Continuous-time based Multiple Model System Control with Reliability Constraint

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Abstract—**In this paper, a new active control framework is developed for non-LTI continuous-time system, which is closer to the real world. Like other control system developments, it consists of two major parts: observer design and controller design. For the observer, the dynamic system is modelled as a jump Markov linear system whose parameters evolve along with the running mode of plant, then the continuous-time based interacting multiple model(CT-IMM) algorithm is proposed to estimate the full state more precisely. For the controller, a dynamic reliability constraint is added and the optimal control problem is reformulated to guarantee the reliability of system. Illustrative examples showed the accuracy of CT-IMM and the validity of controller with reliability constraint.**

Keywords: observer design, controller design, multiple model, reliability constraint, continuous-time

1 Introduction

In linear control theory, separation principle[1] facilitates the control system design by designing the controller and observer respectively in case that not all states are accessible. In the past decades, the researchers have designed plenty of full-state observer or reduced order observer[2]. The applications of those designs must all satisfy the linear-Gaussian assumption. However, most cases in real world are non-linear and non-Gaussian. As stated in [3], state observer design for non-LTI system is still a hot topic with great challenge.

Recently, multiple model approach as an accurate estimation method for hybrid system attracted much attention of observer design and various applications: Brehm[4] investigated and compared profoundly multiple model adaptive estimator(MMAE) based control and multiple model adaptive control(MMAC). Li[5] reviewed detailedly the multiple model approach in passive tracking domain. Luo et.al.[6] employed multiple model approach to track the hidden damage in life prediction of automotive suspension system. Another important advantages of multiple model method are: the parameters can also be roughly estimated by a simple weight-summation. However, as far as we know, quite few researches concerned the continuous-time case of multiple model approach. Dunn[7] and Aguiar et.al.[8] firstly proposed the continuous-time MMAE and Aguiar[9] also discussed a set of typical questions and further research of MMAE. The models in their methods, nevertheless, work individually and this mechanism is too simple to realize the jumping truth of system running modes. Civera[10] designed a pseudo continuous-time based multiple model observer for simultaneous localisation and mapping, however it still updates the model probability in discrete way. Furthermore, LQG optimal controller runs on a well-known linear dynamic problem, which limits its applications. And the simple fixed feedback gain leaves little or no room for system tolerances.

In this paper, there are mainly two contributions. First, a continuous-time based interacting multiple model algorithm is proposed. Different from generic IMM algorithm, we used differential equation, which implies the continuous probability evolution, instead of recursion equation to update the mode probability the system dynamics. In addition, the core part of interacting multiple model — model mixer is redesigned to re-initialize the system state. Second, a plant estimated by observer is used to reformulate the LQG optimal control problem. Further, to guarantee the reliability of system a dynamic reliability constraint is also added to the reformulation in form of implicit probability function.

The rest parts of this paper will be organized as follows. In section 2, continuous-time based multiple model approach is analyzed by means of Bayesian view and then continuous-time based interacting multiple model algorithm is proposed. In section 3, the feedback controller is designed with reliability constraint. Some experimental results are compared in section 4. Finally, the conclusions are drawn and some future works are proposed.

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2 Continuous-time based IMM

In this section, the continuous-time based multiple model in view of Bayesian is introduced and then CT-IMM is designed and analyzed. For simplicity, some of the subscripts are explained as following:

$$\begin{aligned}(\cdot)_t &: \ \textit{variable at time } t,\\ (\cdot)_{t+/t-} &: \ \textit{variable at time } t+\Delta t \ / \ t-\Delta t,\\ (\cdot)^{(i)} &: \ \textit{variable of the ith mode.}\end{aligned}$$

2.1 Bayesian view of CT-MM

We assume that the non-LTI system runs in N LTI modes $m^{(i)}$ whose corresponding models are described as $M(\theta^{(i)}) = \{A(\theta^{(i)}), B(\theta^{(i)}), C(\theta^{(i)}), D(\theta^{(i)}), G(\theta^{(i)})\}$, $i = 1, .., N$, where θ is a parameter vector as a variable of model generating function. For simplicity we denote $A(\theta^{(i)})$ as $A^{(i)}$, as well as $B^{(i)}$, $C^{(i)}$, $D^{(i)}$ and $G^{(i)}$. Then each model satisfies,

$$\dot{x}_t^{(i)} = A^{(i)} x_t^{(j)} + B^{(i)} u_t + G^{(i)} w_t \tag{1a}$$

$$z_t = C^{(i)} x_t^{(i)} + D^{(i)} u_t + v_t \tag{1b}$$

where w_t and v_t are white Gaussian noise with covariance Q_w and Q_v respectively. The system dynamics change the running mode among the set of modes, obeying the property of Markov process,

$$p(M_{t+\Delta t}^{(j)} | M_t^{(i)}) = \begin{cases} T_{ij} \cdot \Delta t + o(\Delta t), & i \neq j \\ 1 + T_{ii} \cdot \Delta t + o(\Delta t), & i = j \end{cases} \tag{2}$$

where matrix T is known as transition rate matrix and it is conservative, $T_{ii} = -\sum_{j=1, j\neq i} T_{ij}$.

The task of observer under consideration is to estimate the state conditioned on the history measurements and on the history control force, or equivalently to approximate the distribution of state space conditioned on the history measurements, i.e. $p(x_t|Z_t)$, where Z_t is the set of history measurements until time t. Notice that this distribution has no more relation with the control, which is a determined part. Further, this expression can be reformulated by full probability distribution,

$$p(x_t|Z_t) \ = \ \int_S p(x_t|S_t, Z_t) \cdot p(S_t|Z_t) dS_t \tag{3}$$

where S_t is mode sequence. In fact the evolution time can be divided into little time pieces such that the interval Δt is small enough, therefore the mode sequence up to time t is denoted by $S_t = M_0, M_{\Delta t}, ..., M_{t-\Delta t}, M_t$.

The problem we encountered is that S_t increases exponentially with time, thus the posterior distribution $p(x_t|Z_t)$ can't be exactly computed. In the past decades, the researchers found some approximation techniques.

If we only take the last time step for scope, $S_t = M_t$, there is no jumping between the modes. We have,

$$p(x_t|Z_t) \ \simeq \ \sum_{M_t} p(x_t|M_t, Z_t) \cdot \mu_t^{(i)}. \tag{4}$$

where $\mu_t^{(i)} = p(M_t^{(i)}|Z_t)$ is the probability of the mode token by system dynamic. And the conditional posterior $p(x_t|M_t, Z_t)$ is estimated respectively by previous estimation, and the overall estimation is obtained via criteria of minimum mean-square error(MMSE), i.e.

$$\hat{x}_t \ = \textstyle\sum_i x_t^{(i)} \cdot \mu_t^{(i)}. \tag{5}$$

CT-MM adaptive estimator[8] fall into this category. Like Eqn.(1), its fusion process is simple and doesn't consider the interaction between models. Notice that, if Eqn.(5) is replaced by $\hat{x}_t = \{x_t^{(j)} | j = arg\ max_i\ \mu_t^{(i)}\}$, the system chooses the most likely mode to run, also called *switching*.

If the last two steps are considered, the modes of dynamics form a first-order markov chain, and $S_t = (M_{t-\Delta t}, M_t)$. Rewrite eqn.(3),

$$\begin{aligned} p(x_t|Z_t) \ \simeq \ & \sum_i \sum_j p(x_t|M_t^{(i)}, M_{t-}^{(j)}, Z_t) \\ & \cdot p(M_{t-}^{(j)}|M_t^{(i)}, Z_t) \cdot p(M_t^{(i)}|Z_t), \end{aligned} \tag{6}$$

where $p(M_{t-}^{(j)}|M_t^{(i)}, Z_t)$ is called merging probability. Eqn.(6) contains two levels of summation, implying the requirement of much more filters.

2.2 Continuous-time based model mixer

To overcome the flaw of computation, interacting multiple model(IMM) was proposed[5]. From the view of Bayesian, a model-conditioned filtering breaks into predition $p(x_t|Z_{t-})$ and updating $p(z_t|x_t)$ as follow,

$$p(x_t|M_t^{(i)}, Z_t) \propto p(x_t|M_t^{(i)}, Z_{t-}) \cdot p(z_t|x_t) \tag{7}$$

where the conditional posterior can be replaced by a re-initialization process,

$$p(x_t|M_t^{(i)}, Z_{t-}) \simeq \sum_j p(x_t|M_t^{(i)}, M_{t-}^{(j)}, \hat{x}_{t-}^{(i)}) \cdot \mu_{t-}^{(i|j)} \tag{8}$$

where $\mu_{t-}^{(i|j)}$ is named mixing probability, indicating the reinitializing contribution of the jth model for the ith model at time $t - \Delta t$, see in Fig.1 model mixer achieves the re-initialization before new time filtering, and

$$\mu_t^{(i|j)} \propto T_{ji} \cdot \mu_t^{(j)}, \quad \sum_i \mu_t^{(i|j)} = 1. \tag{9}$$

See appendix for details of Eqn.(8) and Eqn.(9). Finally, we reconstruct the differential dynamic as following,

$$\dot{x}_t^{(i)} = A^{(i)} \cdot \sum_j \mu_t^{(i|j)} x_t^{(j)} + B^{(i)} \cdot u_t + G^{(i)} w_t. \tag{10}$$

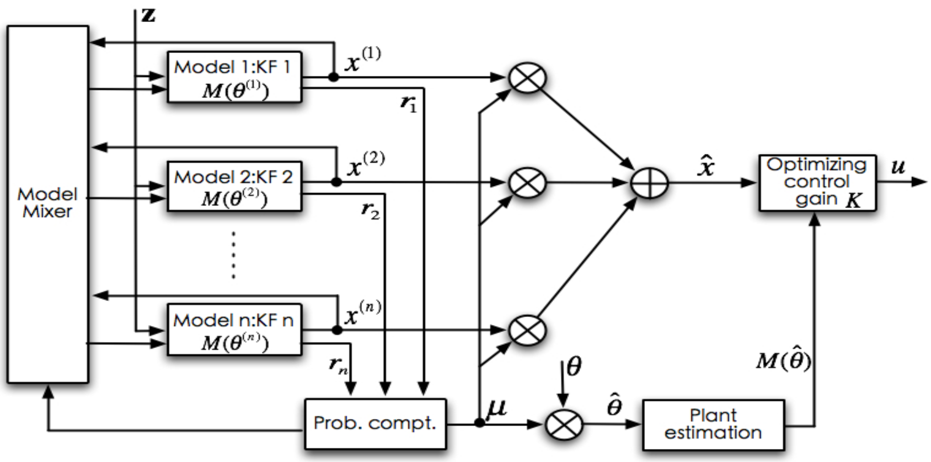


Figure 1: CT-IMM based observer and controller

Finally, the overall estimation of state is also implemented by Eqn.(5). Furthermore, the time-evolution of model probability is yielded by

$$\dot{\mu}_t^{(i)} = \mu_t^{(i)} \cdot (L_t^{(i)} - \sum_j L_t^{(j)} \cdot \mu_t^{(j)}). \tag{11}$$

$\sum_i \mu_t^{(i)} = 1$ can be proved in the same way as Aguiar[9] did. Therein $L_t^{(i)}$ represents the likelihood of measurement when system running in ith mode, i.e. $p(z_t|M_t^{(i)}, Z_{t-})$. In case of Kalman filter the innovation is assumed to obey Gaussian distribution, then the likelihood usually has the form as follow,

$$L_t^{(i)} \quad = \quad \kappa \cdot \exp\Big\{\frac{-\tilde{z}_t'(i) \cdot R^{-1}(i) \cdot \tilde{z}_t(i)}{2}\Big\} \tag{12}$$

where we use a fixed κ to control the evolution speed. $\tilde{z}_t$ is the innovation of measurement, i.e. $\tilde{z}_t(i) = z_t - C^{(i)} x_t^{(i)} - D^{(i)} \cdot u_t$. And covariance of innovation $R(i)$ is equal in case of continuous-time, representing also the measurement noise covariance of model i.

Besides, CT-IMM approximates mathematical model of the plant as $\hat{M} = \{\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{G}\}$, $\hat{A} = A(\hat{\theta})$, $\hat{\theta} = \sum_i \mu^{(i)} \theta^{(i)}$. $\hat{B}$, $\hat{C}$, $\hat{D}$ and $\hat{G}$ have the same form as $\hat{A}$.

3 Reliability constrained controller

Nowadays, most manufacturers attempt to produce their products highly reliable under random loads. The reliability constraints infiltrate each kind of design optimization for products, as well as their dynamic systems.

3.1 Reliability of dynamic system

The most typical failure of dynamic system is that the responses of random load exceed the allowable values. For a one-freedom system, if the state y is in excess of designed two-side threshold $\alpha = (\alpha_1, \alpha_2)$, i.e. $\alpha_1 > y > \alpha_2$, the system will probably break.

As deeply researched, the survival probability of time is

$$r(t) = \exp\{-v(\alpha_1, \alpha_2) \cdot t\} \tag{13}$$

where $v(\alpha_1, \alpha_2)$ is called failing rate or hazard function under certain threshold. According to Rice theory[11], it represents the crossing rate and expressed by,

$$v(\alpha) = \int_0^{+\infty} \dot{y} \cdot p_{y\dot{y}}(\alpha, \dot{y}, t) d\dot{y}. \tag{14}$$

For zero-mean Gaussian process, the joint probability $p_{y\dot{y}}$ is simply the product of two Gaussian distributions due to independency between y and $\dot{y}$, thus

$$p_{y\dot{y}}(y, \dot{y}, t) = \frac{1}{2\pi\sigma_y(t)\sigma_{\dot{y}}(t)} \exp\Big\{-\frac{y^2}{2\sigma_y^2(t)} - \frac{\dot{y}^2}{2\sigma_{\dot{y}}^2(t)}\Big\}, \tag{15}$$

where $\sigma_y(t)$ and $\sigma_{\dot{y}}(t)$ are standard deviation of $y(t)$ and $\dot{y}(t)$ respectively. They can be approximated by signal spectral analysis. For multi-freedom system, mode analysis[12]can be used to decouple the dependency, so that system reliability is achieved by product of individual reliability of each dimension.

3.2 Reliability constrained feedback control

Assuming that the plant runs as piecewise linear Gaussian, the optimal LQG control with reliability constraint is reformulated,

$$min \quad J(K_{tf}) = E\{\int_0^{t_f} x_t' Q x_t + u_t' W u_t \; dt\} \tag{16a}$$

$$s.t. \quad \dot{x}_t = \hat{A} x_t + \hat{B} u_t + \hat{G} w_t \tag{16b}$$

$$u_t = -K_{tf} \cdot x_t \tag{16c}$$

$$\beta \leq P_r(K_{tf}, t_f) \tag{16d}$$

where E represents the expectation, and K_{tf}, called feedback gain, is our design vector. $\hat{A}$, $\hat{B}$ and $\hat{G}$ are taken from mathematical model estimated from CT-IMM. The problem-oriented coefficient matrices Q and W are semi-positive defined and positive defined respectively. Note that system reliability $P_r(K_{tf}, t_f)$ is also a function of feedback gain and is restrained as greater than the threshold β which is usually close to one.

Furthermore, substituting Eqn.(16c) into Eqn.(16b), we obtain a new dynamic system $\dot{x}_t = (\hat{A} - \hat{B}K_{tf})x_t + \hat{G}w_t$. As stated in subsection 3.1, the dynamic reliability is achieved by Rice theory and by decoupling of the new dynamic system. The optimization of Eqn.(16a) may be accomplished in many ways, however, $P_r(K_{tf}, t_f)$ is so implicit that it is impossible to minimize $J(K_{tf})$ by traditional gradient-based optimization techniques. In our following experiments, genetic algorithm(GA) is selected. More investigation for efficient optimization method are required for future work.

4 Simulation examples

For the observer, our algorithm(CT-IMM) gets the better results compared to CT-MM proposed by Aguiar[9]; for the active feedback controller, optimization with reliability constraint trades a little bit larger cost for more system reliability.

4.1 CT-IMM Observer

The model in [9] is reused for simplicity, then we rewrite the state space model as follows,

$$\begin{aligned} \dot{x_t} &= \begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 1 \\ \omega^2 & -\omega^2 & -0.2\omega^2 \end{bmatrix} x_t + \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} w_t \\ y_t &= [0 \;\; 1 \;\; 0]x_t + v_t \end{aligned}$$

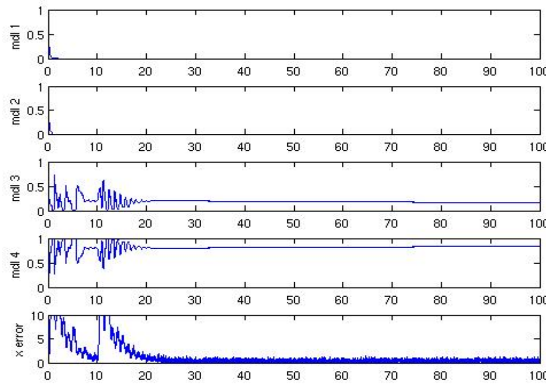
where $u_t = 10 \times sqr(T_u)$, and process noise w_t and measurement noise v_t are set determinedly as $w_t = e^{-0.01t}sqr(T_w)$ and $v_t = e^{-0.01t}sin(50t)$ respectively. Therein $sqr(T_s)$ is a square function from $t = 0$ with amplitude 1.0 and period T_s.

For only one parameter $\theta = \omega$, the set of four models is $M = \{M(0.7), M(1.0), M(2.0), M(4.0)\}$. Initial probability is still even $p_i(0) = 1/4$. In addition, our CT-IMM algorithm takes the transition rate matrix as,

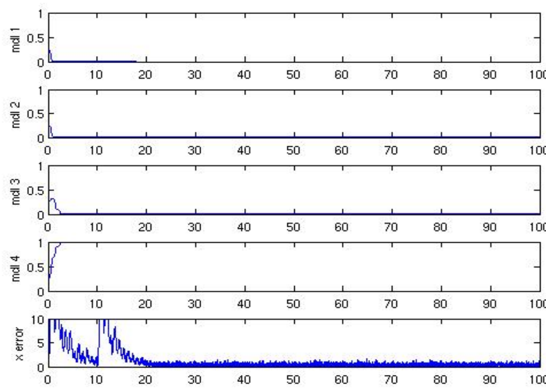
$$T_{ji} = \begin{cases} c_T & \text{if } i = j \\ (1 - c_T)/3 & \text{if } i \neq j \end{cases} . \tag{17}$$

Usually, $c_T = 0.95$ is used in our experiments. Obviously, $c_T = 1$ means CT-IMM works in way of switching mode.

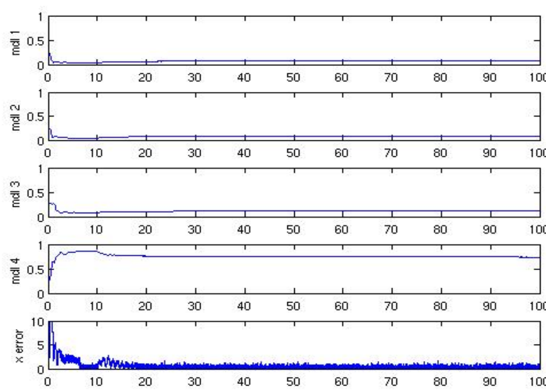
(1). CT-IMM vs. CT-MM
For $\omega = 1.3$ and $\omega = 3.3$ respectively, Fig.2(a) and Fig.3(a) demonstrate the results of CT-MM, and those of CT-IMM are shown in Fig.2(c) and Fig.3(c). Clearly, the mode probabilities of CT-IMM converge faster than those of CT-MM, in addition they have much less vibrations. The estimation errors are also much less. About parameter estimation, CT-MM and CT-IMM get $\hat{\omega} \simeq 1.2913$ and $\hat{\omega} \simeq 1.3052$ respectively, they are both not far from



(a) CT-MM



(b) CT-IMM, switching



(c) CT-IMM, jumping Markov chain $c_T = 0.95$

Figure 2: Comparison of model probability evolution and error with plant $\omega = 3.30$

true value $\omega = 1.3$. When $\omega = 3.3$, we get $\hat{\omega} \simeq 3.5204$ for CT-MM and $\hat{\omega} \simeq 3.3275$ for CT-IMM. They are both acceptable, however, CT-IMM is a little preciser. To sum up, CT-IMM performs much better that CT-MM.

Concerning the computations of CT-MM and CT-IMM, they both run N Kalman filters and a differential-equation based probability updating. Moreover model mixer is much less time-consuming, therefore the efficiency CT-IMM is nearly equal to that of CT-MM.

(2). Switching vs. Mixing

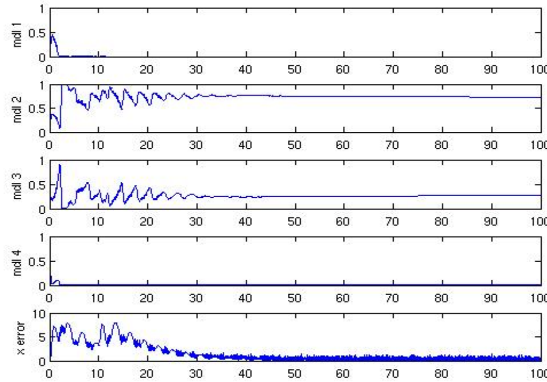
When the transition rate matrix is set specially as a identical matrix, which is also called mode switching and means that there is no model mixing among the models. The results are illustrated in Fig.2(b) and Fig.3(b). Contrast to mode mixing cases (Fig.2(c) and Fig.3(c)), they converge a lot faster than others. There is always only one model that wins all probability. Their errors, however, are larger and the parameter estimated is far from true value: $\hat{\omega} = 4.0$ for 3.30 and $\hat{\omega} = 1.0$ for 1.30.

4.2 Feedback controller

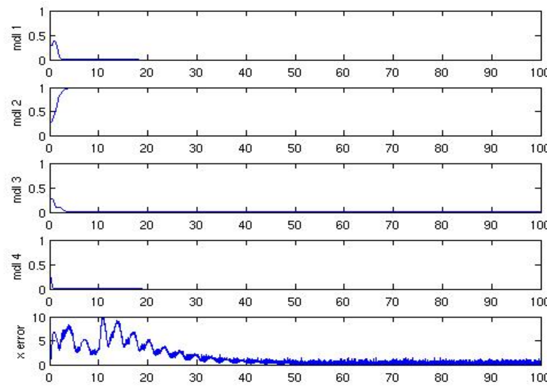
For simplicity, we still use the same state space model with plant $\omega = 1.3$, $Q_w = 0.01$, thus the gain K_{tf} is quite simple with 3 dimensions. The parameters Q and W in Eqn.(16a) are both identity matrix. A long enough time $t_f = 20s$ is used for upper limit of integration.

Without constraint, the optimal feedback gain always takes $K = -W^{-1}B^TP$, where P is the convergenced solution of Riccati equation $A'P + PA - PBW^{-1}B'P + Q = -\dot{P}$. As a result, $K^{no} = [0.5470, -0.2560, 1.5680]$. This result is adequate for infinite time control. For interval $[0, 20s]$, GA optimization yields another result $K_{tf}^{ga} = [1.1960, -0.2971, 1.8745]$, which is a little different from K^{no}. Considering the reliability constraint, we set a reliability lower limit as $\beta = 0.999$ in time interval $[0, 20s]$. The only failure mode presumed for this model is that the third dimension of state exceeds the limit $[\alpha_1, \alpha_2] = [-0.005, 0.005]$. Finally we achieve the new feedback gain $K_{tf}^{re} = [1.4472, -0.4791, 6.1858]$.

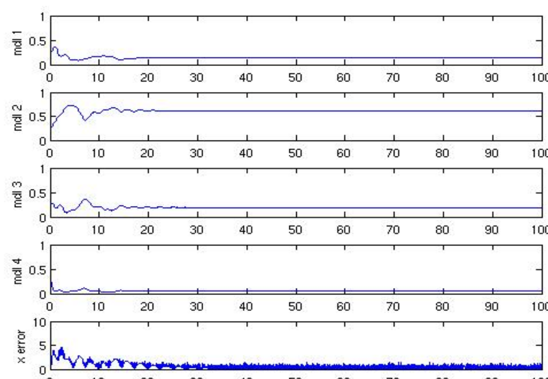
With these two feedback gains obtained by GA, we test two cases of input. First, active control with only random process noise is considered, see Fig.4(a) for response. Obviously, the threshold (α) exceeding frequently happens for the response without active control. Normally, the magnitude of optimal control with reliability constraint is smaller than that of without constraint. In other words, although the optimal control with reliability costs a little bit more than the optimal control without constraint ($Jr = 2.4641e-4$ vs. $Jo = 1.9301e-4$, they are mean value of 20 runs), it can guarantee the required reliability of system. Second, to check convergence speed more



(a) CT-MM



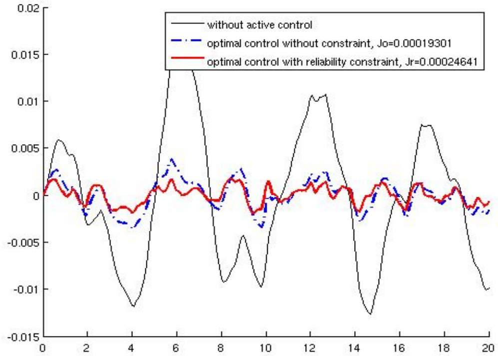
(b) CT-IMM, switching



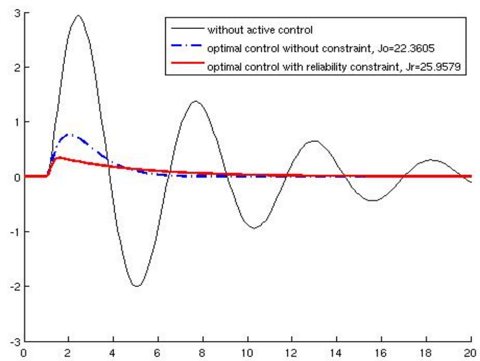
(c) CT-IMM, jumping Markov chain $c_T = 0.95$

Figure 3: Comparison of model probability evolution and error with plant $\omega = 1.30$

clearly, an additional step input ($u_0 = 3$ for time interval $[1, 20s]$) is imported to the process noise, see Fig.4(b) for response. Evidently, control with reliability constraint converge much slower than that without constraint, however, the magnitude is much smaller. Therefore, optimal



(a) input with random process noise



(b) input with a step signal

Figure 4: Response comparison between optimal control with reliability constraint and that without constraint.

control with reliability constraint is "safer".

5 Conclusion and future works

In this paper, a new reliable framework for non-LTI system control is proposed based on two crucial modules: a continuous-time based interacting multiple model observer and an optimal controller with reliability constraint. CT-IMM observer provides not only faster convergence of model fusion but also preciser estimation of parameters. Further, based on piecewise linear dynamic system estimated, an additional reliabilty constraint is introduced into the optimal LQG controller to assure system reliability. Illustrative examples demonstrated some interesting results and proved the effectivenesses of CT-IMM algorithm and the validity of reliability constraint.

However, GA is not applicable to real time running. Efficient optimization method needs more future investigation. In case of realistic application, such as active vehicle suspension, some explicit hazard functions of feedback gain probably exist. Then a close-form solution with reliability constraint will also be researched.

Appendix

A. Extension of Eqn.(8) and Eqn.(9)

$$
\begin{aligned}
p(x_t|M_t^{(i)},Z_{t-}) &= \sum_j p(x_t|M_t^{(i)},M_{t-}^{(j)},Z_{t-})p(M_{t-}^{(j)}|M_t^{(i)},Z_{t-}) \\
&\simeq \sum_j p(x_t|M_t^{(i)},M_{t-}^{(j)},\hat{x}_{t-}^{(i)})\cdot\frac{1}{c}p(M_t^{(i)}|M_{t-}^{(j)},Z_{t-})p(M_{t-}^{(j)}|Z_{t-}) \\
&= \sum_p (x_t|M_t^{(i)},M_{t-}^{(j)},\hat{x}_{t-}^{(i)})\cdot\frac{1}{c}\,T_{ji}\cdot\mu_{t-}^{(j)}.
\end{aligned}
$$

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