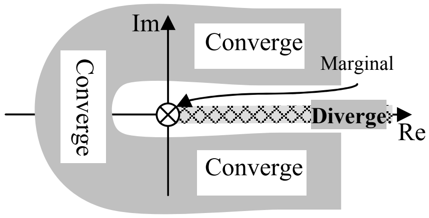

Fig. 2. The regions of λ-plane corresponding to asymptotical behavior of characteristic solutions for positive degree Δ-homogeneous systems (k>0).

Using the information of Table 1, new theorems on the stability analysis of Δ-homogeneous systems could be stated. The first theorem is a generalization of Theorem 1 to the case of Δ-homogeneous systems.

Theorem 3: Consider a Δ-homogeneous system (1) on $\mathbb{R}^n$ and its real **Δ-**eigenvalues corresponding to real **Δ-**eigenvectors defined by (7).

i. A necessary condition for GAS of ZES of (1) is that every real λ defined by (7) is negative.

ii. If a planar Δ-homogeneous system ($n = 2$) has at least one real eigenvector $\underline{v} \in \mathbb{R}^2$ then the ZES of it is GAS iff every real λ defined by (7) is negative. ■

Proof: The part (i) of this theorem is obvious using the Table 1. Using the result of part (i), in part (ii) it is only required to prove the sufficient condition. If the system has at least one real eigenvector $\underline{v} \in \mathbb{R}^2$ then its characteristic solution lies in a real Δ-homogeneous ray. On the other hand the trajectories of Δ-homogeneous system (1) have radial symmetry using Lemma 1. Since (1) is assumed to be a planar system then all the trajectories approach to real Δ-eigenvector rays of system. Then comparing (5) and (7) and applying Theorem 2 for real Δ-eigenvalues completes the proof. ■

It is known from Theorem 3 that the real Δ-eigenvalues provide only necessary conditions for GAS of ZES of Δ-homogeneous systems of higher dimension n>2. Comparing with the LTI systems direct us to consider the role of complex Δ-eigenvalues, however the new definition of homogeneous eigenvalues permits complex valued Δ-eigenvalues.

Let the nonlinear system (1) on $\mathbb{R}^n$ be Δ-homogeneous. Some necessary and sufficient condition for GAS of ZES of (1) using complex Δ-eigenvalues is under consideration. However a generalization of (1) onto $\mathbb{C}^n$ is required for the stability analysis of ZES using complex Δ-eigenvalues. The polynomial Δ-homogeneous systems have such generalization, thus we will consider only polynomial systems.

Against the earlier hopes, there are some drawbacks for the use of complex Δ-eigenvalues for the stability analysis of a Δ-homogeneous system (1), as follows:

1) Using Table 1 in the case of positive degree of homogeneity ($k > 0$) implies that, the role of Δ-eigenvalues in the stability analysis is very complex.

2) Because of nonlinearity of (1) the effect of complex Δ-eigenvalues on behavior of real trajectories of (1) is unknown.

3) The main shortcoming is introduced through the Lemma 3 for Δ-homogeneous system (1) with positive degree of homogeneity ($k > 0$): Let the Δ-homogeneous system (1) has at least one nonzero Δ-eigenvalue $\lambda \neq 0$, then using an arbitrary nonzero $\alpha \in \mathbb{C}$ in Lemma 3 yields that every $\lambda_e \in \mathbb{C} \setminus \{0\}$ is an available Δ-eigenvalue for the Δ-homogeneous system (1). Thus this system has Δ-eigenvalues all over the $\mathbb{C} \setminus \{0\}$. Using Table 1 yields that this system has both types of stable and unstable Δ-eigenvalues.

C. *The Zero Degree Δ-Homogeneous Systems*

Let (1) be a polynomial Δ-homogeneous system with zero degree of homogeneity ($k = 0$).

Fortunately, the previously mentioned shortcomings for complex Δ-eigenvalues do not exist in the case of zero degree Δ-homogeneous systems. Thus good results will be obtained for these systems here:

1) Using $k = 0$ in Lemma 3 implies that if $(\lambda, \underline{v})$ is a pair of Δ-eigenvalues and Δ-eigenvectors for (1) then $(\lambda, \Delta_\alpha^r \underline{v})$ is another such pair for every $\alpha \in \mathbb{K}$. This result is similar to LTI systems.

2) The stability of characteristic solution (10) depends on $\mathrm{Re}\{\lambda\}$ similarly to the LTI systems.

The main theorem of this paper is the following:

Theorem 4: Let (1) be a polynomial Δ-homogeneous system with zero degree of homogeneity ($k = 0$). The ZES of (1) is GAS iff all Δ-eigenvalues of (1) have negative real parts. ■

We need some preliminary assumptions and lemmas to prove this theorem.

Hypothesis 1: Let there are some natural numbers $r_1 < r_2 < \cdots < r_N$ such that the dilation Δ_α^r could be stated using the following block diagonal form:

$$\Delta_\alpha^r \underline{x} = \begin{bmatrix} \alpha^{r_1} I_{n_1} & & \\ & \ddots & \\ & & \alpha^{r_N} I_{n_N} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_N \end{bmatrix} = \begin{bmatrix} \alpha^{r_1} \underline{x}_1 \\ \vdots \\ \alpha^{r_N} \underline{x}_N \end{bmatrix} \tag{12}$$

where $\underline{x} = [\underline{x}_1^T \ \underline{x}_2^T \cdots \underline{x}_N^T]^T$ is a given partition using:

$$\begin{cases} \underline{x}_i \in \mathbb{R}^{n_i} \quad i = 1, 2, \ldots, N \\ \sum_{i=1}^{N} n_i = n \end{cases} \tag{13}$$

■

Hypothesis 1 is not a limiting assumption, because in a given Δ-homogeneous system (1), someone can reorder the state variables to reach ascending r_i numbers in the dilation operator. Then the equal r_i numbers could be grouped as following:

$$\overbrace{\underbrace{r_1 \ r_1 \cdots r_1}_{n_1} \ \underbrace{r_2 \ r_2 \cdots r_2}_{n_2} \cdots \underbrace{r_N \ r_N \cdots r_N}_{n_N}}^{n}$$

Lemma 4: Let (1) be a polynomial Δ-homogeneous system with zero degree of homogeneity (k=0) with respect to the dilation (12). Then (1) has the following lower triangular canonical form:

$$\Sigma : \begin{cases} \dot{\underline{x}}_1 = A_1 \underline{x}_1 \\ \dot{\underline{x}}_2 = P_1(\underline{x}_1) + A_2 \underline{x}_2 \\ \dot{\underline{x}}_3 = P_2(\underline{x}_1, \underline{x}_2) + A_3 \underline{x}_3 \\ \vdots \\ \dot{\underline{x}}_N = P_{N-1}(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_{N-1}) + A_N \underline{x}_N \end{cases} \tag{14}$$

where the $A_i\underline{x}_i$ terms denote the linear parts of system - A_i is a $n_i \times n_i$ matrix for $i = 1,\cdots,N$ - and the $P_i(\cdot)$ polynomials denote the nonlinear parts of system.

Proof: Since $\underline{f} \in \underline{n}_0$ then $\underline{f}(\Delta_\alpha^r \underline{x}) = \Delta_\alpha^r \underline{f}(\underline{x})$. This relationship using (12) and the $\underline{f} = [f_1^T\ f_2^T \cdots f_N^T]^T$ partition is converted to the union of the following relationships:

$$f_i(\Delta_\alpha^r \underline{x}) = \alpha^{r_i} f_i(\underline{x}) \quad , i = 1,2,\ldots,N \tag{15}$$

Using (15) for each i implies that the $f_i(\underline{x})$ is a polynomial vector function independent of $\underline{x}_{i+1},\ldots,\underline{x}_N$, because these variables generate $\alpha^{r_{i+1}},\ldots,\alpha^{r_N}$ when computing $\Delta_\alpha^r \underline{x}$ and their power are greater than α^{r_i}, thus the existence of $\underline{x}_{i+1},\ldots,\underline{x}_N$ in $f_i(\underline{x})$ contradicts with (15). The system has a lower triangular form, because $f_i = f_i(\underline{x}_1,\underline{x}_2,\cdots,\underline{x}_i)$.

Note that the dependence of $f_i(\underline{x})$ on $\underline{x}_i$ could not be nonlinear, because of α^{r_i} factor in (15). Also by the same reason, the dependence of $f_i(\underline{x})$ on $(\underline{x}_1,\underline{x}_2,\ldots,\underline{x}_{i-1})$ must have a nonlinear form. Thus we have the following:

$$\dot{\underline{x}}_i = f_i(\underline{x}) = P_{i-1}(\underline{x}_1,\underline{x}_2,\ldots,\underline{x}_{i-1}) + A_i\underline{x}_i \tag{16}$$

■

Theorem 5: Consider the polynomial Δ-homogeneous system Σ in (14) with zero degree of homogeneity. The ZES of (14) is GAS iff its linearization given below is GAS.

$$\dot{\underline{x}} = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_N \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_N \end{bmatrix} \tag{17}$$

Proof: Sufficiency: Let (17) be GAS, thus the Σ system could be considered as a cascade of several GAS linear subsystems with nonlinear feed forward inputs. The $\dot{\underline{x}}_1 = A_1\underline{x}_1$ subsystem of Σ is GAS and a $\phi_1 \in K$ (a function of class K) exists such that

$$\begin{cases} \| P_1(\underline{x}_1(t)) \| \le \phi_1(\| \underline{x}_1(0) \|) & \forall t \ge 0 \\ \| P_1(\underline{x}_1(t)) \| \to 0 \end{cases} \tag{18}$$

Thus the second subsystem of Σ, i.e. $\dot{\underline{x}}_2 = A_2\underline{x}_2 + P_1(\underline{x}_1)$ is a GAS linear time invariant system $\dot{\underline{x}}_2 = A_2\underline{x}_2$ with a bounded and vanishing input $P_1(\underline{x}_1)$. Thus it is also GAS and we have:

$$\begin{cases} \| \underline{x}_2(t) \| \le k_1 \| \underline{x}_2(0) \| + k_2\phi_1(\| \underline{x}_1(0) \|) \\ \underline{x}_1(t) \to 0 \;\Rightarrow\; P_1(\underline{x}_1(t)) \to 0 \;\Rightarrow\; \underline{x}_2(t) \to 0 \end{cases} \tag{19}$$

The GAS of the other subsystems of Σ could be proved using mathematical induction.

Necessity: Let the ZES of Σ is GAS. All A_i matrices must be Hurwitz. The A_1 matrix is Hurwitz, using the lower triangular form of Σ. To prove the Hurwitz-ness of A_i for $i > 1$, suppose the $(\underline{x}_1 = \underline{0}, \underline{x}_2 = \underline{0},\ldots,\underline{x}_{i-1} = \underline{0})$ initial conditions. Since the ZES of Σ is GAS, the subsystem (16) using $P_{i-1}(\underline{x}_1,\underline{x}_2,\ldots,\underline{x}_{i-1}) = \underline{0}$ is also GAS, thus A_i is Hurwitz. ■

Proof of Theorem 4:

It is sufficient to prove the Theorem 4 for Σ in (14), i.e. the canonical form Δ-homogeneous system of zero degree. The s_i eigenvalues of LTI system (17) are roots of the following equations:

$$\begin{cases} A_i\underline{v}_i = s_i\underline{v}_i \\ \underline{v}_i \ne \underline{0} \end{cases} , \quad i = 1,2,\ldots,N \tag{20}$$

On the other hand, let $(\lambda,\underline{x})$ be a pair of Δ-eigenvalues and Δ-eigenvectors for Σ, thus:

$$f(\underline{x}) = \lambda\nu(\underline{x}) = \lambda \begin{bmatrix} r_1 I_{n_1} & & \\ & \ddots & \\ & & r_N I_{n_N} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_N \end{bmatrix} \tag{21}$$

This equation could be written using (14) as follows:

$$\begin{cases} \lambda r_1\underline{x}_1 = A_1\underline{x}_1 \\ \lambda r_2\underline{x}_2 = P_1(\underline{x}_1) + A_2\underline{x}_2 \\ \lambda r_3\underline{x}_3 = P_2(\underline{x}_1,\underline{x}_2) + A_3\underline{x}_3 \\ \vdots \\ \lambda r_N\underline{x}_N = P_{N-1}(\underline{x}_1,\underline{x}_2,\ldots,\underline{x}_{N-1}) + A_N\underline{x}_N \end{cases} \tag{22}$$

The solutions of (22) could be obtained using the solutions of (20) as follows. The Δ-eigenvector $\underline{x} \ne 0$ thus at least for one i, $\underline{x}_i \ne 0$.

If $\underline{x}_1 \ne 0$ then comparing $\lambda r_1\underline{x}_1 = A_1\underline{x}_1$ and $s_1\underline{v}_1 = A_1\underline{v}_1$ in (22) and (20) yields:

$$\begin{cases} \underline{x}_1 = \underline{v}_1 \ne 0 \\ \lambda = s_1 / r_1 \end{cases} \tag{23}$$

Thus when $\underline{x}_1 \ne 0$, the Δ-eigenvalue λ is equal to eigenvalue of A_1 divided by r_1. The remained $\underline{x}_2,\ldots,\underline{x}_N$ components are determined using the last equations of (22).

Generally for determining a Δ-eigenvector $\underline{x} \ne 0$, let i be the least number such that $\underline{x}_i \ne 0$, thus $\underline{x}_1 = \cdots = \underline{x}_{i-1} = \underline{0}$. Substituting this in (22) leads to $\lambda r_i\underline{x}_i = A_i\underline{x}_i$ which is compared to $s_i\underline{v}_i = A_i\underline{v}_i$ and yields:

$$\begin{cases} \underline{x}_i = \underline{v}_i \ne 0 \\ \lambda = s_i / r_i \end{cases} \tag{24}$$

Thus the Δ-eigenvalue λ is equal to an eigenvalue of A_i divided by r_i. The other components of $\underline{x}$ are computed iteratively using

$$\begin{cases} \text{for} \quad k = i+1 \text{ to } N \\ (\lambda r_k I_{n_k} - A_k)\underline{x}_k = P_{k-1}(0,\ldots,0,\underline{x}_i,\underline{x}_{i+1},\ldots,\underline{x}_{k-1}) \\ \text{next } k \end{cases} \tag{25}$$

Hence all the Δ-eigenvalues for Σ are equal to eigenvalues of (17) multiplied with positive factors. ■

IV. An Example

Example 1: The nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & x_1^2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1^3 + a_{12}x_2 \\ a_{21}x_1^5 + a_{22}x_1^2x_2 \end{bmatrix} \tag{26}$$

is Δ-homogeneous of degree two with respect to weights $r = (r_1, r_2) = (1,3)$, i.e. $\underline{f} \in \underline{n}_2$, because

$$\underline{f}(\Delta_\lambda^r \underline{x}) = \begin{bmatrix} a_{11}(\lambda x_1)^3 + a_{12}(\lambda^3 x_2) \\ a_{21}(\lambda x_1)^5 + a_{22}(\lambda x_1)^2(\lambda^3 x_2) \end{bmatrix} =$$

$$= \lambda^2 \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^3 \end{bmatrix} \begin{bmatrix} a_{11}x_1^3 + a_{12}x_2 \\ a_{21}x_1^5 + a_{22}x_1^2 x_2 \end{bmatrix} = \lambda^2 \Delta_\lambda^r \underline{f}(\underline{x})$$

Let us find the Δ-eigenvalues and Δ-eigenvectors of (26) using the parameter values $A = \begin{bmatrix} -5 & -1 \\ 2 & -4 \end{bmatrix}$. Replacing (26) into (7) yields:

$$\begin{bmatrix} -5v_1^3 - v_2 \\ 2v_1^5 - 4v_1^2 v_2 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow$$

$$\lambda = \frac{-5v_1^3 - v_2}{v_1} = \frac{2v_1^5 - 4v_1^2 v_2}{3v_2} \tag{27}$$

First the Δ-eigenvectors are found eliminating the λ variable and using the $z \triangleq v_1^3 / v_2$ change of variables

$$2z^2 + 11z + 3 = 0 \tag{28}$$

If the equation (28) has real solutions for z, then the real Δ-eigenvector rays are determined by the $zv_2 = v_1^3$ curves. The solution of (28) yields real values $z \in \{-0.29, -5.21\}$. The Fig. 3. shows the real Δ-eigenvector rays and solution curves for (26). It is clear that any solution curves of this nonlinear system approach to the Δ-eigenvector rays. Let us use $v_1 = \pm 1$ and $z = v_1^3 / v_2 \in \{-0.29, -5.21\}$ to obtain:

$$(v_1, v_2) \in \{\pm(1, -0.1919), \pm(1, -3.4748)\} \tag{29}$$

Substituting (29) in (27) yields $\lambda \in \{-4.8, -1.5252\}$, thus all Δ-eigenvalues are negative. Note that if someone uses another values for v_1 then the sign of λ is not affected, because replacing $v_2 = v_1^3 / z$ in (27) yields $\lambda = 0.66 v_1^2 (z - 2) < 0$. This is also implied by Lemma 3.

Using the Theorem 3 implies the GAS of ZES of system.

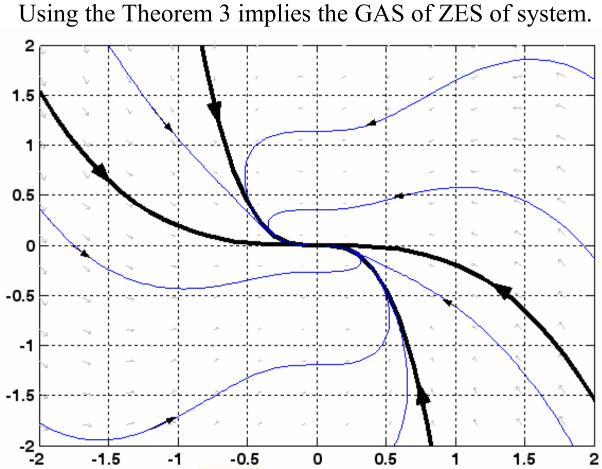

Fig. 3. the real Δ-eigenvector rays and trajectories of (26)

■

V. CONCLUSION

In this paper, the Δ-homogeneous eigenvalues and eigenvectors were introduced for nonlinear Δ-homogeneous systems. The new definition permits the complex valued Δ-homogeneous eigenvalues for such systems. In the case of polynomial Δ-homogeneous systems with zero degree of homogeneity it was shown that ZES of system is GAS iff all the Δ-homogeneous eigenvalues have negative real parts. Finally, an example was given to present the approach.

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